Perelman's entropy in Kähler geometry

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Contents

This talk is based on my 4 articles:

- '19, Constant μ-scalar curvature Kähler metric ...
- '20, Equivariant calculus on μ-character and μK-stability ...
- '21a, Entropies in μ-framework of ..., I
- '21b, Entropies in μ-framework of ..., II (coming soon)

1. Overview of Kähler–Ricci soliton

⋮

What is the limit of Kähler–Ricci flow?

Kähler–Ricci soliton

Let X be a compact Kähler manifold.

Definition (Kähler–Ricci soliton)

A Kähler metric ω on X is called Kähler–Ricci soliton if

$$\mathrm{Ric}\omega - \mathcal{L}_{\partial^\sharp f}\omega = \lambda\omega$$

for some holomorphic $\partial^\sharp f = g^{i\bar{j}}\partial_{\bar{j}} f \partial_i$ and $\lambda \in \mathbb{R}$. Here $f \in C^\infty(X, \mathbb{R})$.

- $\exists \Rightarrow -2\pi K_X = \lambda L$ for $L := [\omega]$. $\rightsquigarrow \lambda \neq 0 \Rightarrow X$ is algebraic.
- $\partial^\sharp f = 0 \Rightarrow$ Kähler–Einstein metric
- $\partial^\sharp f \neq 0 \Rightarrow \lambda > 0 \Rightarrow X$ is a Fano manifold, i.e. $\exists \omega$ with $\mathrm{Ric}\omega > 0$.

The one/two points blowing-up of $\mathbb{C}P^2$ admits KRs, no KE metrics.

$\xi := -2\mathrm{Im}\partial^\sharp f$ is Killing $\Rightarrow \exists$ torus action $X \circlearrowleft T$ with $\xi \in \mathfrak{t}$.
f is a moment map for ξ: $df = -i_\xi \omega$.

Tian–Zhu's volume minimization and uniqueness

Proposition (Tian–Zhu '02)

Let $(X, \omega) \circlearrowleft T$ be a torus action on a Fano manifold with a moment map $\mu : X \to \mathfrak{t}^\vee$. Then the functional

$$-\log \int_X e^{\omega+\mu} : \mathfrak{t} \to \mathbb{R} : \xi \mapsto -\log \int_X e^{\mu_\xi} \omega^n / n!$$

- depends only on the equiv. coh. class $L_T := [\omega + \mu] \in H^2_T(X, \mathbb{R})$,
- admits a unique critical point/maximizer.

If X admits a Kähler–Ricci soliton with $\xi \in \mathfrak{t}$, then ξ must be the critical point.

Theorem (Tian–Zhu '02 , Berndtsson '15)

Kähler–Ricci solitons (ω, ξ) are unique modulo $\mathrm{Aut}(X)$, if it exists.

The existence of Kähler–Ricci soliton

- Wang–Zhu: Every toric Fano manifold admits KRs.
- Delcroix: Every horospherical Fano manifold admits KRs.

Theorem (Berman–Witt-Nyström '14 + Datar–Székelyhidi '16)

The existence of KRs is equivalent to modified K-polystability.

Modified K-polystability is the positivity of modified Futaki invariant $\mathrm{Fut}_\xi(\mathcal{X}, \mathcal{L})$ for non-product special degenerations.
Modified K-semistability is the semi-positivity of modified Futaki invariant.

- Tian: A small perturbation of MU3 does not admit KRs, though it is (modified) K-semistable.
- Delcroix: There exists a Fano manifold which is not even modified K-semistable.

What we should remember here is that modified K-semistability is an algebraic condition depending on the choice of $\xi \in \mathfrak{t}$: to compute $\mathrm{Fut}_\xi(\mathcal{X}, \mathcal{L})$, we must fix ξ.

Test configuration

Let (X, L) be a T-equivariant polarized variety. A T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$ of (X, L) is a $\mathbb{C}^\times \times T$–equivariant flat family of schemes over $\mathbb{A}^1 = \mathbb{C}$ endowed with

- a relatively ample $\mathbb{C}^\times \times T$-equivariant $\mathbb{Q}$-line bundle $\mathcal{L}$ and
- a specific isomorphism $(X, L) \cong (\mathcal{X}_1, \mathcal{L}|_{\mathcal{X}_1})$ of the fibre over $1 \in \mathbb{A}^1$.

We have a natrual compactification $(\bar{\mathcal{X}}, \bar{\mathcal{L}})$ over $\mathbb{P}^1$.

We assign the following filtration $\mathcal{F}_{(\mathcal{X},\mathcal{L})}$ on $R = \bigoplus_m R_m = \bigoplus_m H^0(X, L^{\otimes m})$:

$$\mathcal{F}^\lambda_{(\mathcal{X},\mathcal{L})} R_m := \{ s \in R_m \mid \varpi^{-\lceil \lambda \rceil} \bar{s} \text{ extends to a section of } \mathcal{L}^{\otimes m} \}.$$

For a *normal* test configuration $(\mathcal{X}, \mathcal{L})$ and an irreducible component $E \subset \mathcal{X}_0$, we assign the following valuation v_E on $\mathbb{C}(X)$:

$$v_E(f) := \frac{\mathrm{ord}_E(f \circ p_X)}{\mathrm{ord}_E \mathcal{X}_0}$$

Kähler–Ricci soliton and Kähler–Ricci flow

Kähler–Ricci soliton $\omega \in -K_X$ is a critical point of He's entropy functional

$$H(\omega) := \int_X h e^h \omega^n / n!,$$

where $h \in C^\infty(X, \mathbb{R})$ is characterized by $\sqrt{-1}\partial\bar{\partial}h = \mathrm{Ric}\omega - 2\pi\omega$ and $\int_X e^h \omega^n / n! = 1$.

The gradient flow of this functional is normalized Kähler–Ricci flow:

$$\dot{\omega}_t = \omega_t - \mathrm{Ric}\omega_t.$$

Perelman and Tian–Zhang–Zhang–Zhu showed that if X admits a KRs, then the nKR flow converges to the KRs modulo gauge $\mathrm{Aut}(X)$.

Hamilton–Tian conjecture: nKR flow has a subsequence converging to a length space in the Gromov–Hausdorff topology which is a smooth KRs outside a real codimension 4 closed subset.

The limit of Kähler–Ricci flow

Theorem (Chen–Sun–Wang '18 (cf. Chen–Wang, Bamler))

For any Fano manifold X and any initial Kähler metric ω, the Gromov–Hausdorff limit along Kähler–Ricci flow exists as a Fano variety $\hat{X}(\omega)$ metrized by a Kähler–Ricci soliton $\hat{\omega}$: $\mathrm{Ric}(\hat{\omega}) - L_\xi \hat{\omega} = \hat{\omega}$.

Moreover, there exist the following two step degenerations:

1. $X \rightsquigarrow \bar{X}(\omega)$ via a filtration $\mathcal{F}$ on $R = \bigoplus_m R_m = \bigoplus_m H^0(X, -mK_X)$, which generates $\bar{X}(\omega) \circlearrowleft \mathbb{R} = \exp(\mathbb{R} J\xi_o)$,
2. $\bar{X}(\omega) \rightsquigarrow \hat{X}(\omega)$ via $\mathbb{R}$-equivariant special degeneration, on which $\hat{X}(\omega) \circlearrowleft \mathbb{R} = \exp(\mathbb{R} J\xi_o)$.

Moreover, $\bar{X}(\omega)$ is a modified K-semistable Fano variety w.r.t. $\bar{\xi}$.

Filtration is a generalization of test configuration.

$$\bar{X}(\omega) = \mathrm{Proj}\Big(\bigoplus_m \bigoplus_{\lambda \in \mathbb{R}} t^{-\lambda} \mathcal{F}^\lambda R_m / \mathcal{F}^{\lambda+} R_m \Big)$$

Algebraic characterization of the first degeneration

Dervan–Székelyhidi '20 discovered H-entropy $H_{\mathrm{NA}}(\mathcal{F})$ for finitely generated filtration $\mathcal{F}$, which extends Tian–Zhu's volume functional, and proved

$$\sup_{\mathcal{F}} H_{\mathrm{NA}}(\mathcal{F}) = \inf_{\omega_\phi \in -K_X} H(\omega_\phi).$$

The supremum is attained by the first degeneration $X \rightsquigarrow \bar{X}(\omega)$.

Han–Li '20 proved the maximizer of H_{NA} is unique. As a consequence, $\bar{X}(\omega)$ does not depend on the choice of the initial metric ω. Moreover, they extend H_{NA} to general filtrations and showed the maximizer exists.

Blum–Liu–Xu–Zhuang '21 proved the maximizing filtration is indeed finitely generated.

Remark: The last two results rely on Birkar's theorem on the boundedness of complements (birational geometry).

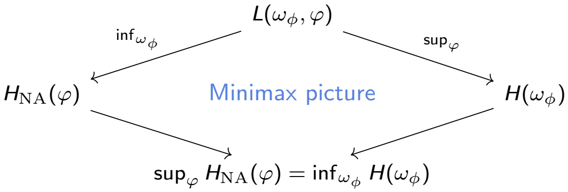
Minimax picture for $\sup_{\mathcal{F}} H_{\mathrm{NA}}(\mathcal{F}) = \inf_{\omega_\phi} H(\omega_\phi)$

Consider

$$L(\omega, f) := \frac{\int_X f e^h \omega^n}{\int_X e^h \omega^n} - \log \int_X e^f \omega^n / n!.$$

By Jensen's inequality, we have $\sup_f L(\omega, f) = H(\omega)$.
On the other hand, $L(\omega_t, -\dot{\phi}_t)$ is monotonically decreasing along $C^{1,\bar{1}}$-geodesic rays ϕ_t, thanks to the convexity of Ding functional.
This shows $H_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \leq \inf_{\omega_\phi} L(\omega_\phi, \dot{\phi}_{(\mathcal{X},\mathcal{L})})$.

$$H_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \leq \inf_{\omega_\phi} L(\omega_\phi, \dot{\phi}_{(\mathcal{X},\mathcal{L})}) \leq \sup_{(\mathcal{X},\mathcal{L})} L(\omega_\phi, -\dot{\phi}_{(\mathcal{X},\mathcal{L})}) \leq H(\omega)$$



Where to go next?

Want to realize analogous picture for general polarized manifold (X, L).

What we want? $\cdots$ a framework of canonical metrics enjoying the following features:

- Moment map picture $\rightsquigarrow$ existence and stability

'19 (moduli paper): $\exists$ moment map picture for Kähler–Ricci soliton

$$\mathrm{Ham}_T(M,\omega) \curvearrowright (\mathcal{J}_T(M,\omega), \Omega_\xi) \xrightarrow{\mathcal{S}_\xi^\lambda} \mathfrak{ham}_T^\vee$$

- Volume minimization $\rightsquigarrow$ optimal degeneration

This is the main topic of this talk.

2. μ-cscK metrics and μK-stability

The theory begins with "moment map picture"

and encounters with Perelman's entropy

via "volume minimization".

μ-cscK metric

(X, L): a compact Kähler manifold and an ample line bundle L
$\xi \in \mathfrak{t}$: a vector in the real lie algebra of a torus T acting on (X, L)

Definition (μ-cscK metric, '19, cf. Y. Nakagawa, generalized KRs '11)

For $\lambda \in \mathbb{R}$, a Kähler metric $\omega \in c_1(L)$ is called $\check{\mu}_\xi^\lambda$-cscK metric if

$$s_{\mu_\xi}^\lambda(\omega) := (s(\omega) + \bar{\square}\mu_\xi) + (\bar{\square}\mu_\xi + |\partial^\sharp \mu_\xi|^2) - \lambda\mu_\xi = \mathrm{const}$$

for $\mu_\xi \in C^\infty(X, \mathbb{R})$ satisfying $d\mu_\xi = -i_\xi\omega \iff \xi = -2\mathrm{Im}\partial^\sharp \mu_\xi$.

- KRs $\mathrm{Ric}(\omega) - \mathcal{L}_{\partial^\sharp \mu_\xi}\omega = \lambda\omega \Leftrightarrow \check{\mu}_\xi^\lambda$-cscK metric in L: $-2\pi K_X = \lambda L$.
- Extremal metric $\Leftrightarrow$ the limit of μ^λ-cscK metrics as $\lambda \to -\infty$.

On $(X, L) = (\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}, -K_X)$, there exists a μ^λ-cscK metric for every $\lambda \in \mathbb{R}$ which gives a smooth path from KRs at $\lambda = 2\pi$ to extremal metric at $\lambda = -\infty$.

μK-stability

Let K_X denote the canonical sheaf on X, which is a T-equiv reflexive sheaf. For a T-equiv *normal* test configuration $(\mathcal{X}, \mathcal{L})$, we define $\check{\mathrm{Fut}}^\lambda_\xi(\mathcal{X}, \mathcal{L})$ by

$$\frac{(2\pi K^T_{\bar{\mathcal{X}}/\mathbb{P}^1} - \lambda \bar{\mathcal{L}}_T . e^{\bar{\mathcal{L}}_T}; \xi) \cdot (e^{L_T}; \xi) - (2\pi K^T_X - \lambda L_T . e^{L_T}; \xi) \cdot (e^{\bar{\mathcal{L}}_T}; \xi)}{(e^{L_T}; \xi)^2} - \lambda \frac{(e^{\bar{\mathcal{L}}_T}; \xi)}{(e^{L_T}; \xi)}.$$

Definition (μK-stability, '20)

A T-equivariant polarized normal variety (X, L) is called $\check{\mu}^\lambda_\xi$K-semistable if $\check{\mathrm{Fut}}^\lambda_\xi(\mathcal{X}, \mathcal{L}) \geq 0$ for every normal T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$.

Theorem (Lahdili '19, I. '20, cf. Apostolov–Lahdili–Jubert '21)

If a polarized manifold (X, L) admits a $\check{\mu}^\lambda_\xi$-cscK metric, it is $\check{\mu}^\lambda_\xi$K-semistable.

Boundedness + the slope of μ-Mabuchi functional along smooth subgeodesic ray subordinate to a smooth test configuration. It is equivariant localization.

You can safely forget the definition of μ-Futaki invariant! This is hard to compute. We have another criterion for μK-semistability as we will see.

Volume minimization (entropy maximization)

Proposition ('19)

$(X, \omega) \circlearrowleft K$: Hamiltonian action. There is a functional

$$\check{\boldsymbol{\mu}}^\lambda : \mathfrak{k} \longrightarrow \mathbb{R}$$
$$\xi \mapsto \check{\mathcal{W}}^\lambda(\omega, \mu_\xi^\omega)$$

depending only on $L_T = [\omega + \mu]$ which satisfies the following.

1. Its derivative at $\xi \in \mathfrak{k}$ is $\check{\mathrm{Fut}}_\xi^\lambda := \int_X s_{\mu_\xi}^\lambda \mu_\bullet e^{\mu_\xi} \omega^n / n! \in \mathfrak{k}^\vee$.
2. It is proper and bounded from above, hence attains a maximum, which implies the existence of ξ with $\check{\mathrm{Fut}}_\xi^\lambda = 0$.
3. The value
$$\lambda_{\mathrm{ice}}(X, L) := \sup\{\lambda \in \mathbb{R} \mid \check{\boldsymbol{\mu}}^{\lambda'} \text{ has a unique crit. pt. for } \forall \lambda' < \lambda\}$$
is finite (never $\pm\infty$). $\rightsquigarrow$ phase transition
4. Let $\xi_\lambda \in \mathfrak{k}$ be the critical point of $\check{\boldsymbol{\mu}}^\lambda$ for $\lambda \ll 0$. Then $\lim_{\lambda \to -\infty} \lambda \xi_\lambda$ is the extremal vector field.

Perelman's entropy and μ-cscK metrics

For a Kähler metric $\omega \in L$ and $f \in C^{0,1}(X)$, we put

$$\check{\mathcal{W}}^{\lambda}(\omega, f) := -\frac{\int_X \left(s(\omega) + |\partial^{\sharp} f|^2 - \lambda(n+f)\right) e^f \omega^n}{\int_X e^f \omega^n} - \lambda \log \int_X e^f \omega^n / n!.$$

We regard $\check{\mathcal{W}}^{\lambda}$ as a functional on the tangent bundle

$$T\mathcal{H}(X, L) = \mathcal{H}(X, L) \times C^{\infty}(X)/\mathbb{R}.$$

Theorem ('21a)

A state (ω, f) is a critical point of $\check{\mathcal{W}}^{\lambda}$ if and only if $\partial^{\sharp} f$ is holomorphic and ω is $\check{\mu}^{\lambda}_{\xi}$-cscK metric for $\xi = \mathrm{Im}\partial^{\sharp} f$.
For $\lambda \leq 0$,

$$\check{\boldsymbol{\mu}}^{\lambda}(\omega) := \sup_f \check{\mathcal{W}}^{\lambda}(\omega, f)$$

gives a smooth functional on $\mathcal{H}(X, L)$ and its critical points are precisely $\check{\mu}^{\lambda}$-cscK metric for some ξ.

Critical points of $\check{\boldsymbol{\mu}}^{\lambda}$ are actually global minimizers (later).

$C^{1,1}$ geodesic rays

A geodesic ray emanating from a Kähler metric $\omega_\phi = \omega + dd^c \phi$ is a $U(1)$-invariant locally bounded $p_X^* \omega$-psh function Φ on $X \times \bar{\Delta}^*$ satisfying

$$\Phi|_{X \times \partial \bar{\Delta}^*} = \phi \quad \text{and} \quad (dd^c_\omega \Phi)^{n+1} = 0.$$

We put $\phi_t := \Phi(\cdot, e^{-t})$ and $\omega_{\phi_t} := \omega + dd^c \phi_t \in \mathrm{PSH}(X, L)$.

For a normal test configuration $(\mathcal{X}, \mathcal{L})$, we can take a smooth function Φ_Ω on $X \times \bar{\Delta}^*$ so that $\omega + dd^c \Phi_\Omega = \Omega|_{X \times \bar{\Delta}^*}$ and $i_\eta d^c \Phi_\Omega = \mu_\eta$ for a smooth equivariant form $\Omega + \mu \in c_{1,U(1)}(\beta^* \bar{\mathcal{L}})$ on $\beta : \tilde{\mathcal{X}} \to \bar{\mathcal{X}}$.

Theorem (Chu–Tossati–Weinkove '18, (cf. Phong–Sturm))

- For a normal test configuration $(\mathcal{X}, \mathcal{L})$, there exists a unique $C^{1,1}_{\mathrm{loc}}$-geodesic ray Φ such that $\Phi - \Phi_\Omega$ is globally bounded.
- If $\mathcal{X}$ is smooth, $\Phi - \Phi_\Omega$ extends to a $C^{1,1}$-function on $\mathcal{X}$ across the central fibre.

Monotonicity along $C^{1,1}$ geodesic rays

$\check{\mathcal{W}}(\omega_{\phi_t}, -\dot{\phi}_t) = -\int_X (s(\omega_{\phi_t}) + |\partial^\sharp \dot{\phi}_t|^2) e^{-\dot{\phi}_t} \omega_{\phi_t}^n / \int_X e^{-\dot{\phi}_t} \omega_{\phi_t}^n$ is not well-defined for $C^{1,1}$-regular ϕ_t.

Observe for smooth ϕ_t, the antiderivative

$$\int_0^t ds \int_X (s(\omega_{\phi_s}) + |\partial^\sharp \dot{\phi}_s|^2) e^{-\dot{\phi}_s} \omega_{\phi_s}^n$$

can be written as

$$\int_X \frac{d\mu_t}{d\mu_0} \log \frac{d\mu_t}{d\mu_0} d\mu_0 - \int_X (\dot{\phi}_t - \dot{\phi}_0) e^{-\dot{\phi}_t} \omega_{\phi_t}^n + \int_0^t ds \int_X n\mathrm{Ric}(\omega_{\phi_0}) \wedge e^{-\dot{\phi}_s} \omega_{\phi_s}^{n-1}.$$

for $\mu_t = e^{-\dot{\phi}_t} \omega_{\phi_t}^n$. We denote it by $\mathcal{A}_\Phi(t)$.

Theorem ('21a, argument analogous to Berman–Berndtsson '17)

For any $C^{1,1}_{\mathrm{loc}}$-geodesic ray Φ emanating from a smooth ω_ϕ, $\mathcal{A}_\Phi(t)$ is convex and continuous up to the boundary of $[0, \infty)$ and we have

$$\check{\mathcal{W}}_\flat(\omega_\phi, -\dot{\phi}_0) := -\frac{\frac{d}{dt_+}|_{t=0} \mathcal{A}_\Phi(t)}{\int_X e^{-\dot{\phi}_0} \omega_\phi^n} \leq \check{\mathcal{W}}(\omega_\phi, -\dot{\phi}_0)$$

.

What appears in the limit?: non-archimedean μ-entropy

Theorem ('21a, argument analogous to Z. Sjöström Dyrefelt '18)

For the geodesic ray Φ subordinate to a *smooth* test configuration $(\mathcal{X}, \mathcal{L})$, we put $\Phi_{;\rho}(x, \tau) := \Phi(x, |\tau|^\rho)$ for $\rho > 0$. Then

$$- \lim_{t \to \infty} \frac{\frac{d}{dt_+} \mathcal{A}_{\Phi_{;\rho}}(t)}{\int_X e^{-\dot{\phi}_{;\rho,t}} \omega^n_{\phi_{;\rho,t}}} = 2\pi \frac{(K_X . e^L) - \frac{\rho}{\pi} (K^{\log, \mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} . e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \frac{\rho}{\pi})}{(e^L) - \frac{\rho}{\pi} (e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \frac{\rho}{\pi})}.$$

Here $K^{\log, \mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} := (K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}} + \mathcal{X}_0^{\mathrm{red}, \mathbb{C}^\times}) - \varpi^* (K^{\mathbb{C}^\times}_{\mathbb{P}^1} + 0^{\mathbb{C}^\times})$.

Now we put $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) := \check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) + \lambda \check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \rho)$

$$\check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) := 2\pi \frac{(K_X . e^L) - \rho (K^{\log, \mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} . e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \rho)}{(e^L) - \rho (e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \rho)}$$

$$\check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \rho) := \frac{(L . e^L) - \rho (\bar{\mathcal{L}}_{\mathbb{C}^\times} . e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \rho)}{(e^L) - \rho (e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \rho)} - \log((e^L) - \rho (e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \rho))$$

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$$-\lim_{t\to\infty} \frac{\frac{d}{dt_+} \mathcal{A}_{\Phi_{;\rho}}(t)}{\int_X e^{-\dot{\phi}_{;\rho,t}} \omega^n_{\phi_{;\rho,t}}} = 2\pi \frac{(K_X . e^L) - \frac{\rho}{\pi}(K^{\log,\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} . e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \frac{\rho}{\pi})}{(e^L) - \frac{\rho}{\pi}(e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \frac{\rho}{\pi})}.$$

Here $K^{\log,\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} := (K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}} + \mathcal{X}_0^{\mathrm{red},\mathbb{C}^\times}) - \varpi^*(K^{\mathbb{C}^\times}_{\mathbb{P}^1} + 0^{\mathbb{C}^\times})$.

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$$\check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \rho) := 2\pi \frac{(\kappa^{\mathbb{C}^\times}_{\mathcal{X}_0} . e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \rho) - \rho(\mathcal{X}_0^{\mathrm{red},\mathbb{C}^\times} - \mathcal{X}_0^{\mathbb{C}^\times} . e^{\mathcal{L}_{\mathbb{C}^\times}}; \rho)}{\int_{\mathbb{R}} e^{-\rho t} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}}$$

$$\check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \rho) := \frac{\int_{\mathbb{R}} (n - \rho t) e^{-\rho t} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}}{\int_{\mathbb{R}} e^{-\rho t} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}} - \log \int_{\mathbb{R}} e^{-\rho t} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}$$

Summary on W-entropy

Let $\mathcal{H}_{\mathrm{NA}}(X, L)$ denote the set of normal test configurations and φ denote its element. Later, we will identify φ with a function on the Berkovich space X^{NA}. Using the geodesic ray $\Phi = \{\phi_t\}$ emanating from ω_ϕ subordinate to φ, we put

$$\check{\mathcal{W}}^\lambda(\omega_\phi, \varphi) := \check{\mathcal{W}}^\lambda_\flat(\omega_\phi, -\dot{\phi}_0)$$

Summary

To sum up, we have

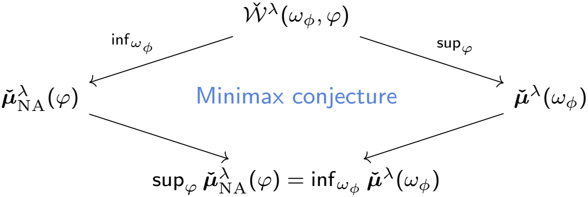
$$\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\varphi) \le \inf_{\omega_\phi \in \mathcal{H}(X,L)} \check{\mathcal{W}}^\lambda(\omega_\phi, \varphi) \le \sup_{\varphi \in \mathcal{H}_{\mathrm{NA}}(X,L)} \check{\mathcal{W}}^\lambda(\omega_\phi, \varphi) \le \check{\boldsymbol{\mu}}^\lambda(\omega_\phi).$$

Theorem ('21a)

If (X, L) admits a $\check{\mu}^\lambda_\xi$-cscK metric ω for $\lambda \le 0$, then

$$\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\xi) = \sup_{\varphi \in \mathcal{H}_{\mathrm{NA}}(X,L)} \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\varphi) = \inf_{\omega_\phi \in \mathcal{H}(X,L)} \check{\boldsymbol{\mu}}^\lambda(\omega_\phi) = \check{\boldsymbol{\mu}}^\lambda(\omega).$$

A conjectural picture: minimax picture



cf. Sion's minimax theorem

Let W be a convex subset of a linear topological space and C be a compact convex subset of a linear topological space. If $f : W \times C \to \mathbb{R}$ is a function satisfying

- $f(w, \cdot) : C \to \mathbb{R}$ is usc and quasi-concave: $f(w, \cdot)^{-1}((-\infty, a))$ is convex,
- $f(\cdot, c) : W \to \mathbb{R}$ is lsc and quasi-convex: $f(\cdot, c)^{-1}((a, \infty))$ is convex,

then we have

$$\sup_{c \in C} \inf_{w \in W} f(w, c) = \inf_{w \in W} \sup_{c \in C} f(w, c).$$

Non-archimedean μ-entropy and μK-semistability

Theorem ('21b, essentially observed in '20)

If a finitely generated filtration $\mathcal{F}$ maximizes $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$, then the central fibre is μ^λK-semistable with respect to the vector ξ generated by the filtration.

We obtain the following criterion for μK-stability

Proposition ('19 + '21b)

Suppose for every test configuration $(\mathcal{X}, \mathcal{L})$ there exists $\xi \in \mathfrak{t}$ such that $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \leq \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\xi)$. Then (X, L) is μ^λK-semistable for some ξ.

and a new proof for the following

Corollary ('21b)

If (X, L) admits a $\check{\mu}^\lambda_\xi$-cscK metric for $\lambda \leq 0$, then it is $\check{\mu}^\lambda_\xi$K-semistable.

Relation to H-entropy

For a Fano manifold $(X, L) = (X, -K_X)$, we have

$$2\pi H(\omega_\phi) \leq \check{\boldsymbol{\mu}}^{2\pi}(\omega_\phi)$$

and

$$\check{\boldsymbol{\mu}}^{2\pi}_{\mathrm{NA}}(\varphi) \leq 2\pi H_{\mathrm{NA}}(\varphi).$$

Theorem (Dervan–Székelyhidi '20 + '21b)

We have

$$\inf_{\omega_\phi \in \mathcal{H}(X,L)} 2\pi H(\omega_\phi) = \inf_{\omega_\phi \in \mathcal{H}(X,L)} \check{\boldsymbol{\mu}}^{2\pi}(\omega_\phi)$$

and

$$\sup_{\varphi \in \mathcal{H}_{\mathrm{NA}}(X,L)} \check{\boldsymbol{\mu}}^{2\pi}_{\mathrm{NA}}(\varphi) = \sup_{\varphi \in \mathcal{H}_{\mathrm{NA}}(X,L)} 2\pi H_{\mathrm{NA}}(\varphi).$$

3. Moment measure and non-archimedean μ-entropy

$$\begin{array}{ccc} \mathcal{H}_{\mathrm{NA}}(X, L) & \xrightarrow{\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}} & [-\infty, \infty) \\ \big\downarrow & \nearrow_{\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}} & \\ \mathcal{E}^{\mathrm{exp}}_{\mathrm{NA}}(X, L) & & \end{array}$$

$$-\frac{\int_{X^{\mathrm{NA}}} (2\pi A_X + \lambda\varphi) \int e^{-t} \mathcal{D}_\varphi + E_{\mathrm{exp}}^{2\pi K_X + \lambda L}(\varphi)}{\iint_{X^{\mathrm{NA}}} e^{-t} \mathcal{D}_\varphi} - \lambda \log \iint_{X^{\mathrm{NA}}} e^{-t} \mathcal{D}_\varphi$$

Global pluripotential theory over trivially valued fields

The Berkovich space X^{NA} is a compactification of $\mathrm{Val}(X)$. It consists of semi-valuations v: $v \in \mathrm{Val}(Y)$ for some irreducible variety $Y \subset X$.

We consider the following function $\varphi_{(\mathcal{X},\mathcal{L})}$ on X^{NA} for a test configuration $(\mathcal{X}, \mathcal{L})$:

$$\varphi_{(\mathcal{X},\mathcal{L})}(v) := \inf\{\sigma \in \mathbb{R} \mid \mathcal{F}_{(\mathcal{X},\mathcal{L})} \subset \mathcal{F}_v[\sigma]\},$$

where $\mathcal{F}_v[\sigma]^\lambda R_m := \{s \in R_m \mid v(s) + m\sigma \geq \lambda\}$.
We have $\varphi_{(\mathcal{X},\mathcal{L})} = \varphi_{(\mathcal{X}',\mathcal{L}')}$ iff the normalization coincides:

$$\mathcal{H}_{\mathrm{NA}}(X, L) = \{\varphi_{(\mathcal{X},\mathcal{L})}\}.$$

Definition (Boucksom–Jonsson)

A non-archimedean psh metric on (X, L) is a function on X^{NA} given as the limit of a decreasing net $\{\varphi_{(\mathcal{X}_i,\mathcal{L}_i)}\}$. We denote the set of NA psh metrics by $\mathrm{PSH}_{\mathrm{NA}}(X, L)$.

Continuous extension to $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$

For $\varphi \in \mathrm{PSH}_{\mathrm{NA}}(X, L)$ and $\rho > 0$, we put

$$E_{\exp}(\varphi_{;\rho}) := \inf \Big\{ -\int_{\mathbb{R}} e^{-\rho t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})} \ \Big|\ \varphi \le \varphi_{(\mathcal{X},\mathcal{L})} \in \mathcal{H}_{\mathrm{NA}}(X, L) \Big\}$$

and

$$\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L) := \{\varphi \in \mathrm{PSH}(X, L) \mid E_{\exp}(\varphi_{;\rho}) > -\infty \text{ for } \forall \rho > 0\}.$$

Theorem ('21b)

There is a metric $d_{\exp}$ on $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$ for which $\varphi \mapsto \int e^{-t} \mathcal{D}_\varphi$ is continuous. Moreover, there is a continuous extension of the following to $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$:

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\varphi) + \frac{\int_{X^{\mathrm{NA}}} A_X \int e^{-t} \mathcal{D}_\varphi}{\int\int_{X^{\mathrm{NA}}} e^{-t} \mathcal{D}_\varphi}.$$

Suppose X is smooth (or the continuity of envelopes holds for (X, L)). Then the metric space $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$ is complete.

The non-archimedean μ-entropy on $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$

Corollary ('21b)

The non-archimedean μ-entropy $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\varphi) =$

$$-\frac{\int_{X^{\mathrm{NA}}} (2\pi A_X + \lambda\varphi) \int e^{-t}\mathcal{D}_\varphi + E^{2\pi K_X + \lambda L}_{\exp}(\varphi)}{\iint_{X^{\mathrm{NA}}} e^{-t}\mathcal{D}_\varphi} - \lambda \log \iint_{X^{\mathrm{NA}}} e^{-t}\mathcal{D}_\varphi$$

is well-defined and upper semi-continuous on $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$.

Conjecture

The subset

$$\left\{ \varphi \in \mathcal{E}^{\exp}_{\mathrm{NA}}(X, L) \;\middle|\; \sup \varphi = 0, \check{\boldsymbol{\mu}}_{\mathrm{NA}}(\varphi) \geq C \right\}$$

is weakly compact and $E_{\exp}(\varphi_{;\rho}) = -\iint_{X^{\mathrm{NA}}} e^{-\rho t}\mathcal{D}_\varphi$ is bounded for any $\rho > 0$.

If the conjecture holds, then there exists a maximizer of $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$ for $\lambda \leq 0$. Let's call it a non-archimedean μ-cscK metric and study the regularity!

Thank you for listening!