Optimal degeneration of algebraic variety and Perelman entropy

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23, May, 2024, CRM, Montreal

Menu

1. From Calabi energy, He entropy ...
2. To Perelman μ-entropy
3. New formula on Non-archimedean μ-entropy
4. Distortion and Moment measure

1. From Calabi energy, He entropy ...

cscK metric

(X, ω): a compact Kähler manifold with a background metric ω.

$$\mathcal{H}(\omega) := \{\phi \in C^\infty(X) \mid \omega + dd^c\phi > 0\}.$$

The Mabuchi functional $M : \mathcal{H}(\omega) \to \mathbb{R}$ is defined as the anti-derivation of $\delta M : T\mathcal{H}(\omega) \to \mathbb{R}$ given by

$$\delta M(\phi, \dot{\phi}) = -\int_X \hat{s}(\omega_\phi)\dot{\phi}\omega_\phi^n.$$

The critical points of M is cscK metric: $s(\omega_\phi) = \bar{s}([\omega])$.

Donaldson–Futaki invariant / Mabuchi invariant

For a normal test configuration $(\mathcal{X}, \mathcal{L})$, we consider

$$M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := (K^{\log}_{\mathcal{X}/\mathbb{P}^1} \cdot \mathcal{L}^{\cdot n}) - \frac{n(K_X \cdot L^{\cdot n-1})}{(L^{\cdot n})} \frac{(\mathcal{L}^{\cdot n+1})}{n+1}.$$

Compare

$$\delta M(\phi, \dot{\phi}) = \int_X (\mathrm{Ric}(\omega_\phi) - \Box\dot{\phi})(\omega_\phi - \dot{\phi})^n - \frac{1}{n+1} \bar{s}([\omega]) \int_X (\omega_\phi - \dot{\phi})^{n+1}.$$

For a test configuration $(\mathcal{X}, \mathcal{L})$, we can assign a "finite energy" geodesic ray $\Phi_{(\mathcal{X},\mathcal{L})}(t) \in C^{1,1}\mathcal{H}(\omega)$. "$\delta M$" makes sense for $C^{1,1}$-geodesic ray.

Theorem (Tian '97, Boucksom–Hisamoto–Jonsson '19, C. Li '22)

$$\lim_{t \to \infty} \delta M(\Phi_{(\mathcal{X},\mathcal{L})}(t), \dot{\Phi}_{(\mathcal{X},\mathcal{L})}(t)) = M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}).$$

YTD conjecture and K-moduli

1. K-polystability (reduced uniform K-stability) $\mathrm{DF} > 0$ is conjectured to be equivalent to the existence of cscK metrics. (YTD conjecture)
2. K-semistability $\mathrm{DF} \geq 0$ is conjectured to be a Zariski open condition, and the moduli stack of K-semistable polarized variety admits a good moduli space of K-polystable polarized variety.

For $L = -\lambda K_X$ (cscK = KE), we have good tools on "regularity":

1. ✓: $\mathrm{Ric} > 0$ / Ding functional (..., Chen–Donaldson–Sun, Tian, ... / Berman–Boucksom–Jonsson)
2. ✓: MMP / boundedness / finite generation (..., Blum–Liu–Xu, ...)

For general L,

1. Reduced to NA entropy regularization conjecture (..., Darvas–Rubinstein, Berman–Darvas–Lu, Chen–Cheng, Chi Li/ Boucksom–Jonsson, Chi Li).
2. Little known (Chen–Sun and Dervan–Naumann)

Today's main interest

Q. What can we expect for K-unstable (X, L)?

A naive expectation is:

There exists a unique test configuration which is **optimal** in some sense.

1. How can we formulate '**optimal**'?
2. What kind of geometric property can we expect?

We hope to

1. introduce a "well-behaved" quantity for test configuration which would be maximized by **optimal** degeneration.
2. show the central fibre of optimal degeneration is K-stable in a generalized sense: we introduce K-stability for polarized dynamics $(X_0, L_0) \circlearrowleft \mathbb{G}_m$ = "well-behaved".

This question is related to algebraic proof of properness of K-moduli space: construction of Θ-stratification of moduli stack (—).

Quick overview of two model frameworks

We have two model frameworks related to extremal metric (Calabi flow) and Kähler–Ricci soliton (Kähler–Ricci flow).

For $\phi \in \mathcal{H}(\omega)$, we consider

$$\boldsymbol{C}(\phi) = \frac{1}{2} \int_X \hat{s}(\omega_\phi)^2 \omega_\phi^n \quad \text{Calabi functional,}$$

$$\boldsymbol{\eta}(\phi) = \int_X \hat{h}_\phi e^{\hat{h}_\phi} \omega_\phi^n \quad \text{He functional,}$$

where $\hat{h}_\phi$ is $\mathrm{Ric}(\omega_\phi) - \omega_\phi = \sqrt{-1}\partial\bar{\partial}\hat{h}_\phi$ with $\int_X e^{\hat{h}_\phi}\omega_\phi^n = [\omega^n]$. We consider $\boldsymbol{\eta}$ for $[\omega] = c_1(X)$ (Fano).

$$\begin{aligned} \text{Critical points} &= \text{Extremal metric } \bar{\partial}\partial^\sharp s(\omega_\phi) = 0 \\ &\quad / \text{ Kähler–Ricci soliton } \bar{\partial}\partial^\sharp h_\phi = 0 \end{aligned}$$

Key observation

We introduce

$$W_{\mathrm{Cal}}(\phi,\dot\phi)=\int_X \hat s(\omega_\phi)\dot\phi\omega_\phi^n-\frac{1}{2}\left(\int_X \dot\phi^2\omega_\phi^n-\frac{1}{[\omega^n]}\Big(\int_X \dot\phi\omega_\phi^n\Big)^2\right)$$

$$W_{\mathrm{He}}(\phi,\dot\phi)=-\frac{\int_X (e^{h_\phi}-1)\dot\phi\omega_\phi^n}{\int_X e^{h_\phi}\omega_\phi^n}-\left(\log\int_X e^{-\dot\phi}\omega_\phi^n+\int_X \dot\phi\omega_\phi^n\right).$$

We can observe

$$\boldsymbol{C}(\phi)=\sup_{\dot\phi\in C^\infty(X)} W_{\mathrm{Cal}}(\phi,\dot\phi)\quad (\text{ compeleting square } \dot\phi=\hat s(\omega_\phi))$$

$$\boldsymbol{\eta}(\phi)=\sup_{\dot\phi\in C^\infty(X)} W_{\mathrm{He}}(\phi,\dot\phi)\quad (\text{ Legendre duality on rel. entropy } \dot\phi=-h_\phi)$$

Limit along geodesic ray

Recall a test configuration $\varphi = (\mathcal{X}, \mathcal{L})$, we can assign a geodesic ray $\Phi_{(\mathcal{X},\mathcal{L})}(t) \in C^{1,1}\mathcal{H}(\omega)$.

Theorem (Hisamoto '16)

$$\dot{\Phi}_{(\mathcal{X},\mathcal{L})}(t)_* \mathrm{MA}(\Phi_{(\mathcal{X},\mathcal{L})}(t)) = \mathrm{DH}_\varphi.$$

Thus for $(\phi(t), \dot{\phi}(t)) = (\Phi_{(\mathcal{X},\mathcal{L})}(t), \dot{\Phi}_{(\mathcal{X},\mathcal{L})}(t))$ we have

$$\lim_{t\to\infty} W_{\mathrm{Cal}}(\phi(t), \dot{\phi}(t)) = -M^{\mathrm{NA}}(\varphi) - \frac{1}{2}\left(\int_{\mathbb{R}} \sigma^2 \mathrm{DH}_\varphi - \frac{1}{(L^{\cdot n})}\left(\int_{\mathbb{R}} \sigma \mathrm{DH}_\varphi\right)^2\right),$$

$$\lim_{t\to\infty} W_{\mathrm{He}}(\phi(t), \dot{\phi}(t)) = -D^{\mathrm{NA}}(\varphi) - \left(\log \int_{\mathbb{R}} e^{-\sigma} \mathrm{DH}_\varphi + \int_{\mathbb{R}} \sigma \mathrm{DH}_\varphi\right).$$

Algebro-geometric quantity

For $\varphi = (\mathcal{X}, \mathcal{L})$, we consider

$$\boldsymbol{C}_{\mathrm{NA}}(\varphi) = -M^{\mathrm{NA}}(\varphi) - \frac{1}{2}\Big(\int_{\mathbb{R}} \sigma^2 \mathrm{DH}_\varphi - \frac{1}{(L^{\cdot n})}\Big(\int_{\mathbb{R}} \sigma \mathrm{DH}_\varphi\Big)^2\Big),$$

$$\boldsymbol{\eta}_{\mathrm{NA}}(\varphi) = -D^{\mathrm{NA}}(\varphi) - \Big(\log \int_{\mathbb{R}} e^{-\sigma} \mathrm{DH}_\varphi + \int_{\mathbb{R}} \sigma \mathrm{DH}_\varphi\Big),$$

respectively, where we consider $\boldsymbol{\eta}_{\mathrm{NA}}$ for $L = -K_X$ (Fano). These are essentially introduced by Donaldson ('05) and Dervan–Székelyhidi ('20), respectively.

By convexity of M and D, we obtain

$$\boldsymbol{C}(\phi) \geq W_{\mathrm{Cal}}(\phi(0), \dot{\phi}(0)) \searrow \lim_{t\to\infty} W_{\mathrm{Cal}}(\phi(t), \dot{\phi}(t)) = \boldsymbol{C}_{\mathrm{NA}}(\varphi)$$

$$\boldsymbol{\eta}(\phi) \geq W_{\mathrm{He}}(\phi(0), \dot{\phi}(0)) \searrow \lim_{t\to\infty} W_{\mathrm{He}}(\phi(t), \dot{\phi}(t)) = \boldsymbol{\eta}_{\mathrm{NA}}(\varphi)$$

$$\rightsquigarrow \sup_{\varphi\in\mathcal{H}_{\mathrm{NA}}(L)} \boldsymbol{C}_{\mathrm{NA}}(\varphi) \leq \inf_{\phi\in\mathcal{H}(\omega)} \boldsymbol{C}(\phi): \text{ Donaldson inequality}$$

Well-behavedness

Theorem (Dervan, Han–C. Li)

1. $\boldsymbol{C}_{\mathrm{NA}}$ (resp. $\boldsymbol{\eta}_{\mathrm{NA}}$) is maximized by the trivial test configuration if and only if (X, L) is K-semistable.
2. If $\boldsymbol{C}_{\mathrm{NA}}$ (resp. $\boldsymbol{\eta}_{\mathrm{NA}}$) is maximized by a "product test configuration" induced by "$(X, L) \circlearrowleft \mathbb{G}_m$", then "$(X, L) \circlearrowleft \mathbb{G}_m$" is relatively K-semistable (resp. modified K-semistable). ($\Leftarrow$ is known for $\boldsymbol{\eta}_{\mathrm{NA}}$)
3. If a "normal test configuration" $(\mathcal{X}, \mathcal{L})$ maximizes $\boldsymbol{C}_{\mathrm{NA}}$ (resp. $\boldsymbol{\eta}_{\mathrm{NA}}$), then the central fibre "$(\mathcal{X}_0, \mathcal{L}_0) \circlearrowleft \mathbb{G}_m$" is relatively K-semistable (resp. modified K-semistable).

Relative K-stability is related to the existence of extremal metric.
Modified K-stability is equivalent to the existence of Kähler–Ricci soliton.
These stability notions are defined for polarized dynamics
"$(X, L) \circlearrowleft \mathbb{G}_m$" $= (X, L; \xi)$, not only for pol. variety $(X, L) = (X, L; 0)$.

Existence

For each test configuration $(\mathcal{X}, \mathcal{L})$, we can assign a (maximal, finite energy) geodesic ray $\Phi = \{\Phi_t\}_{t \in [0,\infty)} \in \mathcal{R}^1_{\mathcal{I}}(\omega)$ of Kähler potentials. We can extend $\boldsymbol{C}_{\mathrm{NA}}$ to the space of (maximal finite energy) geodesic rays $\mathcal{R}^1_{\mathcal{I}}(\omega) \cong \mathcal{E}^1_{\mathrm{NA}}(L)$ by putting

$$\boldsymbol{C}_{\mathrm{ray}}(\Phi) = -M'(\Phi) - \frac{1}{2}\left(\int_{\mathbb{R}} \sigma^2 \mathrm{DH}_\varphi - \left(\int_{\mathbb{R}} \sigma \mathrm{DH}_\varphi\right)^2\right).$$

Theorem (Xia + C. Li (cf. Székelyhidi, A-M. Li–Lian–Sheng))

There exists a maximal finite energy geodesic ray $\Phi \in \mathcal{R}^2_{\mathcal{I}}$ which maximizes $\boldsymbol{C}_{\mathrm{ray}}$.

On the other hand, for $L = -K_X$ ($\mathbb{Q}$-Fano) and $\boldsymbol{\eta}_{\mathrm{NA}}$,

Theorem (Chen–Sun–Wang, Dervan–Székelyhidi, Blum–Liu–Xu–Zhuang)

There exists a finitely generated filtration $\mathcal{F}$ which maximizes $\boldsymbol{\eta}_{\mathrm{NA}}$. Its central fibre is modified K-semistable $\mathbb{Q}$-Fano variety.

Optimal degeneration | canonical metric

Theorem (Donaldson, Xia / Dervan–Székelyhidi)

$$\sup_{\varphi \in \mathcal{E}^2_{\mathrm{NA}}(L)} \boldsymbol{C}_{\mathrm{NA}}(\varphi) = \inf_{\phi \in \mathcal{E}^2(\omega)} \boldsymbol{C}(\phi)$$

$$\sup_{\varphi \in \mathcal{H}^{\mathbb{R}}_{\mathrm{NA}}(L)} \boldsymbol{\eta}_{\mathrm{NA}}(\varphi) = \inf_{\phi \in \mathcal{H}(\omega)} \boldsymbol{\eta}(\phi).$$

Furthermore, optimal destabilizers are assymptotic to Calabi flow and Kähler–Ricci flow, respectively.

Beautiful results!

Sakasa-Fuji

2. To Perelman μ-entropy

Perelman W-entropy (in Ricci flow)

Perelman's original convention: for a manifold M with $\dim_{\mathbb{R}} M = m$,

$$\mathcal{W}(g, f; \tau) = \tau \int_M (R(g) + |\nabla f|^2) \frac{e^{-f}}{(4\pi\tau)^{m/2}} \mathrm{vol}_g - \int_M (m - f) \frac{e^{-f}}{(4\pi\tau)^{m/2}} \mathrm{vol}_g$$

for $\tau > 0$, a metric g and $f \in C^\infty(M)$ with $\int_M \frac{e^{-f}}{(4\pi\tau)^{m/2}} \mathrm{vol}_g = 1$. He proved monotonicity along the following evolving equation

$$\dot{g}_t = -2\mathrm{Ric}(g_t), \quad f_t = -\Delta_{g_t} f_t + |\nabla f_t|^2_{g_t} - R(g_t) + \frac{n}{2\tau_t}, \quad \dot{\tau}_t = -1.$$

On Fano manifold $c_1(X) > 0$, we consider normalized Kähler–Ricci flow $\dot{\omega}_t = \omega_t - \mathrm{Ric}(\omega_t)$. For this, $\mathcal{W}(\omega_t, f_t; \tau_t)$ is monotonic along

$$\dot{\omega}_t = \omega_t - \mathrm{Ric}(\omega_t), \quad f_t = \ldots, \quad \tau_t \equiv 1/2.$$

Perelman W-entropy in Kähler geometry

In Kähler geometry, we use the following convention

$$W^\lambda_{\mathrm{Per}}(\phi,\dot\phi) = -\int_X (s(\omega_\phi) + \frac{1}{2}|\nabla\hat{\dot\phi}|^2) e^{-\hat{\dot\phi}} \omega_\phi^n + \lambda \int_X (n - \hat{\dot\phi}) e^{-\hat{\dot\phi}} \omega_\phi^n$$

for $(\phi,\dot\phi) \in T\mathcal{H}(\omega)$ and $\lambda \in \mathbb{R}$, where $\int_X e^{-\hat{\dot\phi}} \omega_\phi^n = 1$. Note

$$W^\lambda_{\mathrm{Per}}(\phi,\dot\phi) = -\lambda(2\pi\lambda^{-1})^n \cdot (\mathcal{W}(g_\phi,\dot\phi;\frac{1}{2\lambda}) + n).$$

Today, we focus on $\lambda = 0$ (for simplicity): $W_{\mathrm{Per}}(\phi,\dot\phi) := W^0_{\mathrm{Per}}(\phi,\dot\phi)$.

Theorem (arXiv:2101.11197)

For a smooth test configuration $(\mathcal{X},)$ and its geodesic ray $\Phi(t) = \Phi_{(\mathcal{X},\mathcal{L})}(t)$, we can define "$W^\lambda_{\mathrm{Per}}(\Phi,\dot\Phi)(t)$" and we have

$$W^\lambda_{\mathrm{Per}}(\Phi,\dot\Phi)(t) \searrow \boldsymbol{\mu}_{\mathrm{NA}}(\mathcal{X},\mathcal{L}) \quad \text{as } t \to \infty.$$

Non-archimedean μ-entropy | Perelman μ-entropy

For a ntc $(\mathcal{X}, \mathcal{L})$, we define

$$\boldsymbol{\mu}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := \frac{\frac{1}{(n-1)!}(K_X . L^{\cdot n-1}) - (K_{\mathcal{X}/\mathbb{P}^1}^{\log, \mathbb{G}_m} \cdot e^{\mathcal{L}_{\mathbb{G}_m}})}{\frac{1}{n!}(L^{\cdot n}) - (e^{\mathcal{L}_{\mathbb{G}_m}})} - \frac{n(K_X . L^{\cdot n-1})}{(L^{\cdot n})}.$$

Compare

$$W_{\mathrm{Per}}(\phi, \dot{\phi}) = -n! \int_X (\mathrm{Ric}(\omega_\phi) - \Box\dot{\phi}) e^{\omega_\phi - \dot{\phi}}.$$

$$\boldsymbol{\mu}^\lambda_{\mathrm{Per}}(\omega_\phi) := \sup_f W^\lambda(\omega_\phi, f).$$

Critical points = μ-cscK metric

Corollary (arXiv:2101.11197)

$$\sup_{(\mathcal{X}, \mathcal{L})} \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \leq \inf_{\omega_\phi} \boldsymbol{\mu}^\lambda_{\mathrm{Per}}(\omega_\phi).$$

Conjecture

Conjecture (For $\lambda \leq 0$)

$$\max_{\varphi \in \mathcal{E}^{\exp,1}_{\mathrm{NA}}(L)} \boldsymbol{\mu}^{\lambda}_{\mathrm{NA}}(\varphi) = \inf_{\omega_\phi \in \mathcal{H}(\omega)} \boldsymbol{\mu}^{\lambda}_{\mathrm{Per}}(\omega_\phi)$$

Non-archimedean μ-entropy is well-behaved

Theorem (Well-behavedness, arXiv:2202.12168)

If a normal test configuration $(\mathcal{X}, \mathcal{L})$ maximizes $\boldsymbol{\mu}_{\mathrm{NA}}$, then its central fibre "$(\mathcal{X}_0, \mathcal{L}_0) \circlearrowleft \mathbb{G}_m$" is μK-semistable.

μK-stability is related to the existence of μ-cscK metric: a generalization of cscK metric to polarized dynamics "$(X, L) \circlearrowleft \mathbb{G}_m$" which unifies the framework of extremal metric and Kähler–Ricci soliton.

3. New formula on Non-archimedean μ-entropy

Main applications I: New formula for $\boldsymbol{\mu}_{\mathrm{NA}}$

Theorem A (To appear)

There exists a map $\mathrm{Dist} : \mathcal{H}_{\mathrm{NA}}(L) \to \mathcal{E}^1_{\mathrm{NA}}(L)$ such that

$$\boldsymbol{\eta}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \log \left(\frac{D^{\mathrm{NA}}(\mathrm{Dist}(\mathcal{X}, \mathcal{L}))}{E^{\mathrm{NA}}(\mathrm{Dist}(\mathcal{X}, \mathcal{L}))} + 1 \right)$$

$$\boldsymbol{\mu}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \frac{M^{\mathrm{NA}}(\mathrm{Dist}(\mathcal{X}, \mathcal{L}))}{E^{\mathrm{NA}}(\mathrm{Dist}(\mathcal{X}, \mathcal{L}))}$$

with

$$E^{\mathrm{NA}}(\mathrm{Dist}(\mathcal{X}, \mathcal{L})) = -\int_{\mathbb{R}} e^{-\sigma} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}(\sigma) < 0.$$

By this formula, we can extend

$$\boldsymbol{\mu}_{\mathrm{NA}} : \mathcal{E}^{\mathrm{exp},1}_{\mathrm{NA}}(L) \to [-\infty, \infty) : \varphi \mapsto \frac{M^{\mathrm{NA}}(\mathrm{Dist}(\varphi))}{E^{\mathrm{NA}}(\mathrm{Dist}(\varphi))}.$$

Main applications II: Existence of optimal destabilizing ray

Let $\mathcal{R}^1$ denote the space of finite energy geodesic rays and $\mathcal{R}^1_{\mathcal{I}}$ denote the space of ($\mathcal{I}$-)maximal finite energy geodesic rays (cf. Berman–Boucksom–Jonsson, Darvas–Xia). We consider

$$\boldsymbol{\mu}_{\mathrm{ray}} : \mathcal{E}^{\exp,1}(L) \to [-\infty, \infty) : \varphi \mapsto \mathrm{Dist}(\varphi)_{\mathrm{ray}} \mapsto \frac{M'(\mathrm{Dist}(\varphi)_{\mathrm{ray}})}{E'(\mathrm{Dist}(\varphi)_{\mathrm{ray}})}$$

By C. Li's inequality $M^{\mathrm{NA}} \leq M'$, we have

$$\boldsymbol{\mu}_{\mathrm{ray}} \leq \boldsymbol{\mu}_{\mathrm{NA}}.$$

Theorem B (To appear)

Assume X is smooth. There exists $\varphi \in \mathcal{E}^{\exp,1}_{\mathrm{NA}}(L)$ which maximizes $\boldsymbol{\mu}_{\mathrm{ray}}$.

If NA entropy regularization conjecture is true, the above theorem can be read as "there exists $\varphi \in \mathcal{E}^{\exp,1}_{\mathrm{NA}}(L)$ which maximizes $\boldsymbol{\mu}_{\mathrm{NA}}$".

Main applications III: Miscellaneous

Conjecture (For $\lambda \leq 0$)

$$\max_{\varphi \in \mathcal{E}^{\mathrm{exp},1}_{\mathrm{NA}}(L)} \boldsymbol{\mu}_{\mathrm{ray}}(\varphi) = \sup_{\varphi \in \mathcal{E}^{\mathrm{exp},1}_{\mathrm{NA}}(L)} \boldsymbol{\mu}_{\mathrm{NA}}(\varphi) = \inf_{\phi \in \mathcal{H}(\omega)} \boldsymbol{\mu}_{\mathrm{Per}}(\omega_\phi)$$

Theorem (For $\lambda \leq 0$, to appear)

If $\max_{\varphi \in \mathcal{E}^{\mathrm{exp},1}_{\mathrm{NA}}(L)} \boldsymbol{\mu}_{\mathrm{ray}}(\varphi) \leq \inf_{\phi \in \mathcal{H}(\omega)} \boldsymbol{\mu}_{\mathrm{Per}}(\omega_\phi)$, μ-cscK metrics are unique modulo $\mathrm{Aut}(X, L)$.

Theorem (to appear)

The following are equivalent:

1. (X, L) is K-semistable over $\mathcal{H}_{\mathrm{NA}}(L)$ (resp. $\mathcal{E}^1_{\mathrm{NA}}(L)$)
2. Trivial configuration maximizes $\boldsymbol{\mu}_{\mathrm{NA}}$ over $\mathcal{H}_{\mathrm{NA}}(L)$ (resp. $\mathcal{E}^1_{\mathrm{NA}}(L)$).

4. Distortion and Moment measure

Distortion of Non-archimedean L-psh function

Recall for a tc $(\mathcal{X}, \mathcal{L})$, we have a continuous function $\varphi_{(\mathcal{X},\mathcal{L})} : X^{\beth} \to \mathbb{R}$. For $v_E \in X^{\mathrm{div}} \subset X^{\beth}$ associated to $E \subset \mathcal{Y} \to \mathcal{X}$, we have

$$\varphi_{(\mathcal{X},\mathcal{L})}(v_E) = \frac{\mathrm{ord}_E(\mathcal{L} - p^* L)}{\mathrm{ord}_E(\mathcal{X}_0)}.$$

A function $\varphi : X^{\beth} \to [-\infty, \infty)$ is called L-psh function if there exists a net of tc's $(\mathcal{X}_i, \mathcal{L}_i)$ such that $\varphi_{(\mathcal{X}_i,\mathcal{L}_i)}$ is a decreasing net converging to φ pointwisely. Let $\mathrm{PSH}_{\mathrm{NA}}(L)$ denote the set of L-psh functions.

For an increasing concave $\chi : \mathbb{R} \to \mathbb{R}$, we introduce χ-distortion of φ:

$$\mathrm{Dist}_\chi(\varphi) := \inf_{t>0} (t \triangleright \varphi + \chi^{\blacktriangle}(t)),$$

where

$$(t \triangleright \varphi)(v) := t\varphi(t^{-1} v), \quad \chi^{\blacktriangle}(t) := \sup_{\sigma \in \mathbb{R}} (\chi(\sigma) - t\sigma).$$

Let's get a feeling of Distortion

Proposition (To appear)

For $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(L)$, we have $\mathrm{Dist}_\chi(\varphi) \in \mathrm{PSH}_{\mathrm{NA}}(L)$.

Suppose (X, L) is a toric variety. Let $P \subset M_{\mathbb{R}} = \mathfrak{t}^\vee$ be the moment polytope.

$$\{T\text{-invariant } \varphi\} \longleftrightarrow \{f : N_{\mathbb{R}} \to \mathbb{R}\}$$
$$\varphi \longmapsto f_\varphi(\xi) := \varphi(v_\xi) + \sup_{\mu \in P} \langle \mu, \xi \rangle$$

For the Legendre transform $g_\varphi(\mu) := \inf_{\xi \in N_{\mathbb{R}}} (\langle \mu, \xi \rangle + f_\varphi(\xi))$, we have

$$g_{\mathrm{Dist}_\chi(\varphi)} = \chi(g_\varphi)$$

thanks to the duality:

$$\inf_{t>0} (t\sigma + \chi^\blacktriangle(t)).$$

Profile of Distortion

For a continuous L-psh function $\varphi \in \mathrm{CPSH}_{\mathrm{NA}}(L)$, we can assign a filtration $\mathcal{F}_\varphi$ by putting

$$\mathcal{F}^\lambda_{\varphi,m} := \{ s \in H^0(X, L^{\otimes m}) \mid \inf_{v \in X^\beth} (\varphi(v) + v(s/s_v)/m) \geq \lambda/m \}.$$

For a normal test configuration $(\mathcal{X}, \mathcal{L})$, we have $\mathcal{F}_\varphi = \widehat{\mathcal{F}}_{(\mathcal{X},\mathcal{L})}$.

We can easily observe

$$\mathcal{F}^\lambda_{\mathrm{Dist}_\chi(\varphi),m} = \mathcal{F}^{m\chi^{-1}(\lambda/m)}_{\varphi,m}.$$

(cf. Reboulet–Witt Nyström)

e.g. For $\chi(\sigma) = -e^{-\sigma}$,

$$\mathcal{F}^\lambda_{\mathrm{Dist}_\chi(\varphi),m} := \begin{cases} \mathcal{F}^{-m\log(-\lambda/m)}_{\varphi,m} & \lambda < 0 \\ 0 & \lambda \geq 0 \end{cases}$$

Distortion and Duistermaat–Heckman measure

Proposition (arXiv:2202.12168, see also M. Xia)

For each $\sigma \in \mathbb{R}$, the map

$$\varphi_{(\mathcal{X},\mathcal{L})} \mapsto \int_{[\sigma,\infty)} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}$$

is monotonic. For general $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(L)$, we can define a unique Radon measure DH_φ satisfying $\int_{\mathbb{R}} \mathrm{DH}_\varphi \leq 1$ and

$$\int_{[\sigma,\infty)} \mathrm{DH}_\varphi = \inf \left\{ \int_{[\sigma,\infty)} \mathrm{DH}_{(\mathcal{X},\mathcal{L})} \;\middle|\; \varphi \leq \varphi_{(\mathcal{X},\mathcal{L})} \right\}.$$

Proposition (To appear)

For $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(L)$, we have $\mathrm{DH}_{\mathrm{Dist}_\chi(\varphi)} = \chi_* \mathrm{DH}_\varphi$.

Distortion and Ding invariant

Let us recall

$$D_{\mathrm{NA}}(\mathcal{X},\mathcal{L}) := \inf_{v \in X^{\beth}} (A_X(v) + \varphi_{(\mathcal{X},\mathcal{L})}(v)) - \int_{\mathbb{R}} \sigma \mathrm{DH}_{\varphi_{(\mathcal{X},\mathcal{L})}}$$
$$= \mathrm{lct}_{(\mathcal{X},\mathcal{X}_0 - (\mathcal{L}+K_{\mathcal{X}/\mathbb{A}^1}))}(\mathcal{X}_0) - \frac{n!}{(L^{\cdot n})} \frac{(\mathcal{L}^{\cdot n+1})}{(n+1)!}.$$

For $L_{\mathrm{NA}}(\varphi) := \inf_{v \in X^{\beth}} (A_X(v) + \varphi(v))$, we compute

$$\begin{aligned} L_{\mathrm{NA}}(\mathrm{Dist}_\chi(\varphi)) &= \inf_{v \in X^{\beth}} \inf_{t>0} (A_X(v) + t\varphi(t^{-1}v) + \chi^{\blacktriangle}(t)) \\ &= \inf_{t>0} \inf_{w \in X^{\beth}} (tA_X(w) + t\varphi(w) + \chi^{\blacktriangle}(t)) \\ &= \inf_{t>0} (t \inf_{w \in X^{\beth}} (A_X(w) + \varphi(w)) + \chi^{\blacktriangle}(t)) \\ &= \chi(L_{\mathrm{NA}}(\varphi)), \end{aligned}$$

using the duality

$$\inf(t\sigma + \chi^{\blacktriangle}(t)) = \chi(\sigma).$$

Let's observe $\boldsymbol{\mu}_{\mathrm{NA}}$

Recall

$$\boldsymbol{\mu}_{\mathrm{NA}}(\mathcal{X},\mathcal{L}) := \frac{n(K_X.L^{\cdot n-1})}{(L^{\cdot n})} - \frac{\frac{1}{(n-1)!}(K_X.L^{\cdot n-1}) - (K^{\log,\mathbb{G}_m}_{\mathcal{X}/\mathbb{P}^1} \cdot e^{\mathcal{L}_{\mathbb{G}_m}})}{\frac{1}{n!}(L^{\cdot n}) - (e^{\mathcal{L}_{\mathbb{G}_m}})}.$$

Using

$$K^{\log,\mathbb{G}_m}_{\mathcal{X}/\mathbb{P}^1} - p^* K_X = \sum_{E\subset\mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) A_X(v_E) E^{\mathbb{G}_m},$$

we can write

$$\begin{aligned}
(K^{\log,\mathbb{G}_m}_{\mathcal{X}/\mathbb{P}^1} \cdot e^{\mathcal{L}_{\mathbb{G}_m}}) &= ((K^{\log,\mathbb{G}_m}_{\mathcal{X}/\mathbb{P}^1} - p^* K_X) \cdot e^{\mathcal{L}_{\mathbb{G}_m}}) + (p^* K_X \cdot e^{\mathcal{L}_{\mathbb{G}_m}}) \\
&= \sum_{E\subset\mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) A_X(v_E) (E^{\mathbb{G}_m} \cdot e^{\mathcal{L}_{\mathbb{G}_m}}) \\
&\qquad + \frac{d}{ds}\Big|_{s=0} (e^{p^*(sK_X+L)+(\mathcal{L}_{\mathbb{G}_m}-p^*L)}) \\
&= \sum_{E\subset\mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) A_X(v_E) \frac{(L^{\cdot n})}{n!} \int_{\mathbb{R}} e^{-\sigma} \mathrm{DH}_{(E,\mathcal{L}|_E)} + \cdots
\end{aligned}$$

Moment measure

For a normal test configuration $(\mathcal{X}, \mathcal{L})$, we introduce the moment measure $\mathcal{D}_{(\mathcal{X},\mathcal{L})}$ on $X^{\beth} \times \mathbb{R}$ by

$$\mathcal{D}_{(\mathcal{X},\mathcal{L})} := \frac{n!}{(L^{\cdot n})} \sum_{E \subset \mathcal{X}_0} \mathrm{ord}_E(\mathcal{X}_0) \delta^{X^{\beth}}_{v_E} \otimes \mathrm{DH}_{(E, \mathcal{L}|_E) \circlearrowleft \mathbb{G}_m}.$$

Theorem (arXiv:2202.12168 + to appear)

We can define a measure $\mathcal{D}_\varphi$ on $X^{\beth} \times \mathbb{R}$ for $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(L)$ with $\int_{\mathbb{R}} \mathrm{DH}_\varphi = 1$ so that

$$\mathcal{D}_{\varphi_i} \to \mathcal{D}_\varphi \text{ weakly}$$

for any $\varphi_i \searrow \varphi$.

Distortion and Moment measure

For an increasing concave function χ, consider the map

$$\Delta^{\chi} : X^{\beth} \times \mathbb{R} \to X^{\beth} \times \mathbb{R} : (v, \sigma) \mapsto (\dot{\chi}(\sigma).v, \chi(\sigma)).$$

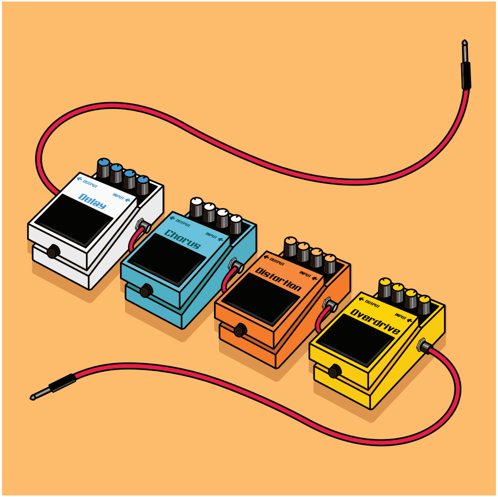
Theorem (To appear)

For $\varphi \in \mathcal{E}_{\mathrm{NA}}(L)$, we have

$$\mathcal{D}_{\mathrm{Dist}_{\chi}(\varphi)} = \Delta^{\chi}_{*} \mathcal{D}_{\varphi}.$$

Using $A_X(t.v) = tA_X(v)$, we compute

$$\begin{aligned} \int_{X^{\beth}} A_X(v) \int_{\mathbb{R}} \dot{\chi}(\sigma) \mathcal{D}_{\varphi}(v, \sigma) &= \int_{X^{\beth} \times \mathbb{R}} A_X(\dot{\chi}(\sigma).v) \mathcal{D}_{\varphi}(v, \sigma) \\ &= \int_{X^{\beth} \times \mathbb{R}} (\Delta^{\chi})^{*} A_X \mathcal{D}_{\varphi} = \int_{X^{\beth} \times \mathbb{R}} A_X \Delta^{\chi}_{*} \mathcal{D}_{\varphi} \\ &= \int_{X^{\beth} \times \mathbb{R}} A_X \mathcal{D}_{\mathrm{Dist}_{\chi}(\varphi)} = \int_{X^{\beth} \times \mathbb{R}} A_X \mathrm{MA}(\mathrm{Dist}_{\chi}(\varphi)). \end{aligned}$$



The name “Distortion” is inspired by effect pedal for electric guitar

Thank you!