An overview of non-archimedean pluripotential theory and its outlook

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Contents

Global pluripotential structure

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- X: a compact Hausdorff space.
- $\mathcal{C}^+(X)$: the set of usc functions on X valued in $[-\infty, \infty)$.
- $\mathcal{M}(X)$: the set of Radon probability measures on X.

We call the following structure $(\mathcal{PSH}, E, E^\vee, \mathrm{MA})$ a global pluripotential (g.p.p.) structure on X. This terminology is used only in this talk.

- A convex subset $\mathcal{PSH} \sqcup \{-\infty\} \subset \mathcal{C}^+(X)$ which includes constants and is closed under pointwise decreasing limit.
- A monotone concave function $E : \mathcal{PSH} \to [-\infty, \infty)$ such that $E(t) = t$.
- A convex function $E^\vee : \mathcal{M} \to [0, \infty]$.

$$\mathcal{E}^1(X, E) := \{\phi \in \mathcal{PSH} \mid E(\phi) > -\infty\},$$
$$\mathcal{M}^1(X, E^\vee) := \{\mu \in \mathcal{M}(X) \mid E^\vee(\mu) < \infty\}.$$

- A map $\mathrm{MA} : \mathcal{E}^1(X, E)/\mathbb{R} \to \mathcal{M}^1(X, E^\vee)$.

Axiom

These $E, E^\vee$ are to satisfy the following (Legendre duality):

$$E^\vee(\mu) = \sup\{E(\phi) - \int_X \phi d\mu \mid \phi \in \mathcal{E}^1(X, E)\}$$
$$E(\phi) = \inf\{E^\vee(\mu) + \int_X \phi d\mu \mid \mu \in \mathcal{M}^1(X, E^\vee)\}$$

and MA is to satisfy the subdifferential relation

$$\int_X \phi \mathrm{MA}(\phi) = E(\phi) - E^\vee(\mathrm{MA}(\phi)).$$

- Topology on $\mathcal{M}^1(X, E^\vee)$: the coarsest topology which makes $E^\vee : \mathcal{M} \to [0, \infty)$ and $\mathcal{M}^1 \to \mathbb{R} : \mu \mapsto \int_X f d\mu$ continuous for every continuous $f \in C^0(X)$.
- Topology on $\mathcal{E}^1(X, E)$: the coarsest topology which makes $E : \mathcal{E}^1 \to \mathbb{R}$ and $\mathcal{E}^1 \to \mathbb{R} : \phi \mapsto \int_X \phi \mathrm{MA}(\psi)$ continuous for every $\psi \in \mathcal{PSH} \cap C^0(X)$. (cf. Chern–Levine–Nirenberg)

Basic properties

We will see three examples of g.p.p. structure: real MA, complex MA and non-archimedean MA.

Before going on, let us check some basic properties derived from the minimal assumptions in the first page.

1. For $\phi \in \mathcal{PSH}$, $\phi + t \in \mathcal{PSH}$.
2. If $\phi \in \mathcal{PSH}$ is bounded, then $\phi \in \mathcal{E}^1$.
3. $\lim_i E(\phi_i) = E(\phi)$ for every decreasing limit $\phi_i \searrow \phi \in \mathcal{PSH}$.
4. $\lim E(\phi_i) = E(\phi)$ for every uniform limit $\phi_i \to \phi \in \mathcal{PSH}$.
5. $\lim_i \int_X \phi_i d\mu = \int_X \phi d\mu$ for every decreasing limit $\phi_i \searrow \phi \in \mathcal{PSH}$.

In particular, decreasing limit $\phi_i \searrow \phi$ is a convergent net in $\mathcal{E}^1(X, E)$.
3 is a consequence of Dini's lemma (compactness).

Complex analytic case

How to define $\mathcal{PSH}$? – complex analytic case

$B^n \subset \mathbb{C}^n$: a (poly)disc

Definition

An usc function $\phi : B^n \to [-\infty, \infty) \not\equiv -\infty$ is called psh if $f^*\phi : B^1 \to [-\infty, \infty)$ is subharmonic (at $0 \in B^1$) or identically $-\infty$ for every holomorphic map $f : B^1 \to B^n$: namely,

$$\phi(f(0)) \leq \frac{1}{\pi r^2} \int_{B^1_r} \phi(f(z)) d\mu(z) \quad (\leq \frac{1}{2\pi r} \int_0^{2\pi} \phi(f(re^{i\theta})) d\theta)$$

for every $0 < r < 1$.

Note: This definition a priori uses the Lebesgue measure, which is a special structure we can use in the archimedean setup.

For a compact Kähler manifold $X = (\varphi_\alpha : B^n \to X)_\alpha$ and a Kähler metric $\omega = (dd^c f_\alpha)_\alpha$, we define

$$\mathrm{PSH}(X, \omega) := \{\phi : X \to [-\infty, \infty) \mid \phi \circ \varphi_\alpha + f_\alpha : B^n \to [-\infty, \infty) \text{ is psh } \}$$

Global regularization

Consider $\rho_t(z) := t^{-2}\rho(|z|/t)$ on $\mathbb{C}$ defined by $\rho : [0,\infty) \to [0,\infty)$: C^∞, $\rho|_{[0,0.5)} \equiv 1$, $\rho|_{[1,\infty)} \equiv 0$ with $2\pi \int_0^1 r\rho(r)dr = 1$.
Then for $\rho_t * \phi(z) = \int_{\mathbb{C}} \rho_t(z-w)\phi(w)d\mu(w)$,

$$\begin{aligned}\frac{1}{\pi r^2}\int_{B_r^1} \rho_t * \phi(z)d\mu(z) &= \int_{B_t^1} d\mu(w)\rho_t(w)\frac{1}{\pi r^2}\int_{B_r^1}\phi(z-w)d\mu(z)\\ &\geq \int_{B_t^1} d\mu(w)\rho_t(w)\phi(0-w) = \rho_t * \phi(0)\end{aligned}$$

and since $\frac{1}{2\pi r}\int_0^{2\pi} \phi(re^{i\theta})d\theta \searrow \phi(0)$ for psh,

$$\rho_t * \phi(0) = 2\pi \int_0^1 r\rho(r)\Big(\frac{1}{2\pi rt}\int_0^{2\pi}\phi(rte^{i\theta})d\theta\Big)dr \searrow \phi(0).$$

Theorem (Błocki–Kołdziej)

For every $\phi \in \mathrm{PSH}(X,\omega)$, there exists a sequence $\phi_i \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$ which decreases to ϕ pointwisely.

Approximation by Fubini–Study metrics

Let (X, L) be a polarized manifold. For smooth $\omega_\phi \in c_1(L)$, take the hermitian metric h_ϕ on L so that $-\sqrt{-1}\partial\bar{\partial}\log h_\phi = \omega_\phi$. Consider Kodaira embedding $f_k^\phi : X \to \mathbb{P}^{N_k}$ defined by an orthogonal basis $s_0^\phi, \ldots, s_{N_k}^\phi$ on $H^0(X, L^{\otimes k})$ w.r.t. the inner product induced by $h_\phi^{\otimes k}$. Then $\omega_{\phi,k} := k^{-1}(f_k^\phi)^*\omega_{\mathrm{FS}}$ is independent of the choice of the o.n.b. and

Theorem (Tian)

$\omega_{\phi,k}$ converges to ω_ϕ in the C^∞-topology (in the order $O(k^{-1})$).

Thus $\mathrm{PSH}(X, \omega)$ is the decreasing limit closure of

$$\mathrm{FS}(X, \omega) := \{\phi : X \to \mathbb{R} \mid \exists h \text{ s.t. } \phi = k^{-1}\log\sum_{i=0}^{N_k}\frac{|s_i|_{h_\omega}^2}{\|s_i\|_h^2}\},$$

where (s_i) is a orthogonal basis w.r.t. $\|\cdot\|_h$.
We will construct $\mathcal{PSH}$ on Berkovich space X^{NA} in analogy of this fact.

How to define $E : \mathrm{PSH}(X, \omega) \to [-\infty, \infty)$

For smooth $\phi \in \mathrm{PSH}(X, \omega) \cap C^\infty(X)$, we put

$$E(\phi) := -\frac{1}{(n+1)!} \int_0^1 dt \int_X (\omega_{\phi_t} - \dot{\phi}_t)^{n+1} = \frac{1}{n!} \int_0^1 dt \int_X \dot{\phi}_t \omega_{\phi_t}^n$$

by taking smooth path $\omega_{\phi_t} := \omega + dd^c \phi_t$ connecting ω and ω_ϕ.
For $\phi_t = t\phi$, we compute

$$E(\phi) = \sum_{k=0}^{n} \frac{\int_0^1 t^k dt}{(n-k)!k!} \int_X \phi\, \omega^{n-k} (dd^c \phi)^k = \frac{1}{(n+1)!} \sum_{\ell=0}^{n} \int_X \phi\, \omega^{n-\ell} \omega_\phi^\ell.$$

In particular, E is monotonic: $E(\phi) \le E(\psi)$ if $\phi \le \psi$.
For general $\phi \in \mathrm{PSH}(X, \omega)$, we put

$$E(\phi) := \inf\{E(\psi) \mid \psi \in \mathrm{PSH}(X, \omega) \cap C^\infty(X)\}.$$

The extended functional is monotonic and convex.

How to define $\mathrm{MA} : \mathcal{E}^1(X,\omega) \to \mathcal{M}(X)$?

Now we have

$$\mathcal{E}^1(X,\omega) := \{\phi \in \mathrm{PSH}(X,\omega) \mid E(\phi) > -\infty\}.$$

For $\phi \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$, we put

$$\mathrm{MA}(\phi) := \frac{1}{\int_X \omega^n}(\omega + dd^c\phi)^n.$$

For bounded $\phi \in \mathrm{PSH}(X,\omega)$, we can use Bedford–Taylor product:

$$\begin{aligned}\int_X \varphi \wedge (\omega + dd^c\phi)^{k+1} := &\int_X \varphi \wedge \omega \wedge (\omega + dd^c\phi)^k \\ &+ \int_X \phi dd^c\varphi \wedge (\omega + dd^c\phi)^k,\end{aligned}$$

using the fact that $(\omega + dd^c\phi)^k \geq 0$ gives a measure valued form. For $\phi \in \mathrm{PSH}(X,\omega)$, we take the limit $\mathrm{MA}(\phi) := \lim_{\tau\to\infty} \mathrm{MA}(\max\{\phi,\tau\})$.

Another approach: energy pairing, I

To define $\mathrm{MA}(\phi)$ for general $\phi \in \mathcal{E}^1(X, \omega)$, we introduce the following symmetric pairing: for $\phi_i \in \mathrm{PSH}(X, \omega_i) \cap C^\infty(X)$, we put

$$\langle \phi_0, \ldots, \phi_n \rangle_{\omega_0, \ldots, \omega_n} := -\int_0^1 dt \int_X \prod_{i=0}^n (\omega_{i,\phi_i(t)} - \dot{\phi}_i(t)).$$

Using $\phi_i(t) = t\phi_i$, we can compute

$$\langle \phi_0, \ldots, \phi_n \rangle = \sum_{\ell=0}^n \int_X \phi_i \omega_0 \wedge \cdots \widehat{\omega_\ell} \wedge \omega_{\ell+1, \phi_{\ell+1}} \wedge \cdots \wedge \omega_{n, \phi_n}.$$

In particular, $\langle \phi_0, \ldots, \phi_n \rangle$ is monotonic and affine in each variable. Therefore, we can define $\langle \phi_0, \ldots, \phi_n \rangle$ for $\phi_i \in \mathrm{PSH}(X, \omega_i)$ by putting

$$\langle \phi_0, \ldots, \phi_n \rangle := \inf\{\langle \psi_0, \ldots, \psi_n \rangle \mid \phi_i \leq \psi_i \in \mathrm{PSH}(X, \omega_i) \cap C^\infty(X)\}$$

Another approach: energy pairing, II

Now we consider the case $\omega_0 = \cdots = \omega_n = \omega$. By the previous formula, we have

$$\int_X (\phi_0 - \phi_0')\mathrm{MA}(\phi) = \frac{1}{\int_X \omega^n}(\langle \phi_0, \phi^{\cdot n}\rangle_{\omega^{\cdot n+1}} - \langle \phi_0', \phi^{\cdot n}\rangle_{\omega^{\cdot n+1}}).$$

for $\phi_0, \phi_0', \phi \in \mathrm{PSH}(X, \omega) \cap C^\infty(X)$.
We can define the difference for general $\phi_0, \phi_0', \phi \in \mathcal{E}^1(X, \omega)$ thanks to the following lemma:

Lemma

We have

$$\langle \phi_0, \ldots, \phi_n \rangle \geq \frac{1}{(n+1)!}\langle (\sum_{\ell=0}^n \phi_\ell)^{n+1}\rangle \geq (2^n n!)^n \sum_{\ell=0}^n E(\phi_\ell).$$

Apply Riesz representation theorem

The set $\mathcal{D} := \{\phi_0 - \phi_0' \mid \phi_0, \phi_0' \in \mathrm{PSH}(X, \omega) \cap C^\infty(C)\}$ is dense in $C^0(X)$. For $f = \phi_0 - \phi_0' \in \mathcal{D}$ and $\phi \in \mathcal{E}^1(X, \omega)$, we put

$$T_\phi(f) := \frac{1}{\int_X \omega^n}(\langle \phi_0, \phi^{\cdot n} \rangle - \langle \phi_0', \phi^{\cdot n} \rangle),$$

which is well defined by the affinity on the first variable.
Since $|T_\phi(f)| \leq \|f\|_{C^0}$ for $\phi \in \mathrm{PSH}(X, \omega) \cap C^\infty(X)$, the same inequality holds for general $\phi \in \mathcal{E}^1(X, \omega)$.
By the monotonicity, $T_\phi(f) \geq 0$ for $f \in \mathcal{D}$. Therefore, T_ϕ extends to a positive bounded linear map $C^0(X) \to \mathbb{R}$.
By Riesz–Markov–Kakutani theorem, there exists a Radon measure μ_ϕ on X such that

$$T_\phi(f) = \int_X f d\mu_\phi.$$

We denote it by $\mathrm{MA}_\omega(\phi)$.

MA is subdifferential

We put $E_\natural(\phi) := (\int_X \omega^n/n!)^{-1} E(\phi)$. For $\mu \in \mathcal{M}(X)$, we put

$$E^\vee(\mu) := \sup\{E_\natural(\phi) - \int_X \phi d\mu \mid \phi \in \mathcal{E}^1(X,\omega)\} \in [0,\infty],$$

$$\mathcal{M}^1(X,\omega) := \{\mu \in \mathcal{M}(X) \mid E^\vee(\mu) < \infty\}.$$

We have

$$E_\natural(\psi) - \int_X \psi \mathrm{MA}_\omega(\phi) \leq E_\natural(\phi) - \int_X \phi \mathrm{MA}_\omega(\phi)$$

For instance when $n = 1$,

$$\begin{aligned}\Big(\frac{1}{2}\langle\psi,\psi\rangle - (\langle\psi,\phi\rangle - \langle 0,\phi\rangle)\Big) - \Big(\frac{1}{2}\langle\phi,\phi\rangle - (\langle\phi,\phi\rangle - \langle 0,\phi\rangle)\Big) \\ = \frac{1}{2}\langle(\psi-\phi),(\psi-\phi)\rangle \\ = \int_X (\psi-\phi) dd^c(\psi-\phi) = -\int_X d(\psi-\phi)\wedge d^c(\psi-\phi) \leq 0.\end{aligned}$$

Finally, we get ...

Therefore, we get

$$\int_X \phi \mathrm{MA}_\omega(\phi) = E(\phi) - E^\vee(\mathrm{MA}_\omega(\phi)).$$

Since

$$\int_X \phi \mathrm{MA}_\omega(\phi) = \frac{1}{\int_X \omega^n}(\langle \phi, \dots, \phi \rangle - \langle 0, \phi, \dots, \phi \rangle) > -\infty,$$

we have $\mathrm{MA}_\omega(\phi) \in \mathcal{M}^1(X, \omega)$.

In conclusion, for every Kähler metric ω on X, $(\mathrm{PSH}(X, \omega), E_\natural, E^\vee, \mathrm{MA}_\omega)$ is a g.p.p. structure on X.

pluripotential Calabi–Yau theorem

A g.p.p. structure $(\mathcal{PSH}, E, E^\vee, \mathrm{MA})$ on X is said to satisfy pluripotential Calabi–Yau theorem if the map

$$\mathrm{MA} : \mathcal{E}^1(X, E)/\mathbb{R} \to \mathcal{M}^1(X, E^\vee)$$

is a homemomorphism.

Theorem

For every compact Kähler manifold (X, ω), the g.p.p. structure $(\mathrm{PSH}(X, \omega), E_\natural, E^\vee, \mathrm{MA}_\omega)$ satisfy pluripotential Calabi–Yau theorem.

Injectivity and continuity: Introduce quasi-distances $I, I^\vee$ on $\mathcal{E}^1(X, \omega)$ and $\mathcal{M}^1(X, \omega)$ compatible with the topology and show

$$I(\varphi, \psi) \approx I^\vee(\mathrm{MA}_\omega(\varphi), \mathrm{MA}_\omega(\psi)).$$

Surjectivity: Find maximizer of $\phi \mapsto E(\phi) - \int_X \phi d\mu$, using the compactness of $\mathrm{PSH}(X, \omega)/\mathbb{R}$.

Use in the study of cscK metric

Extension of Mabuchi functional

Consider the Mabuchi functional defined on $\mathrm{PSH}(X,\omega) \cap C^\infty(X)$:

$$\mathcal{M}(\phi) := -\frac{1}{\int_X \omega^n} \int_0^1 dt \int_X \dot{\phi}_t (s(\omega_{\phi_t}) - \underline{s}) \omega_{\phi_t}^n.$$

Using
$s(\omega_\phi)\omega_\phi^n = n\mathrm{Ric}(\omega_\phi) \wedge \omega_\phi^{n-1} = n\sqrt{-1}\bar{\partial}\partial \log \frac{\omega_\phi^n}{\omega^n} \wedge \omega_\phi^{n-1} + n\mathrm{Ric}(\omega) \wedge \omega_\phi^{n-1}$,
we get Chen–Tian formula:

$$\begin{aligned}\mathcal{M}(\phi) = &\int_X \frac{\mathrm{MA}_\omega(\phi)}{\mathrm{MA}_\omega(0)} \log \frac{\mathrm{MA}_\omega(\phi)}{\mathrm{MA}_\omega(0)} \mathrm{MA}_\omega(0) \\ &- \frac{1}{\int_X \omega^n} (\langle 0.\phi^{\cdot n} \rangle_{\rho_+,\omega^{\cdot n}} - \langle 0, \phi^{\cdot n} \rangle_{\rho_-,\omega^{\cdot n}}) + \underline{s} E_\natural(\phi),\end{aligned}$$

where smooth Kähler metrics ρ_+, ρ_- are chosen so that
$\rho_+ - \rho_- = \mathrm{Ric}(\omega)$.

Entropy and Compactness

The first term

$$\int_X \frac{\mathrm{MA}_\omega(\phi)}{\mathrm{MA}_\omega(0)} \log \frac{\mathrm{MA}_\omega(\phi)}{\mathrm{MA}_\omega(0)} \mathrm{MA}_\omega(0)$$

is called the relative entropy and denoted by $\mathrm{Ent}(\mathrm{MA}_\omega(0), \mathrm{MA}_\omega(\phi))$.

The relative entropy $\mathrm{Ent}(\mu, \nu)$ is defined for general $\mu, \nu \in \mathcal{M}(X)$ using the Radon–Nikodym derivative $d\nu/d\mu$ when ν is absolutely continuous w.r.t. μ and put $\mathrm{Ent}(\mu, \nu) = +\infty$ otherwise. This is lower semi-continuous w.r.t. the topology on $\mathcal{M}^1(X, \omega)$ (Legendre duality).

Now we get the (greatest) lsc extension

$$\mathcal{M} : \mathcal{E}^1(X, \omega) \to (-\infty, \infty].$$

Theorem (Berman–Boucksom–Eyssidieux–Guedj–Zeriahi)

Suppose a sequence $\{\phi_i\}$ has bounded $\sup \phi$, $E(\phi)$ and $\mathrm{Ent}(\mathrm{MA}_\omega(0), \mathrm{MA}_\omega(\phi))$. Then $\{\phi_i\}$ contains a convergent subsequence in the topology of $\mathcal{E}^1(X, \omega)$.

Darvas–Rubinstein's result

Conjecture (Darvas–Rubinstein)

The minimizers of $\mathcal{M} : \mathcal{E}^1(X, \omega) \to (-\infty, \infty]$ are smooth.

Theorem (Darvas–Rubinstein)

Suppose the conjecture holds, then the following are equivalent:

1. X admits a cscK metric.
2. $\mathcal{M}$ is $G = \mathrm{Aut}(X)$-invariant and there exists $\sigma, C > 0$ such that

$$\mathcal{M}(\phi) \geq \sigma \inf_{g \in G} J(g^* \phi) - C,$$

where

$$J(\phi) = \int_X \phi \mathrm{MA}_\omega(0) - E_\natural(\phi) \geq 0.$$

Chen–Cheng's result

Darvas–Rubinstein's conjecture is solved by Chen–Cheng:

Theorem (Chen–Cheng)

The minimizers of $\mathcal{M} : \mathcal{E}^1(X, \omega) \to (-\infty, \infty]$ are smooth.

Darvas introduced a complete distance d_1 on $\mathcal{E}^1(X, \omega)$. A continuous path $\Phi : [0, \tau) \to \mathcal{E}^1(X, \omega)$ is called a psh geodesic if $\Phi(e^{-\log |z|})(x)$ on $\bar{\Delta} \times X$ is psh in the interior and $E(\Phi(t))$ is affine, which is known to satisfy $d_1(\Phi(t), \Phi(t')) = c|t - t'|$. It is known by Berman–Darvas–Lu that $\mathcal{M} : \mathcal{E}^1(X, \omega) \to (-\infty, \infty]$ is convex along psh geodesics.

Theorem (Chen–Cheng)

The following are equivalent

- There are no cscK metrics.
- There is a psh geodesic ray $\Phi : [0, \infty) \to \mathcal{E}^1(X, \omega)$ with $\lim_{t \to \infty} t^{-1} \mathcal{M}(\Phi(t)) \leq 0$ which is not parallel to a geodesic ray generated by a holomorphic vector field X.

Non-archimedean (trivially valued) case

Semivaluations

Let X^{sch} be an irreducible projective scheme over $\mathbb{C}$.
For each scheme theoretic point $\mathfrak{p} \in X$, consider the irreducible subvariety $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ and its rational function field $\mathbb{C}(V(\mathfrak{p}))$. We put

$$X(\mathfrak{p}) := \left\{ v : \mathbb{C}(V(\mathfrak{p})) \to (-\infty, \infty] \;\middle|\; \begin{array}{c} v(fg) = v(f) + v(g) \\ v(f+g) \geq \min\{v(f), v(g)\} \\ v|_{\mathbb{C}^*} \equiv 0, v(0) = \infty \end{array} \right\}$$

and for a Zariski open set $U^{\mathrm{sch}} \subset X^{\mathrm{sch}}$ we put

$$U^{\mathrm{NA}} := \bigcup_{\mathfrak{p} \in U^{\mathrm{sch}}} X(\mathfrak{p}).$$

For $f \in \mathcal{O}_{X^{\mathrm{sch}}}(U^{\mathrm{sch}})$, we define

$$|f| : U^{\mathrm{NA}} \to [0, \infty)$$

by $|f|(v) := \exp(-v_{\mathfrak{p}}(f|_{V(\mathfrak{p})}))$.

Berkovich space

We then endow X^{NA} with the coarsest topology for which the maps $\mathfrak{home} : X^{\mathrm{NA}} \to X^{\mathrm{sch}} : v_{\mathfrak{p}} \mapsto \mathfrak{p}$ and $|f| : U^{\mathrm{NA}} \to [0, \infty)$ are continuous. This is the Berkovich space.

The Berkovich space X^{NA} is a compact Hausdorff space (not first countable).

Non-archimedean norm and diagonal basis

Consider a non-archimedean norm $\|\cdot\|$ on a fin. dim vector space V over the trivially valued $\mathbb{C}$: $\|\cdot\| : V \to [0, \infty)$ which satisfy

- $\|u + v\| \leq \|u\| \oplus \|v\| := \max\{\|u\|, \|v\|\}$,
- $\|c.v\| = |c|_{\mathrm{triv}} \|v\|$, where

$$|c|_{\mathrm{triv}} := \begin{cases} 0 & c = 0 \\ 1 & c \neq 0 \end{cases} \quad \text{(trivially valued)}$$

It is known there exists a basis $v_1, \ldots v_n$ of V satisfying

$$\|\sum_{i=1}^{n} c_i v_i\| = \bigoplus_{i=1}^{n} |c_i| \|v_i\| = \max\{\|v_i\| \mid c_i \neq 0\},$$

which we call a diagonal basis w.r.t. $\|\cdot\|$.

Non-archimedean L-psh

Let (X, L) be a polarized variety. For a non-archimedean norm $\| \cdot \|$ on $H^0(X, L^{\otimes k})$, we put

$$\phi_{\|\cdot\|} := k^{-1} \log \bigoplus_i |s_i|/\|s_i\| = k^{-1} \max_i (\log |s_i| - \log \|s_i\|),$$

using the diagonal basis (s_i) of $H^0(X, L^{\otimes k})$.
We introduce the following space of functions

$$\mathrm{FS}_{\mathrm{NA}}(X, L) := \left\{ \phi_{\|\cdot\|} : X^{\mathrm{NA}} \to \mathbb{R} \;\middle|\; \exists k \|\cdot\| \text{ is a NA norm on } H^0(X, L^{\otimes k}) \right\}$$

and define $\mathrm{PSH}_{\mathrm{NA}}(X, L)$:

Definition

We define $\mathrm{PSH}_{\mathrm{NA}}(X, L)$ as the set of all usc functions $\phi : X^{\mathrm{NA}} \to [-\infty, \infty)$ which can be written as the pointwise limit $\phi = \lim_i \phi_i$ of some decreasing net ϕ_i in $\mathrm{FS}_{\mathrm{NA}}(X, L)$.

Next, to define non-archimedean Monge–Ampere measure, we relate $\phi \in \mathrm{FS}_{\mathrm{NA}}(X, L)$ with a geometric object.

Polyhedral configuration

We put $B_p := \mathbb{C}^p$, $B_p^\circ := (\mathbb{C} \setminus \{0\})^p$ and $T_p := (\mathbb{G}_m)^p$
A polyhedral configuration of (X, L) is a triple $(\mathcal{X}/B_p, \mathcal{L}; \xi)$:

- $\mathcal{X}/B_p = (\varpi : \mathcal{X} \to B_p)$ is a T-equivariant projective flat family of projective schemes endowed with the T-equivariant isomorphism $\Theta : B_p^\circ \times X \to \varpi^{-1}(B_p^\circ)$ over B_p°.
- $\mathcal{L}$ is a T-equivariant ϖ-semiample $\mathbb{Q}$-line bundle on $\mathcal{X}$ endowed with a T-equivariant isomorphism $\theta : p_X^* L \to \Theta^* \mathcal{L}$, where $p_X \to B_p^\circ \times X \to X$ is the projection.
- $\xi \in [0, \infty)^p$.

For a polyhedral configuration, we define a norm $\| \cdot \|_k$ on $H^0(X, L^{\otimes k})$ by

$$\|s\|_k := \inf\{e^{-\langle \mu, \xi \rangle} \mid z^{-\mu} \bar{s} \text{ extends to a section of } \mathcal{L}^{\otimes k}\},$$

where $\mu \in \mathbb{Z}^p$. We have $\phi_{\|\cdot\|_k} = \phi_{\|\cdot\|'_k}$ for sufficiently large k, k'.
The assignment

$$\mathrm{PC}(X, L) \to \mathrm{FS}_{\mathrm{NA}}(X, L) : (\mathcal{X}/B_p, \mathcal{L}; \xi) \mapsto \phi_{(\mathcal{X}/B_p, \mathcal{L}; \xi)} := \phi_{\|\cdot\|_k}$$

is onto.

Non-archimedean energy pairing (Deligne pairing)

A test configuration $(\mathcal{X}/\mathbb{A}^1, \mathcal{L})$ is nothing but the polyhedral configuration for $p = 1$ and $\xi = 1 \in [0, \infty)$. Namely, it consists of

- $\mathcal{X}/\mathbb{A}^1 = (\varpi : \mathcal{X} \to \mathbb{A}^1)$ is a T-equivariant projective flat family of projective schemes endowed with the T-equivariant isomorphism $\Theta : (\mathbb{A}^1 \setminus \{0\}) \times X \to \varpi^{-1}(\mathbb{A}^1 \setminus \{0\})$ over $\mathbb{A}^1 \setminus \{0\}$.
- $\mathcal{L}$ is a $\mathbb{G}_m$-equivariant ϖ-semiample $\mathbb{Q}$-line bundle on $\mathcal{X}$ endowed with a $\mathbb{G}_m$-equivariant isomorphism $\theta : p_X^* L \to \Theta^* \mathcal{L}$.

For $\phi_\ell \in \mathrm{FS}_{\mathrm{NA}}(X, L_\ell)$ defined by test configurations ($\Leftrightarrow \log \|s_i\| \in \mathbb{Q}$), the non-archimedean energy pairing can be written by the usual intersection product

$$\langle \phi_0, \ldots, \phi_n \rangle = (\bar{\mathcal{L}}_0 . \cdots . \bar{\mathcal{L}}_n)$$

on the compactfication $\overline{\mathcal{X}}/\mathbb{P}^1$ of a tc $\mathcal{X}$ dominating every $\mathcal{X}_\ell$.

g.p.p. structure $(\mathcal{PSH}, E, E^\vee, \mathrm{MA})$ on X^{NA}, I

We have $\phi \leq \psi$ iff $\mathcal{L}_\psi - \mathcal{L} \geq 0$, so that the energy pairing is monotonic. Using the monotonicity, we can define the energy pairing for general $\phi_\ell \in \mathrm{PSH}(X, L_\ell)$.

We put

$$E_{\mathrm{NA}}(\phi) := \frac{n!}{(L^{\cdot n})} \frac{1}{(n+1)!} \langle \phi^{\cdot n+1} \rangle_{L^{\cdot n+1}}$$

and

$$\mathcal{E}^1_{\mathrm{NA}}(X, L) := \{\phi \in \mathrm{PSH}(X, L) \mid E_{\mathrm{NA}}(\phi) > -\infty\}.$$

The energy pairing $\langle \phi_0, \ldots, \phi_n \rangle$ is finite if $\phi_\ell \in \mathcal{E}^1(X, L)$.

g.p.p. structure $(\mathcal{PSH}, E, E^\vee, \mathrm{MA})$ on X^{NA}, II

We put $\mathcal{D} := \{\phi_0 - \phi_0' \mid \phi_0 - \phi_0' \in \mathrm{FS}_{\mathrm{NA}}(X, L)\}$, then $\mathcal{D}$ is dense in $C^0(X^{\mathrm{NA}})$. The operator $T_\phi : \mathcal{D} \to \mathbb{R}$ given by

$$T_\phi(f) := \langle \phi_0 . \phi^{\cdot n} \rangle - \langle \phi_0' . \phi^{\cdot n} \rangle$$

extends to a positive bounded linear map $T_\phi : C^0(X^{\mathrm{NA}}) \to \mathbb{R}$. Thus we get a measure μ_ϕ satisfying

$$T_\phi(f) = \int_X f d\mu_\phi.$$

We denote it by $\mathrm{MA}_{\mathrm{NA}}(\phi)$.

We can similarly define $E^\vee_{\mathrm{NA}}$ by Legendre duality and $\mathcal{M}^1_{\mathrm{NA}}(X, L)$. Then

$$(\mathrm{PSH}_{\mathrm{NA}}(X, L), E_{\mathrm{NA}}, E^\vee_{\mathrm{NA}}, \mathrm{MA}_{\mathrm{NA}})$$

gives the g.p.p. structure on X^{NA}.

Non-archimedean Calabi–Yau theorem

Theorem (Boucksom–Jonsoon)

When X is smooth, the g.p.p structure $(\mathrm{PSH}_{\mathrm{NA}}(X, L), E_{\mathrm{NA}}, E^{\vee}_{\mathrm{NA}}, \mathrm{MA}_{\mathrm{NA}})$ on X^{NA} satisfies the Calabi–Yau theorem.

K-stability

K-stability

Suppose X is klt. There is a (greatest) lsc functional $A_X : X^{\mathrm{NA}} \to (-\infty, \infty]$ called log discrepancy: for $x = c.\mathrm{ord}_E \in X^{\mathrm{NA}}$ given by $E \subset Y \to X$, we put

$$A_X(x) := c(\mathrm{ord}_E(K_Y/K_X) + 1).$$

The non-archimedean Mabuchi invariant $\mathcal{M}_{\mathrm{NA}}$ for a tc $\phi \in \mathrm{FS}^{\mathbb{Q}}_{\mathrm{NA}}(X, L)$ is defined by

$$\int_{X^{\mathrm{NA}}} A_X \mathrm{MA}_{\mathrm{NA}}(\phi) + \frac{1}{(L^{\cdot n})}(p_X^* K_X.\bar{\mathcal{L}}^{\cdot n}) - \frac{n(K_X.L^{\cdot n-1})}{(L^{\cdot n})} E_{\mathrm{NA}}(\phi).$$

Take an ample line bundle ρ_+, ρ_- on X so that $\rho_+ - \rho_- = -p_X^* K_X$, then we can write

$$(p_X^* K_X.\bar{\mathcal{L}}^{\cdot n}) = -(\langle 0, \phi^{\cdot n}\rangle_{\rho_+} - \langle 0, \phi^{\cdot n}\rangle_{\rho_-}).$$

Therefore, we get the lsc extension

$$\mathcal{M}_{\mathrm{NA}} : \mathcal{E}^1_{\mathrm{NA}}(X, L) \to (-\infty, \infty].$$

Chi–Li's result, I

Some psh geodesic rays are induced by non-archimedean psh metrics. Such geodesic ray is called maximal.

Theorem (Chi Li)

If a psh geodesic ray $\Phi : [0, \infty) \to \mathcal{E}^1(X, \omega)$ satisfies $\lim_{t\to\infty} t^{-1}\mathcal{M}(\Phi(t)) < \infty$, it is maximal.

Chi Li's result, II

Theorem (Chi Li)

When $\Phi : [0, \infty) \to \mathcal{E}^1(X, \omega)$ is a geodesic ray generated by a test configuration ϕ, then we have

$$\lim_{t \to \infty} t^{-1} \mathcal{M}(\Phi(t)) = \mathcal{M}_{\mathrm{NA}}(\phi).$$

Conjecture

For any $\phi \in \mathcal{E}^1_{\mathrm{NA}}(X, L)$, there exists a convergent sequence $\phi_i \to \phi$ in $\mathcal{E}^1_{\mathrm{NA}}(X, L)$ with ϕ_i given by test configurations such that $\lim_{i \to \infty} \mathcal{M}(\phi_i) = \mathcal{M}(\phi)$.

If the conjecture is true, then the above formula is known to hold for all maximal geodesic rays. It then follows that $\mathcal{M}_{\mathrm{NA}}(\phi) > 0$ for every non-trivial $\phi \in \mathcal{E}^1_{\mathrm{NA}}(X, L)$ (the uniform K-stability w.r.t. test configurations) implies the existence of (unique) cscK metric.

A comment on my recent work, I

The NA Monge–Ampere measure can express intersection numbers.

My recent work constructs a functional $E_{\mathsf{exp}} : \mathrm{PSH}_{\mathrm{NA}}(X, L) \to [-\infty, \infty)$ and a measure $\int e^{-t}\mathcal{D}_\phi$ on X^{NA} for $\phi \in \mathrm{PSH}_{\mathrm{NA}}(X, L)$ with $E_{\mathsf{exp}}(\phi) > -\infty$, which can express higher equivariant intersections.

Using this, we can introduce a usc functional $\mu_{\mathrm{NA}} : \mathcal{E}^{\mathsf{exp}}_{\mathrm{NA}}(X, L) \to [-\infty, \infty)$ called non-archimedean μ-entropy, which is a 'dual' of Perelman's μ-entropy.

The existence of maximizer of this functional is my current interest, which gives a destabilizing geodesic ray. The question is important in moduli theory of polarized varieties.

A comment on my recent work, II

On toric variety, we can consider torus invariant subspace $\mathcal{E}^{\mathrm{exp}}_{\mathrm{NA}}(X, L)^T$ and we can identify it with the space of lsc convex functions on a polytope P with the integrability condition $\int_P e^q d\mu < \infty$. For such q, μ_{NA} can be written as

$$\mu_{\mathrm{NA}}(q) = -2\pi \frac{\int_{\partial P} e^q d\sigma}{\int_P e^q d\mu}.$$

Theorem

There exists a maximizer q of μ_{NA}. When P is 2-dimensional, it is bounded continuous.

Thank you!