Distortion and Moment measure of L-psh function over trivially valued field

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Menu

1. Motivation and Prior researches

K-stability

Let $(X, L)/k = \mathbb{C}$ be a polarized manifold. For a normal tc $(\mathcal{X}, \mathcal{L}) \circlearrowleft \mathbb{G}_m$,

$$\frac{(L^{\cdot n})}{n!}\mathrm{DF}(\mathcal{X}, \mathcal{L}) = \frac{(K_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{L}^{\cdot n})}{n!} - \frac{n(K_X \cdot L^{\cdot n-1})}{(L^{\cdot n})} \frac{(\mathcal{L}^{\cdot n+1})}{(n+1)!}$$

1. K-polystability (reduced uniform K-stability) $\mathrm{DF} > 0$ is conjectured to be equivalent to the existence of cscK metrics. (YTD conjecture)
2. K-semistability $\mathrm{DF} \geq 0$ is conjectured to be a Zariski open condition, and the moduli stack of K-semistable polarized variety admits a good moduli space of K-polystable polarized variety.

For $L = -\lambda K_X$ (cscK = KE), we have good tools on "regularity":

1. ✓: $\mathrm{Ric} > 0$ / Ding functional (..., Chen–Donaldson–Sun, Tian, ... / Berman–Boucksom–Jonsson)
2. ✓: MMP / boundedness / finite generation (..., Blum–Liu–Xu, ...)

Quick review of recent developments on K-stability

For general L,

1. Reduced to NA entropy regularization conjecture (..., Chen–Cheng, Berman–Darvas–Lu / Boucksom–Jonsson, Chi Li).
2. Little known (Chen–Sun and Dervan–Naumann)

$$\begin{aligned} K_{\mathcal{X}/\mathbb{P}^1} \overset{\text{replace}}{\rightsquigarrow} K^{\log}_{\mathcal{X}/\mathbb{P}^1} &:= K_{\mathcal{X}/\mathbb{P}^1} - (\mathcal{X}_0 - \mathcal{X}_0^{\mathrm{red}}) \\ &= p^* K_X + \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) A_X(v_E) E. \end{aligned}$$

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&= p^* K_X + \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) A_X(v_E) E.
\end{aligned}
$$

$$
\begin{aligned}
\frac{(L^{\cdot n})}{n!} M_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &= \frac{1}{n!} (K^{\log}_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{L}^{\cdot n}) - \cdots \\
&= \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) A_X(v_E) \frac{1}{n!} (E \cdot \mathcal{L}^{\cdot n}) \\
&\qquad + \frac{1}{n!} (p^* K_X \cdot \mathcal{L}^{\cdot n}) - \cdots .
\end{aligned}
$$

Non-archimedean entropy

For a normal test configuration $(\mathcal{X}, \mathcal{L})$, we put

$$\mathrm{Ent}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \frac{n!}{(L^{\cdot n})} \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) A_X(v_E) \frac{1}{n!} (E \cdot \mathcal{L}^{\cdot n}).$$

Let us consider

$$X^{\beth} := \{ \text{ valuations on some irreducible subvariety } Y \subset X \} \circlearrowleft \mathbb{R}_+$$

and the measure

$$\mathrm{MA}_{(\mathcal{X}, \mathcal{L})} := \frac{n!}{(L^{\cdot n})} \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) \frac{1}{n!} (E \cdot \mathcal{L}^{\cdot n}) \delta^{X^{\beth}}_{v_E}$$

on $X^{\beth}$. We observe

$$\mathrm{Ent}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \int_{X^{\beth}} A_X(v) \mathrm{MA}_{(\mathcal{X}, \mathcal{L})}(v).$$

NA entropy regularization conjecture

For a test configuration $(\mathcal{X}, \mathcal{L})$, we can assign a function $\varphi_{(\mathcal{X},\mathcal{L})} : X^{\beth} \to \mathbb{R}$ by

$$\varphi(v) = m^{-1} \max\{-v(s/s_v) - \log \|s\|_{(\mathcal{X},\mathcal{L})} \mid s \in R_m \setminus \{0\}\}.$$

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Topology: We endow the set $X^{\beth}$ with the weakest topology which makes $\varphi_{(\mathcal{X},\mathcal{L})}$ continuous for every $(\mathcal{X}, \mathcal{L})$. It turns out that the topology is independent of the choice of L and $X^{\beth}$ becomes a compact Hausdorff space. The log discrepancy A_X extends to an lsc function on $X^{\beth}$.

$\exists \mathrm{MA} : \mathcal{E}^1_{\mathrm{NA}}(L) \to \mathcal{M}(X^{\beth})$: $\mathcal{E}^1_{\mathrm{NA}}(L)$ consisting of functions on $X^{\beth}$ which is nicely approximated by $\varphi_{(\mathcal{X}_i,\mathcal{L}_i)}$.

Conjecture (NA entropy regularization conjecture)

For $\varphi \in \mathcal{E}^1_{\mathrm{NA}}(L)$, there exists a sequence of test configurations $(\mathcal{X}_i, \mathcal{L}_i)$ such that $\varphi_{(\mathcal{X}_i,\mathcal{L}_i)} \xrightarrow{E^1} \varphi$ and $\int_{X^{\beth}} A_X \mathrm{MA}_{(\mathcal{X}_i,\mathcal{L}_i)} \to \int_{X^{\beth}} A_X \mathrm{MA}_{\varphi}$.

The main question

Q. What can we expect for K-unstable (X, L)?

A naive expectation is:

There exists a unique test configuration which is **optimal** in some sense.

1. How can we formulate '**optimal**'?
2. What kind of geometric property can we expect?

We hope to

1. introduce a "well-behaved" quantity for test configuration which would be maximized by **optimal** degeneration.
2. show the central fibre of optimal degeneration is K-stable in a generalized sense: we introduce K-stability for polarized dynamics $(X_0, L_0) \circlearrowleft \mathbb{G}_m =$ "well-behaved".

Quick overview of two model frameworks

We have two model frameworks related to extremal metric (Calabi flow) and Kähler–Ricci soliton (Kähler–Ricci flow).

For $\varphi = (\mathcal{X}, \mathcal{L})$, the quantities are:

$$\boldsymbol{C}_{\mathrm{NA}}(\varphi) = -M^{\mathrm{NA}}(\varphi) - \frac{1}{2}\left(\int_{\mathbb{R}} \sigma^2 \mathrm{DH}_\varphi - \left(\int_{\mathbb{R}} \sigma \mathrm{DH}_\varphi\right)^2\right),$$

$$\boldsymbol{\eta}_{\mathrm{NA}}(\varphi) = -D^{\mathrm{NA}}(\varphi) - \left(\log \int_{\mathbb{R}} e^{-\sigma} \mathrm{DH}_\varphi + \int_{\mathbb{R}} \sigma \mathrm{DH}_\varphi\right),$$

respectively, where we consider $\boldsymbol{\eta}_{\mathrm{NA}}$ for $L = -K_X$ (Fano).
We note

$$\sup_{t>0} \boldsymbol{C}_{\mathrm{NA}}(t \triangleright \varphi) = \begin{cases} 0 & M^{\mathrm{NA}}(\varphi) \geq 0 \\ -\frac{M^{\mathrm{NA}}(\varphi)}{\int_{\mathbb{R}} \sigma^2 \mathrm{DH}_\varphi - (\int_{\mathbb{R}} \sigma \mathrm{DH}_\varphi)^2} & M^{\mathrm{NA}}(\varphi) < 0 \end{cases}$$

These are essentially introduced by Donaldson ('05) and Dervan–Székelyhidi ('20), respectively.

Well-behavedness

Theorem (Dervan, Han–C. Li)

1. $\boldsymbol{C}_{\mathrm{NA}}$ (resp. $\boldsymbol{\eta}_{\mathrm{NA}}$) is maximized by the trivial test configuration if and only if (X, L) is K-semistable.
2. If $\boldsymbol{C}_{\mathrm{NA}}$ (resp. $\boldsymbol{\eta}_{\mathrm{NA}}$) is maximized by a "product test configuration" induced by "$(X, L) \circlearrowleft \mathbb{G}_m$", then "$(X, L) \circlearrowleft \mathbb{G}_m$" is relatively K-semistable (resp. modified K-semistable). ($\Leftarrow$ is known for $\boldsymbol{\eta}_{\mathrm{NA}}$)
3. If a "normal test configuration" $(\mathcal{X}, \mathcal{L})$ maximizes $\boldsymbol{C}_{\mathrm{NA}}$ (resp. $\boldsymbol{\eta}_{\mathrm{NA}}$), then the central fibre "$(\mathcal{X}_0, \mathcal{L}_0) \circlearrowleft \mathbb{G}_m$" is relatively K-semistable (resp. modified K-semistable).

Relative K-stability is related to the existence of extremal metric.
Modified K-stability is equivalent to the existence of Kähler–Ricci soliton.
These stability notions are defined for polarized dynamics "$(X, L) \circlearrowleft \mathbb{G}_m$" $= (X, L; \xi)$, not only for pol. variety $(X, L) = (X, L; 0)$.

Existence

For each test configuration $(\mathcal{X}, \mathcal{L})$, we can assign a (maximal, finite energy) geodesic ray $\Phi = \{\Phi_t\}_{t \in [0,\infty)} \in \mathcal{R}^1_{\mathcal{I}}(\omega)$ of Kähler potentials. We can extend $\boldsymbol{C}_{\mathrm{NA}}$ to the space of (maximal finite energy) geodesic rays $\mathcal{R}^1_{\mathcal{I}}(\omega) \cong \mathcal{E}^1_{\mathrm{NA}}(L)$ by putting

$$\boldsymbol{C}_{\mathrm{ray}}(\Phi) = -M'(\Phi) - \frac{1}{2}\left(\int_{\mathbb{R}} \sigma^2 \mathrm{DH}_\varphi - \left(\int_{\mathbb{R}} \sigma \mathrm{DH}_\varphi\right)^2\right).$$

Theorem (Xia + C. Li (cf. Székelyhidi, A-M. Li–Lian–Sheng))

There exists a maximal finite energy geodesic ray $\Phi \in \mathcal{R}^2_{\mathcal{I}}$ which maximizes $\boldsymbol{C}_{\mathrm{ray}}$.

On the other hand, for $L = -K_X$ ($\mathbb{Q}$-Fano) and $\boldsymbol{\eta}_{\mathrm{NA}}$,

Theorem (Chen–Sun–Wang, Dervan–Székelyhidi, Blum–Liu–Xu–Zhuang)

There exists a finitely generated filtration $\mathcal{F}$ which maximizes $\boldsymbol{\eta}_{\mathrm{NA}}$. Its central fibre is modified K-semistable $\mathbb{Q}$-Fano variety.

Optimal degeneration | canonical metric

Let $\boldsymbol{C}, \boldsymbol{\eta} : \mathcal{H}(\omega) \to \mathbb{R}$ denote the Calabi functional the He functional, respectively.

Critical points = extremal metric / Kähler–Ricci soliton

Theorem (Donaldson, Xia / Dervan–Székelyhidi)

$$\sup_{\varphi \in \mathcal{E}^2_{\mathrm{NA}}(L)} \boldsymbol{C}_{\mathrm{NA}}(\varphi) = \inf_{\phi \in \mathcal{E}^2(\omega)} \boldsymbol{C}(\phi)$$

$$\sup_{\varphi \in \mathcal{H}^{\mathbb{R}}_{\mathrm{NA}}(L)} \boldsymbol{\eta}_{\mathrm{NA}}(\varphi) = \inf_{\phi \in \mathcal{H}(\omega)} \boldsymbol{\eta}(\phi).$$

Furthermore, optimal destabilizers are assymptotic to Calabi flow and Kähler–Ricci flow, respectively.

Beautiful results!

Sakasa-Fuji = upside-down Fuji = optimal degeneration

2. Non-archimedean μ-entropy

Duistermaat–Heckman measure

Let (Z, A) be a polarized scheme with $\mathbb{G}_m$-action. Using the weight decomposition $H^0(Z, A^{\otimes m}) = \bigoplus_{\lambda \in \mathbb{Z}} H^0(Z, A^{\otimes m})_\lambda$, we define the probability measure $\mathrm{DH}_{(Z,A) \circlearrowleft \mathbb{G}_m}$ on $\mathbb{R}$ by

$$\mathrm{DH}_{(Z,A)\circlearrowleft \mathbb{G}_m} := \lim_{m \to \infty} \frac{1}{\dim H^0(Z, A^{\otimes m})} \sum_{\lambda \in \mathbb{Z}} \dim H^0(Z, A^{\otimes m})_\lambda \delta^{\mathbb{R}}_{\lambda/m}.$$

For a test configuration $(\mathcal{X}, \mathcal{L})$, using the following associated filtrations

$$\widehat{\mathcal{F}}^\lambda_{(\mathcal{X},\mathcal{L}),m} := \{ s \in H^0(X, L^{\otimes m}) \mid \exists d \in \mathbb{N} \text{ s.t. } p^* s \cdot \varpi^{-\lceil d\lambda \rceil} \in H^0(\mathcal{X}_d, \mathcal{L}_d^{\otimes m}) \}$$

we can compute

$$\mathrm{DH}_{(\mathcal{X},\mathcal{L})} := \mathrm{DH}_{(\mathcal{X}_0,\mathcal{L}_0)\circlearrowleft \mathbb{G}_m} = \lim_{m\to\infty} \frac{1}{\dim H^0(X, L^{\otimes m})} \sum_{\lambda \in \mathbb{Z}} \dim \frac{\widehat{\mathcal{F}}^\lambda_{(\mathcal{X},\mathcal{L}),m}}{\widehat{\mathcal{F}}^{\lambda+}_{(\mathcal{X},\mathcal{L}),m}} \delta^{\mathbb{R}}_{\lambda/m}.$$

Equivariant cohomology

Let Z be a scheme with $\mathbb{G}_m$-action (e.g. tc $\mathcal{X}$ / central fibre $\mathcal{X}_0$). We consider the $\mathbb{G}_m$-equivariant cohomology:

$$H^p_{\mathbb{G}_m}(Z) := H^p([Z/\mathbb{G}_m]) = H^p(Z \times_{\mathbb{G}_m} E\mathbb{G}_m).$$

Example

$$H^p_{\mathbb{G}_m}(\mathrm{pt}) = H^p(\mathbb{P}^\infty) = \begin{cases} \mathbb{Z} & p \geq 0 \text{ even} \\ 0 & p \text{ otherwise} \end{cases}$$

We have the cup product

$$\smile : H^p_{\mathbb{G}_m}(Z) \otimes H^q_{\mathbb{G}_m}(Z) \to H^{p+q}_{\mathbb{G}_m}(Z).$$

For a proper scheme Z of pure (complex) dimension n, we have the push-forward

$$\int_Z : H^p_{\mathbb{G}_m}(Z) \to H^{p-2n}_{\mathbb{G}_m}(\mathrm{pt}).$$

Self equivariant intersection

For $L_{\mathbb{G}_m} \in H^2_{\mathbb{G}_m}(Z, \mathbb{R})$ and $\ell \geq 0$, we have

$$\int_Z L_{\mathbb{G}_m}^{\smile \ell} \in H^{2(\ell-n)}_{\mathbb{G}_m}(\mathrm{pt}).$$

In general, this can be a non-zero number for each $\ell \geq n$.

Proposition (cf. Ginzburg–Guillemin–Karshon, BHJ)

For an ample A, putting $A_{\mathbb{G}_m} := c_{1,\mathbb{G}_m}(A)$, we have

$$\frac{1}{\ell!} \int_Z A_{\mathbb{G}_m}^{\smile \ell} = \frac{(L^{\cdot n})}{n!} \frac{1}{(\ell-n)!} \int_{\mathbb{R}} (-\sigma)^{\ell-n} \mathrm{DH}_{(Z,A)}(\sigma).$$

For an entire holomorphic function $f(x) = \sum_{\ell=0}^{\infty} \frac{a_\ell}{\ell!} x^\ell$ and $L_{\mathbb{G}_m} \in H^2_{\mathbb{G}_m}(Z, \mathbb{R})$, we define f-self equivariant intersection of $L_{\mathbb{G}_m}$ by

$$(f(L_{\mathbb{G}_m})) := \sum_{\ell=0}^{\infty} \frac{a_\ell}{\ell!} \int_Z L_{\mathbb{G}_m}^{\smile \ell} = \frac{(L^{\cdot n})}{n!} \int_{\mathbb{R}} f^{(n)}(-\sigma) \mathrm{DH}_{(Z,A)}(\sigma) \in \mathbb{C}.$$

Equivariant localization

For a test configuration $(\mathcal{X}, \mathcal{L})$, putting $\mathcal{L}_{\mathbb{G}_m} := c_{1,\mathbb{G}_m}(\mathcal{L})$ and $\mathcal{L}_{\mathbb{G}_m}|_{\mathcal{X}_0} := c_{1,\mathbb{G}_m}(\mathcal{L}|_{\mathcal{X}_0})$, we have

$$\int_{\mathcal{X}} \overline{\mathcal{L}}_{\mathbb{G}_m}^{\ell} + \int_{\mathcal{X}_0} \mathcal{L}_{\mathbb{G}_m}|_{\mathcal{X}_0}^{\ell} = (L^{\cdot \ell}) = \begin{cases} (L^{\cdot n}) & \ell = n \\ 0 & \text{otherwise} \end{cases}$$

Example

$$\frac{1}{(n+1)!}(\mathcal{L}^{\cdot n+1}) = -\frac{1}{(n+1)!}(\mathcal{L}_{\mathbb{G}_m}|_{\mathcal{X}_0}^{\cdot n+1}) = \frac{(L^{\cdot n})}{n!} \int_{\mathbb{R}} \sigma \mathrm{DH}_{(\mathcal{X},\mathcal{L})}$$

$$\frac{1}{(n+2)!}(\mathcal{L}_{\mathbb{G}_m}^{\cdot n+2}) = -\frac{1}{(n+2)!}(\mathcal{L}_{\mathbb{G}_m}|_{\mathcal{X}_0}^{\cdot n+2}) = -\frac{(L^{\cdot n})}{n!} \frac{1}{2} \int_{\mathbb{R}} \sigma^2 \mathrm{DH}_{(\mathcal{X},\mathcal{L})}$$

$$(e^{\mathcal{L}_{\mathbb{G}_m}}) = \frac{1}{n!}(L^{\cdot n}) - (e^{\mathcal{L}_{\mathbb{G}_m}|_{\mathcal{X}_0}}) = \frac{1}{n!}(L^{\cdot n}) - \frac{(L^{\cdot n})}{n!} \int_{\mathbb{R}} e^{-\sigma} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}.$$

Twist

For $M_{\mathbb{G}_m}, L_{\mathbb{G}_m} \in H^2_{\mathbb{G}_m}(Z, \mathbb{Q})$, we have

$$\frac{1}{(\ell-1)!} \int_Z M_{\mathbb{G}_m} \smile L_{\mathbb{G}_m}^{\smile \ell-1} = \frac{d}{ds}\Big|_{s=0} \frac{1}{\ell!} \int_Z (L_{\mathbb{G}_m} + sM_{\mathbb{G}_m})^{\smile \ell} \in H^{2(\ell-n)}_{\mathbb{G}_m}(\mathrm{pt}, \mathbb{Q}).$$

For an entire holomorphic function $f(x) = \sum_{\ell=0}^{\infty} \frac{a_\ell}{\ell!} x^\ell$, we have

$$(M_{\mathbb{G}_m} \cdot f'(\mathcal{L}_{\mathbb{G}_m})) = \frac{d}{ds}\Big|_{s=0} (f(\mathcal{L}_{\mathbb{G}_m} + sM_{\mathbb{G}_m})).$$

We can also define $(M_{\mathbb{G}_m} \cdot f'(\mathcal{L}_{\mathbb{G}_m}))$ for an $\mathbb{G}_m$-equivariant Borel–Moore homology class $M_{\mathbb{G}_m} \in H^{BM, \mathbb{G}_m}_{2n-2}(Z, \mathbb{Q})$. The right hand side of the above formula may not make sense for this general $M_{\mathbb{G}_m}$.

Non-archimedean μ-entropy

Now we introduce the non-archimedean μ-entropy: for a ntc $(\mathcal{X}, \mathcal{L})$,

$$\boldsymbol{\mu}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \frac{n(K_X . L^{\cdot n-1})}{(L^{\cdot n})} - \frac{\frac{1}{(n-1)!}(K_X . L^{\cdot n-1}) - (K_{\mathcal{X}/\mathbb{P}^1}^{\log, \mathbb{G}_m} \cdot e^{\mathcal{L}_{\mathbb{G}_m}})}{\frac{1}{n!}(L^{\cdot n}) - (e^{\mathcal{L}_{\mathbb{G}_m}})}.$$

Theorem (Well-behavedness, arXiv:2202.12168)

If a normal test configuration $(\mathcal{X}, \mathcal{L})$ maximizes $\boldsymbol{\mu}_{\mathrm{NA}}$, then its central fibre "$(\mathcal{X}_0, \mathcal{L}_0) \circlearrowleft \mathbb{G}_m$" is μK-semistable.

μK-stability is related to the existence of μ-cscK metric: a generalization of cscK metric to polarized dynamics "$(X, L) \circlearrowleft \mathbb{G}_m$" which unifies the framework of extremal metric and Kähler–Ricci soliton.

Perelman entropy

$$W^\lambda(\omega_\phi, f) := -\frac{1}{2\pi}\frac{\int_X (s(\omega_\phi) + \frac{1}{2}|\nabla f|^2) e^{-f}\omega_\phi^n}{\int_X e^{-f}\omega_\phi^n} + \frac{\lambda}{2\pi}\cdots.$$

$$\boldsymbol{\mu}^\lambda_{\mathrm{Per}}(\omega_\phi) := \sup_f W^\lambda(\omega_\phi, f).$$

Critical points $=$ μ-cscK metric

Theorem (arXiv:2101.11197)

$$\sup_{(\mathcal{X},\mathcal{L})} \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X},\mathcal{L}) \leq \inf_{\omega_\phi} \boldsymbol{\mu}^\lambda_{\mathrm{Per}}(\omega_\phi).$$

Sketch of Proof (psh-ity of Bergman kernel / Cartan model and equivariant localization) For the $C^{1,1}$-geodesic ray ϕ_t associated to a smooth test configuration $(\mathcal{X},\mathcal{L})$, we have

$$W(\omega_0, \dot\phi_0) \geq W(\omega_{\phi_t}, \dot\phi_t) \searrow \lim_{t\to\infty} W(\omega_{\phi_t}, \dot\phi_t) = \boldsymbol{\mu}_{\mathrm{NA}}(\mathcal{X},\mathcal{L}) \quad \square$$

Conjecture

Conjecture (For $\lambda \leq 0$)

$$\max_{\varphi \in \mathcal{E}^{\exp,1}_{\mathrm{NA}}(L)} \boldsymbol{\mu}^{\lambda}_{\mathrm{NA}}(\varphi) = \inf_{\omega_\phi \in \mathcal{H}(\omega)} \boldsymbol{\mu}^{\lambda}_{\mathrm{Per}}(\omega_\phi)$$

3. Application of distortion and moment measure

Main applications I: New formula for $\boldsymbol{\mu}_{\mathrm{NA}}$

Theorem A (To appear)

There exists a map $\mathrm{Dist} : \mathcal{H}_{\mathrm{NA}}(L) \to \mathcal{E}^1_{\mathrm{NA}}(L)$ such that

$$\boldsymbol{\eta}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \log \left(\frac{D^{\mathrm{NA}}(\mathrm{Dist}(\mathcal{X}, \mathcal{L}))}{E^{\mathrm{NA}}(\mathrm{Dist}(\mathcal{X}, \mathcal{L}))} + 1 \right)$$

$$\boldsymbol{\mu}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{M^{\mathrm{NA}}(\mathrm{Dist}(\mathcal{X}, \mathcal{L}))}{E^{\mathrm{NA}}(\mathrm{Dist}(\mathcal{X}, \mathcal{L}))}$$

with

$$E^{\mathrm{NA}}(\mathrm{Dist}(\mathcal{X}, \mathcal{L})) = - \int_{\mathbb{R}} e^{-\sigma} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}(\sigma) < 0.$$

By this formula, we can extend

$$\boldsymbol{\mu}_{\mathrm{NA}} : \mathcal{E}^{\exp,1}_{\mathrm{NA}}(L) \to [-\infty, \infty) : \varphi \mapsto \frac{M^{\mathrm{NA}}(\mathrm{Dist}(\varphi))}{E^{\mathrm{NA}}(\mathrm{Dist}(\varphi))}.$$

Main applications II: Existence of optimal destabilizing ray

Let $\mathcal{R}^1$ denote the space of finite energy geodesic rays and $\mathcal{R}^1_{\mathcal{I}}$ denote the space of ($\mathcal{I}$-)maximal finite energy geodesic rays (cf. Berman–Boucksom–Jonsson, Darvas–Xia). We consider

$$\boldsymbol{\mu}_{\mathrm{ray}} : \mathcal{E}^{\exp,1}(L) \to [-\infty, \infty) : \varphi \mapsto \mathrm{Dist}(\varphi)_{\mathrm{ray}} \mapsto \frac{M'(\mathrm{Dist}(\varphi)_{\mathrm{ray}})}{E'(\mathrm{Dist}(\varphi)_{\mathrm{ray}})}$$

By C. Li's inequality $M^{\mathrm{NA}} \leq M'$, we have

$$\boldsymbol{\mu}_{\mathrm{ray}} \leq \boldsymbol{\mu}_{\mathrm{NA}}.$$

Theorem B (To appear)

Assume X is smooth. There exists $\varphi \in \mathcal{E}^{\exp,1}_{\mathrm{NA}}(L)$ which maximizes $\boldsymbol{\mu}_{\mathrm{ray}}$.

If NA entropy regularization conjecture is true, the above theorem can be read as "there exists $\varphi \in \mathcal{E}^{\exp,1}_{\mathrm{NA}}(L)$ which maximizes $\boldsymbol{\mu}_{\mathrm{NA}}$".

Main applications III: Miscellaneous

Conjecture (For $\lambda \leq 0$)

$$\max_{\varphi \in \mathcal{E}^{\mathrm{exp},1}_{\mathrm{NA}}(L)} \boldsymbol{\mu}_{\mathrm{ray}}(\varphi) = \sup_{\varphi \in \mathcal{E}^{\mathrm{exp},1}_{\mathrm{NA}}(L)} \boldsymbol{\mu}_{\mathrm{NA}}(\varphi) = \inf_{\phi \in \mathcal{H}(\omega)} \boldsymbol{\mu}_{\mathrm{Per}}(\omega_\phi)$$

Theorem (For $\lambda \leq 0$, to appear)

If $\max_{\varphi \in \mathcal{E}^{\mathrm{exp},1}_{\mathrm{NA}}(L)} \boldsymbol{\mu}_{\mathrm{ray}}(\varphi) \leq \inf_{\phi \in \mathcal{H}(\omega)} \boldsymbol{\mu}_{\mathrm{Per}}(\omega_\phi)$, μ-cscK metrics are unique modulo $\mathrm{Aut}(X, L)$.

Theorem (to appear)

The following are equivalent:

1. (X, L) is K-semistable over $\mathcal{H}_{\mathrm{NA}}(L)$ (resp. $\mathcal{E}^1_{\mathrm{NA}}(L)$)
2. Trivial configuration maximizes $\boldsymbol{\mu}_{\mathrm{NA}}$ over $\mathcal{H}_{\mathrm{NA}}(L)$ (resp. $\mathcal{E}^1_{\mathrm{NA}}(L)$).

4. Distortion and Moment measure

Distortion of Non-archimedean L-psh function

Recall for a tc $(\mathcal{X}, \mathcal{L})$, we have a continuous function $\varphi_{(\mathcal{X},\mathcal{L})} : X^{\beth} \to \mathbb{R}$. For $v_E \in X^{\mathrm{div}} \subset X^{\beth}$ associated to $E \subset \mathcal{Y} \to \mathcal{X}$, we have

$$\varphi_{(\mathcal{X},\mathcal{L})}(v_E) = \frac{\mathrm{ord}_E(\mathcal{L} - p^* L)}{\mathrm{ord}_E(\mathcal{X}_0)}.$$

A function $\varphi : X^{\beth} \to [-\infty, \infty)$ is called L-psh function if there exists a net of tc's $(\mathcal{X}_i, \mathcal{L}_i)$ such that $\varphi_{(\mathcal{X}_i,\mathcal{L}_i)}$ is a decreasing net converging to φ pointwisely. Let $\mathrm{PSH}_{\mathrm{NA}}(L)$ denote the set of L-psh functions.

For an increasing concave $\chi : \mathbb{R} \to \mathbb{R}$, we introduce χ-distortion of φ:

$$\mathrm{Dist}_\chi(\varphi) := \inf_{t>0} (t \triangleright \varphi + \chi^{\blacktriangle}(t)),$$

where

$$(t \triangleright \varphi)(v) := t\varphi(t^{-1} v), \quad \chi^{\blacktriangle}(t) := \sup_{\sigma \in \mathbb{R}} (\chi(\sigma) - t\sigma).$$

Let's get a feeling of Distortion

Proposition (To appear)

For $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(L)$, we have $\mathrm{Dist}_\chi(\varphi) \in \mathrm{PSH}_{\mathrm{NA}}(L)$.

Suppose (X, L) is a toric variety. Let $P \subset M_{\mathbb{R}} = \mathfrak{t}^\vee$ be the moment polytope.

$$\{T\text{-invariant } \varphi\} \longleftrightarrow \{f : N_{\mathbb{R}} \to \mathbb{R}\}$$
$$\varphi \longmapsto f_\varphi(\xi) := \varphi(v_\xi) + \sup_{\mu \in P} \langle \mu, \xi \rangle$$

For the Legendre transform $g_\varphi(\mu) := \inf_{\xi \in N_{\mathbb{R}}} (\langle \mu, \xi \rangle + f_\varphi(\xi))$, we have

$$g_{\mathrm{Dist}_\chi(\varphi)} = \chi(g_\varphi)$$

thanks to the duality:

$$\inf_{t>0} (t\sigma + \chi^{\blacktriangle}(t)).$$

Profile of Distortion

For a continuous L-psh function $\varphi \in \mathrm{CPSH}_{\mathrm{NA}}(L)$, we can assign a filtration $\mathcal{F}_\varphi$ by putting

$$\mathcal{F}^\lambda_{\varphi,m} := \{s \in H^0(X, L^{\otimes m}) \mid \inf_{v \in X^\beth} (\varphi(v) + v(s/s_v)/m) \geq \lambda/m\}.$$

For a normal test configuration $(\mathcal{X}, \mathcal{L})$, we have $\mathcal{F}_\varphi = \widehat{\mathcal{F}}_{(\mathcal{X},\mathcal{L})}$.

We can easily observe

$$\mathcal{F}^\lambda_{\mathrm{Dist}_\chi(\varphi),m} = \mathcal{F}^{m\chi^{-1}(\lambda/m)}_{\varphi,m}.$$

(cf. Reboulet–Witt Nyström)

e.g. For $\chi(\sigma) = -e^{-\sigma}$,

$$\mathcal{F}^\lambda_{\mathrm{Dist}_\chi(\varphi),m} := \begin{cases} \mathcal{F}^{-m\log(-\lambda/m)}_{\varphi,m} & \lambda < 0 \\ 0 & \lambda \geq 0 \end{cases}$$

Distortion and Duistermaat–Heckman measure

Proposition (arXiv:2202.12168, see also M. Xia)

For each $\sigma \in \mathbb{R}$, the map

$$\varphi_{(\mathcal{X},\mathcal{L})} \mapsto \int_{[\sigma,\infty)} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}$$

is monotonic. For general $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(L)$, we can define a unique Radon measure DH_φ satisfying $\int_{\mathbb{R}} \mathrm{DH}_\varphi \leq 1$ and

$$\int_{[\sigma,\infty)} \mathrm{DH}_\varphi = \inf \left\{ \int_{[\sigma,\infty)} \mathrm{DH}_{(\mathcal{X},\mathcal{L})} \;\middle|\; \varphi \leq \varphi_{(\mathcal{X},\mathcal{L})} \right\}.$$

Proposition (To appear)

For $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(L)$, we have $\mathrm{DH}_{\mathrm{Dist}_\chi(\varphi)} = \chi_* \mathrm{DH}_\varphi$.

Distortion and Ding invariant

Let us recall

$$\begin{aligned} D_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) &:= \inf_{v \in X^{\beth}} (A_X(v) + \varphi_{(\mathcal{X}, \mathcal{L})}(v)) - \int_{\mathbb{R}} \sigma \mathrm{DH}_{\varphi_{(\mathcal{X}, \mathcal{L})}} \\ &= \mathrm{lct}_{(\mathcal{X}, \mathcal{X}_0 - (\mathcal{L} + K_{\mathcal{X}/\mathbb{A}^1}))}(\mathcal{X}_0) - \frac{n!}{(L^{\cdot n})} \frac{(\mathcal{L}^{\cdot n+1})}{(n+1)!}. \end{aligned}$$

For $L_{\mathrm{NA}}(\varphi) := \inf_{v \in X^{\beth}} (A_X(v) + \varphi(v))$, we compute

$$\begin{aligned} L_{\mathrm{NA}}(\mathrm{Dist}_\chi(\varphi)) &= \inf_{v \in X^{\beth}} \inf_{t > 0} (A_X(v) + t\varphi(t^{-1} v) + \chi^{\blacktriangle}(t)) \\ &= \inf_{t>0} \inf_{w \in X^{\beth}} (t A_X(w) + t\varphi(w) + \chi^{\blacktriangle}(t)) \\ &= \inf_{t>0} (t \inf_{w \in X^{\beth}} (A_X(w) + \varphi(w)) + \chi^{\blacktriangle}(t)) \\ &= \chi(L_{\mathrm{NA}}(\varphi)), \end{aligned}$$

using the duality

$$\inf(t\sigma + \chi^{\blacktriangle}(t)) = \chi(\sigma).$$

Let's observe μ_{NA}

Recall

$$\mu_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \frac{n(K_X . L^{\cdot n-1})}{(L^{\cdot n})} - \frac{\frac{1}{(n-1)!}(K_X . L^{\cdot n-1}) - (K_{\mathcal{X}/\mathbb{P}^1}^{\log, \mathbb{G}_m} \cdot e^{\mathcal{L}_{\mathbb{G}_m}})}{\frac{1}{n!}(L^{\cdot n}) - (e^{\mathcal{L}_{\mathbb{G}_m}})}.$$

Using

$$K_{\mathcal{X}/\mathbb{P}^1}^{\log, \mathbb{G}_m} - p^* K_X = \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) A_X(v_E) E^{\mathbb{G}_m},$$

we can write

$$\begin{aligned}
(K_{\mathcal{X}/\mathbb{P}^1}^{\log, \mathbb{G}_m} \cdot e^{\mathcal{L}_{\mathbb{G}_m}}) &= ((K_{\mathcal{X}/\mathbb{P}^1}^{\log, \mathbb{G}_m} - p^* K_X) \cdot e^{\mathcal{L}_{\mathbb{G}_m}}) + (p^* K_X \cdot e^{\mathcal{L}_{\mathbb{G}_m}}) \\
&= \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) A_X(v_E)(E^{\mathbb{G}_m} \cdot e^{\mathcal{L}_{\mathbb{G}_m}}) \\
&\quad + \frac{d}{ds}\Big|_{s=0} (e^{p^*(sK_X + L) + (\mathcal{L}_{\mathbb{G}_m} - p^* L)}) \\
&= \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) A_X(v_E) \frac{(L^{\cdot n})}{n!} \int_{\mathbb{R}} e^{-\sigma} \mathrm{DH}_{(E, \mathcal{L}|_E)} + \cdots
\end{aligned}$$

Moment measure

For a normal test configuration $(\mathcal{X}, \mathcal{L})$, we introduce the moment measure $\mathcal{D}_{(\mathcal{X},\mathcal{L})}$ on $X^\beth \times \mathbb{R}$ by

$$\mathcal{D}_{(\mathcal{X},\mathcal{L})} := \frac{n!}{(L^{\cdot n})} \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) \delta^{X^\beth}_{v_E} \otimes \mathrm{DH}_{(E, \mathcal{L}|_E) \circlearrowleft \mathbb{G}_m}.$$

Theorem (arXiv:2202.12168 + to appear)

We can define a measure $\mathcal{D}_\varphi$ on $X^\beth \times \mathbb{R}$ for $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(L)$ with $\int_{\mathbb{R}} \mathrm{DH}_\varphi = 1$ so that

$$\mathcal{D}_{\varphi_i} \to \mathcal{D}_\varphi \text{ weakly}$$

for any $\varphi_i \searrow \varphi$.

Distortion and Moment measure

For an increasing concave function χ, consider the map

$$\Delta^\chi : X^\beth \times \mathbb{R} \to X^\beth \times \mathbb{R} : (v, \sigma) \mapsto (\dot{\chi}(\sigma).v, \chi(\sigma)).$$

Theorem (To appear)

For $\varphi \in \mathcal{E}_{\mathrm{NA}}(L)$, we have

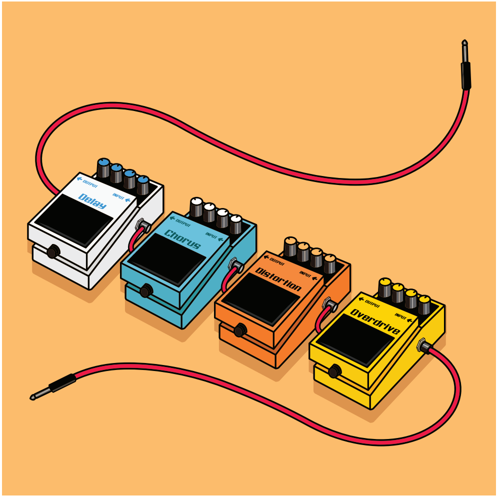
$$\mathcal{D}_{\mathrm{Dist}_\chi(\varphi)} = \Delta^\chi_* \mathcal{D}_\varphi.$$

Using $A_X(t.v) = tA_X(v)$, we compute

$$\int_{X^\beth} A_X(v) \int_{\mathbb{R}} \dot{\chi}(\sigma) \mathcal{D}_\varphi(v, \sigma) = \int_{X^\beth \times \mathbb{R}} A_X(\dot{\chi}(\sigma).v) \mathcal{D}_\varphi(v, \sigma)$$
$$= \int_{X^\beth \times \mathbb{R}} (\Delta^\chi)^* A_X \mathcal{D}_\varphi = \int_{X^\beth \times \mathbb{R}} A_X \Delta^\chi_* \mathcal{D}_\varphi$$
$$= \int_{X^\beth \times \mathbb{R}} A_X \mathcal{D}_{\mathrm{Dist}_\chi(\varphi)} = \int_{X^\beth \times \mathbb{R}} A_X \mathrm{MA}(\mathrm{Dist}_\chi(\varphi)).$$

Questions

1. Is optimal destabilizer for $\boldsymbol{\mu}_{\mathrm{NA}}$ unique? (Toric $\lambda < 0$ ✓)
2. Is the optimal destabilizer φ for φ bounded and continuous?
3. If it is, is the filtration $\mathcal{F}_\varphi$ can be written as $\mathcal{F}_\varphi = \bigcup_{i=1}^k \mathcal{F}_{v_i}[\sigma_i]$, using finitely many quasi-monomial valuations v_i and $\sigma_i \in \mathbb{R}$? (cf. Székelyhidi's work on ruled surface)
4. Is $\mathcal{F}_\varphi$ finitely generated? (I speculate this would be not true.)
5. Can we extend the theory of scheme to get a geometric realization of "the central fibre of $\mathcal{F}_\varphi$"? ("almost ring" in Mingchen's talk?)
6. Let φ_1, φ_2 be the optimal destabilizer for $(X_1, L_1), (X_2, L_2)$ respectively. Is $\varphi_1 \times \varphi_2$ the optimal destabilizer for $(X_1 \times X_2, L_1 \boxtimes L_2)$? (Toric $\lambda \leq 0$ ✓)
   This would imply that the framework of μ-entropy possesses a thermodynamical structure. (cf. arXiv:2303.09090)
7. In relation with Carlo's talk, is it possible to understand Perelman entropy / NA μ-entropy as "limit of Einstein–Hilbert functional" / NA EH by $\lim_{N\to\infty} X^{\times N}$? This is inspired by statistical mechanical interpretation of Perelman entropy. (We are discussing this now.)



The name “Distortion” is inspired by effect pedal for electric guitar

Thank you!