Introduction to Optimal degeneration problems

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I want to explain three analogous frameworks on optimal degeneration

1. The framework of extremal metric
2. The framework of Kähler–Ricci soliton
3. The framework of μ-cscK metric

From K-stability viewpoint, all of these are covered in Lahdili's weighted cscK framework.

Today, we focus K-instability aspect, which cannot be discussed in weighted cscK framework (too general).

Canonical metric of space

Let X be a compact Kähler manifold and L be a Kähler class.

$$\mathcal{H}(X, L) := \{\omega \in L \mid \omega > 0\}$$

Let $s(\omega) = -g^{i\bar{j}}\partial_i\partial_{\bar{j}} \log \det g$ be the scalar curvature. We put

$$\hat{s}(\omega) = s(\omega) - \frac{\int_X s(\omega)\omega^n}{\int_X \omega^n}.$$

A Kähler metric ω is called a cscK metric if

$$\hat{s}(\omega) = 0.$$

Uniqueness: any two cscK metrics ω, ω' in the same Kähler class L is isometric by some holomorphic map $f : X \to X$.

Conjecture (Yau–Tian–Donaldson conjecture)

Given (X, L), the existence of a cscK metric in $\mathcal{H}(X, L)$ is equivalent to K-stability of (X, L).

Degeneration of space

A pair $(\mathcal{X}, \mathcal{L})$ is called a test configuration if

1. $\mathcal{X}$ is a scheme/complex analytic space flat over $\mathbb{A}^1$ endowed with a $\mathbb{G}_m$-action compatible with the scale action on $\mathbb{A}^1$.
2. $\mathbb{G}_m$-equivariant isomorphism $X \times \mathbb{G}_m \cong \mathcal{X}^* = \mathcal{X} \setminus \mathcal{X}_0$ away from the fibre $\mathcal{X}_0$ over $0 \in \mathbb{A}^1$.
3. $\mathcal{L} \in H^2_{\mathbb{G}_m}(\mathcal{X}, \mathbb{R})$ is a $\mathbb{G}_m$-equivariant relatively ample cohomology class whose restriction to $\mathcal{X}^*$ coincides with $p_X^* L$ as $\mathbb{G}_m$-equivariant class via the above isomorphism.

Canonical compactification using $\mathbb{G}_m$-action: $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$
We put

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := (K_{\bar{\mathcal{X}}/\mathbb{P}^1}.\bar{\mathcal{L}}^n) - \frac{n}{n+1}\frac{(K_X.L^{n-1})}{(L^n)}(\bar{\mathcal{L}}^{n+1}).$$

(X, L) is called K-semistable if $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$ for every test configuration.

Question

It is known that if (X, L) admits a cscK metric, then it is K-semistable (more strongly K-polystable).

In view of GIT analogue, K-semistable varieties are believed to form an Artin moduli stack which admits a good moduli space of K-polystable varieties.

... Good!

Question: What we can say for K-unstable (X, L)?

normalized base change

We want to formulate the notion of “test configuration with the least Donaldson–Futaki invariant”.

For a normalized base change $\mathcal{X}_d$ of $\mathcal{X}$ along $z^d : \mathbb{A}^1 \to \mathbb{A}^1$, we have

$$\mathrm{DF}(\mathcal{X}_d, \mathcal{L}_d) \leq d.\mathrm{DF}(\mathcal{X}, \mathcal{L})$$

and

$$d^{-1}\mathrm{DF}(\mathcal{X}_d, \mathcal{L}_d) = (d')^{-1}\mathrm{DF}(\mathcal{X}_{d'}, \mathcal{L}_{d'}) =: M(\mathcal{X}, \mathcal{L})$$

for sufficiently divisible $d, d' \in \mathbb{N}$.

When $M(\mathcal{X}, \mathcal{L}) < 0$, $M(\mathcal{X}_d, \mathcal{L}_d) = d.M(\mathcal{X}, \mathcal{L})$ gets arbitrary small by taking $d \to \infty$.

$\rightsquigarrow$ Want to define a “norm” $\|(\mathcal{X}, \mathcal{L})\|$ to normalize $M(\mathcal{X}, \mathcal{L})$.

normalized Donaldson–Futaki invariant

We consider the weak limit of measures on $\mathbb{R}$:

$$\mathrm{DH}_{(\mathcal{X},\mathcal{L})} := \lim_{m\to\infty} \frac{1}{m^n} \sum_{\lambda\in\mathbb{Z}} \dim H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes m})_\lambda . \delta_{\lambda/m}.$$

We put

$$E(\mathcal{X},\mathcal{L}) := \int_{\mathbb{R}} t\mathrm{DH}_{(\mathcal{X},\mathcal{L})}(t) / \int_{\mathbb{R}} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}$$

and use

$$\|(\mathcal{X},\mathcal{L})\| := \left(\int_{\mathbb{R}} (t - E(\mathcal{X},\mathcal{L}))^2 \mathrm{DH}_{(\mathcal{X},\mathcal{L})} \right)^{1/2}$$

as the norm. It satisfies $\|(\mathcal{X},\mathcal{L})\| \geq 0$ and

$$\|(\mathcal{X}_d,\mathcal{L}_d)\| = d.\|(\mathcal{X},\mathcal{L})\|.$$

Optimal degeneration problem

Show the existence and the uniqueness of the minimizer of $M(\mathcal{X},\mathcal{L})/\|(\mathcal{X},\mathcal{L})\|$ modulo scaling.

Donaldson's inequality

The Calabi energy $Ca : \mathcal{H}(X, L) \to \mathbb{R}$

$$Ca(\omega) = \int_X \hat{s}(\omega)^2 \omega^n.$$

Theorem (Donaldson '05)

$$\left(\frac{2\pi M(\mathcal{X}, \mathcal{L})}{\|(\mathcal{X}, \mathcal{L})\|} \right)^2 \leq \inf_{\omega \in \mathcal{H}(X,L)} Ca(\omega)$$

for any $(\mathcal{X}, \mathcal{L})$ with $M(\mathcal{X}, \mathcal{L}) \leq 0$.

Minimizer of Ca is extremal metric:

$$\partial^{\#} s(\omega)$$

is holomorphic.

Extension

It is NOT known if there exists a tc with minimal $M(\mathcal{X},\mathcal{L})/\|(\mathcal{X},\mathcal{L})\|$.
We can find a minimizer on a larger space $\mathcal{E}^2_{\mathrm{NA}} \supset \{$ tc $\}$ (Xia '21).

There are two ways to extend

$$\hat{\mu}^{\mathrm{ext}}(\mathcal{X},\mathcal{L}) := \begin{cases} \left(\frac{2\pi M(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|}\right)^2 & M(\mathcal{X},\mathcal{L}) \leq 0 \\ 0 & M(\mathcal{X},\mathcal{L}) > 0 \end{cases}$$

to $\mathcal{E}^2_{\mathrm{NA}}$:

1. use geometric pluripotential theory $\hat{\mu}^{\mathrm{ext}}_{\mathrm{geo}}$ or
2. use non-archimedean pluripotential theory $\hat{\mu}^{\mathrm{ext}}_{\mathrm{NA}}$.

These are expected to coincide ($=$ regularization of NA entropy).

Similarly, there exists an extension of $Ca : \mathcal{H}(X,\omega) \to \mathbb{R}$ to a larger space $\mathcal{E}^2 \supset \mathcal{H}(X,\omega)$.

Xia's result

The Calabi flow

$$\frac{d}{dt}\omega_t = \sqrt{-1}\partial\bar{\partial}s(\omega_t)$$

decreases the Calabi energy.

Long time existence of smooth solution is not known, but there exists $[0,\infty) \to \mathcal{E}^2$ which can be regarded as Calabi flow.

Using this and compactness on $\mathcal{E}^2$, Xia proved the following.

Theorem (Xia '21)

There exists $\phi_{\mathrm{ext}} \in \mathcal{E}^2_{\mathrm{NA}}$ which maximizes $\hat{\mu}^{\mathrm{ext}}_{\mathrm{geo}}$ and we have

$$\max_{\phi\in\mathcal{E}^2_{\mathrm{NA}}} \hat{\mu}^{\mathrm{ext}}_{\mathrm{geo}}(\phi) = \inf_{\varphi\in\mathcal{E}^2} Ca(\varphi).$$

Li–Lian–Sheng: Toric ϕ_{ext} is bounded $\rightsquigarrow$ filtration on $\bigoplus_m H^0(X, L^{\otimes m})$.

Framework of Kähler–Ricci soliton

$$\hat{\mu}^{\mathrm{ext}}(\mathcal{X},\mathcal{L}) = \max_{\rho\geq 0} \mu^{\mathrm{ext}}(\mathcal{X},\mathcal{L};\rho) := \max_{\rho\geq 0}(-4\pi M(\mathcal{X},\mathcal{L}).\rho - \|(\mathcal{X},\mathcal{L})\|^2.\rho^2)$$

For $(X,L) = (X,-2\pi K_X)$ is a Fano manifold, there is another framework on optimal degeneration: Framework of Kähler–Ricci soliton.

$$H(\mathcal{X},\mathcal{L};\rho) = -L(\mathcal{X},\mathcal{L}).\rho - \log \int_{\mathbb{R}} e^{\rho.t}\mathrm{DH}_{(\mathcal{X},\mathcal{L})}(t)$$

μ^{ext}	H: H-entropy
Ca: Calabi energy	He: He entropy
Extremal metric	Kähler–Ricci soliton
Calabi flow	Kähler–Ricci flow

Geometric property of optimal degeneration

A good thing in the framework of Kähler–Ricci soliton is that the optimal degeneration has a geometric realization (cf. Chen–Sun–Wang, Dervan–Székelyhidi, Han–Li, Blum–Liu–Xu–Zhuang):

$$\mathcal{X} \to \mathbb{A}^k \circlearrowleft (\mathbb{G}_m)^k \text{ with } \xi \in [0, \infty)^k.$$

Theorem (Han–Li '20)

The central fibre $\mathcal{X}_0$ is a $\mathbb{Q}$-Fano variety which is modified K-semistable with respect to the vector field generated by ξ.

Existence of Kähler–Ricci soliton $\Rightarrow$ modified K-semistable

Optimal degeneration is associated to the following filtration:

$$\mathcal{F}_m^\lambda = \{s \in H^0(X, mL) \mid \lim_{t \to \infty} t^{-1} \log \|s\|_{g(t)} \leq \lambda\},$$

where $g(t)$ is a Kähler–Ricci flow.

Framework of μ-cscK metric

There is a family of similar framework parametrized by $\lambda \in \mathbb{R}$:

μ^λ: μ-entropy
$\mu^\lambda_{\mathrm{Pe}}$: Perelman μ-entropy
μ^λ-cscK metric
"μ^λ-flow"

For Fano manifold $(X, L) = (X, -2\pi\lambda K_X)$, μ^λ-cscK metric is equivalent to Kähler–Ricci soliton. (Here $\lambda^{-1} = 2\pi(-K_X.L^{n-1})/(L^n) > 0$)

For general (X, L), a rescaling limit of the framework of μ^λ-cscK metric as $\lambda \to -\infty$ is the framework of extremal metric.

Perelman entropy

For $f \in C^\infty(X)$ normalized as $\int_X e^f \omega^n = \int_X \omega^n$, we consider

$$W^\lambda(\omega, f) := -\frac{1}{\int_X \omega^n} \left(\int_X (s(\omega) + \Box f) e^f \omega^n - \lambda \int_X f e^f \omega^n \right).$$

If we put $\nu_\omega = \omega^n / \int_X \omega^n$, $\nu_{\omega,f} := e^f \omega^n / \int_X \omega^n$ and $\rho(\nu) := \sqrt{-1}\bar{\partial}\partial \log \nu$ for $\nu \in \Omega^{n,n}(X)$ then

$$W^\lambda(\omega, f) = -\left(\int_X \mathrm{tr}_\omega(\rho(\nu_{\omega,f}))\nu_{\omega,f} - \lambda \int_X \frac{\nu_{\omega,f}}{\nu_\omega} \log \frac{\nu_{\omega,f}}{\nu_\omega} \nu_\omega \right).$$

$$\left(\text{cf. free energy } \sum_{x \in \Omega} E(x) p(x) + T \sum_{x \in \Omega} p(x) \log p(x) \right)$$

We put

$$\mu^\lambda_{\mathrm{Pe}}(\omega, f) := \sup_{f : \int_X e^f \omega^n = \int_X \omega^n} W^\lambda(\omega, f).$$

μ-cscK metric

Theorem (arXiv:2101.11197)

The critical points of $W^\lambda : \mathcal{H}(X, L) \times C^\infty(X)/\mathbb{R} \to \mathbb{R}$ are precisely those pairs (ω, f) satisfying the following couple of equations:

$$\bar{\partial}\partial^\# f = 0$$

$$s(\omega) + \Delta f - \frac{1}{2}|\nabla f|^2 = \lambda f + \mathrm{const.}$$

We call ω a μ^λ-cscK metric if there exists some f satisfying this equation.

Proposition

For $\lambda \leq 0$, the functional $\mu^\lambda_{\mathrm{Pe}} : \mathcal{H}(X, \omega) \to \mathbb{R}$ is smooth and its critical points are all minimizers and are μ^λ-cscK metrics.

Conjecture

For $\lambda \leq 0$, μ^λ-cscK metrics are unique modulo $\mathrm{Aut}(X)$.

μ-entropy

Theorem (arXiv:2101.11197)

There exists an algebro-geometric quantity

$$\hat{\mu}^{\lambda}(\mathcal{X}, \mathcal{L}) = \sup_{\rho \geq 0} \mu^{\lambda}(\mathcal{X}, \mathcal{L}; \rho)$$

which satisfies Donaldson type inequality

$$\sup_{(\mathcal{X}, \mathcal{L})} \hat{\mu}^{\lambda}(\mathcal{X}, \mathcal{L}) \leq \inf_{\omega} \mu^{\lambda}_{\mathrm{Pe}}(\omega).$$

Conjecture

For $\lambda \leq 0$,

$$\sup_{(\mathcal{X}, \mathcal{L})} \hat{\mu}^{\lambda}(\mathcal{X}, \mathcal{L}) = \inf_{\omega} \mu^{\lambda}_{\mathrm{Pe}}(\omega).$$

This is true when there exists a μ^{λ}-cscK metric, i.e. minimizer of RHS.

Toric description

Let (X, L) be a toric variety and $(\mathcal{X}, \mathcal{L})$ be a toric tc associated to a convex function q on the moment polytope $P \subset \mathfrak{t}^\vee$. Put $u_{\rho.q} := e^{\rho \cdot q} / \int_P e^{\rho \cdot q} d\mu$, then

$$\mu^\lambda(\mathcal{X}, \mathcal{L}; \rho) = -(2\pi \int_{\partial P} u_{\rho.q} d\sigma - \lambda \int_P u_{\rho.q} \log u_{\rho.q} d\mu) =: \mu^\lambda(\rho.q)$$

Theorem (arXiv:2303.09090)

For every $\lambda \in \mathbb{R}$, there exists a lsc convex function $q : P \to (-\infty, \infty]$ which maximizes μ^λ.
When $\lambda \leq 0$, the maximizer is unique up to the addition of constant.
Moreover, q is L^p for every $p \in [1, \infty)$.

Conjecture

The maximizer is piecewise affine.

Where we are now

Set out nice framework

$\downarrow$

Construct optimal degeneration

$\downarrow$

Analyze asymptotic behavior of geometric flow

$\downarrow$

Analyze the regularity of optimal degeneration