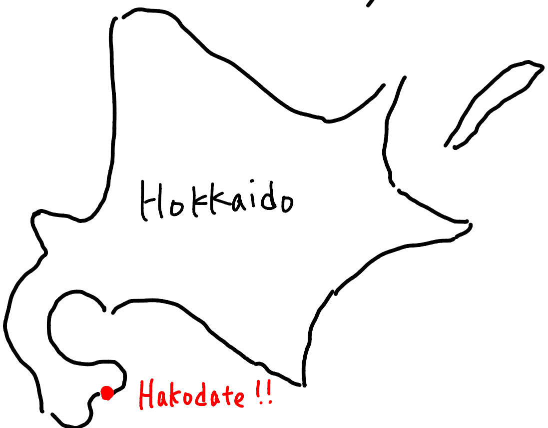
Kähler-Ricci soliton and K-st@bility, beyond Kähler-Einstein



Contents

References

[Tian-Zhu '02] A new holomorphic invariant and uniqueness of Kähler-Ricci solitons
Fundamental results

[Berman & Witt-Nyström '14] Complex optimal transport and the pluricanonical theory of Kähler-Ricci solitons $^{\exists}$KR ⇒ mK-ps
def of ^{w}DF

[Datar-Székelyhidi '16] Kähler-Einstein metrics along the smooth continuity method mK-ps ⇒ $^{\exists}$KR

[Chen-Sun-Wang '15 arXiv] Kähler-Ricci flow, Kähler-Einstein metric and K-stability

[Wang-Zhu '04] Kähler-Ricci solitons on toric manifolds with positive first Chern class toric Fano ⇒ $^{\exists}$KR

[Delcroix '18] K-stability of Fano spherical varieties horospherical Fano ⇒ $^{\exists}$KR

[Inoue '18 arXiv] The moduli space of Fano manifolds with Kähler-Ricci solitons

I. Modified K-stability

- X : $\mathbb{Q}$-Fano variety $/\mathbb{C}$

Consider some torus action (not nec maximal)

$$X \curvearrowleft T \simeq (\mathbb{C}^*)^s .$$

- $M = \mathrm{Hom}(T, \mathbb{C}^*)$: the character lattice $\simeq \mathbb{Z}^s$
- $N = \mathrm{Hom}(\mathbb{C}^*, T)$: the 1-psg lattice. $\simeq \mathbb{Z}^s$

The Duistermaat-Heckman measure associated to $X \curvearrowleft T$ is a measure on $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ def'd by

$$DH_{X \curvearrowleft T} := \lim_{g \to \infty} \frac{1}{g\, h^0(X, -g\ell K_X)} \sum_{u \in M} h^0_u(X, -g\ell K_X)\, \delta_{g^{-1}u}$$

, where $h^0(X, -g\ell K_X) = \sum_{u \in M} h^0_u(X, -g\ell K_X)$ is the wt decomp.

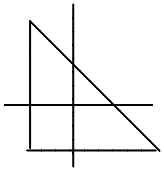
<u>Rem.</u>

DH measure is supported on a polytope, containing $0 \in M_{\mathbb{R}}^{\vee}$ called the moment polytope,
which is the image of a continuous (w.r.t. analytic topology) map
$\mu : X \to M_{\mathbb{R}}$. ($\mu$ is called the moment map)

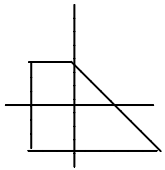
Actually, $DH = \mu_* \omega^n / \int_X \omega^n$ $\quad \omega \in 2\pi c_1(-\ell K_X)$: Kähler met.

<u>eg.</u> If $X \curvearrowleft T$ is toric, DH is the (normalized) restriction of the Euclidean measure on $M_{\mathbb{R}}$ (def'd by the lattice $M \subset M_{\mathbb{R}}$) to the canonical polytope.

$\mathbb{C}P^2$



1 pt blow up of $\mathbb{C}P^2$

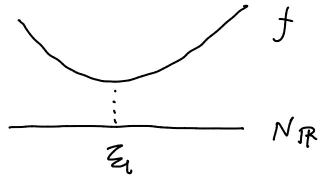


Consider a fct'l (smooth)

$$f : N_{\mathbb{R}} \to \mathbb{R} : \xi \longmapsto \int_{M_{\mathbb{R}}} e^{\langle x . \xi \rangle} DH(x) .$$

This is strictly convex (as $e^{\langle x . \xi \rangle}$ is) and proper (as $0 \in \mathrm{Supp}(DH)$)

Hence $\exists! \ \xi \in N_{\mathbb{R}}$ the minimizer of f.



$$\Leftrightarrow \quad Df_{\xi} = \int_{M_{\mathbb{R}}} \langle x , \bullet \rangle e^{\langle x . \xi \rangle} DH(x) \equiv 0$$

$Df_{\xi} \in N_{\mathbb{R}}^* = M_{\mathbb{R}}$

(weighted barycenter)

Completeness of $\mathbb{R}$.

We call this $\xi \in N_{\mathbb{R}}$ the K-optimal vector of $X \curvearrowleft T$.

Def A special degeneration of $X \curvearrowleft T$ consists of the following data

- $\begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ \mathbb{C} \end{array}$: $\mathcal{X}$ is a normal $\mathbb{Q}$-Gorenstein variety w/ a $T \times \mathbb{C}^*$-action
  π is a $\mathbb{C}^*$-equiv proj flat morphism w/ $\mathbb{Q}$-Fano fibers.
  which is trivial away from the origin.

- $$\begin{array}{ccc} X \times \mathbb{C}^\times & \xrightarrow[\sim]{\theta} & \pi^{-1}(\mathbb{C}^\times) \\ \downarrow & & \downarrow \\ \mathbb{C}^\times & = & \mathbb{C}^\times \end{array}$$: θ is a $T \times \mathbb{C}^*$-equiv isom $/ \mathbb{C}^\times$

I'd like to use "weighted"
as "modified" sounds "secondary"

For a sp dg (π, θ), we define the modified Donaldson-Futaki inv

stands for "weighted"

$$\mathrm{DF}^{w}(\pi, \theta) := -\int_{M_{\mathbb{R}} \times \mathbb{R}} \langle x, (0,1)^{N_{\mathbb{R}} \times \mathbb{R}} \rangle \, e^{\langle x, \xi \rangle} \, DH_{\mathcal{X}_0 \curvearrowleft T \times \mathbb{C}^*}(x),$$

where $\xi \in N_{\mathbb{R}}$ is the K-optimal vector of $X \curvearrowleft T$.
(it is also the K-op vec of $\mathcal{X}_0 \curvearrowleft T$
as DH is T-equiv deform invariant)

Def (modified K-stability)

$X \curvearrowleft T$ is called modified K-semistable if for any sp dg (π, θ)

$^{w}DF(\pi, \theta) \geq 0$.

It is called modified K-polystable if moreover

$^{w}DF(\pi, \theta) = 0 \Rightarrow \mathcal{X} \simeq X \times \mathbb{C}$: $T \times \mathbb{C}^*$-equiv'ly.

($\Leftarrow$ if mKss)

modified K-stable if mKps & $\underbrace{\mathrm{Aut}^{\circ}_{T}(X)}_{\text{the id comp of the centralizer of } T} = T$.

Rem

mK stability is equivalent to (T-equiv) Kstability iff the K-op vec $\xi = 0$.

$\Updownarrow$

$DF = \int x \, DH(x) = 0 \in M_{\mathbb{R}}$

$\Updownarrow$

$DF(\pi, \theta) = 0$ for product dg.

II. Why modified K-stability? — relation with Kähler-Ricci soliton

For any smooth X, the modified K-polystability of X w.r.t. some torus action (e.g. take a maximal $X \curvearrowleft T$) is proved to be equivalent to the existence of some canonical metrics on X.

[Datar-Székelyhidi '16]

— BASIC FACTS [Tian-Zhu '02] —

- KR soliton is unique up to $\mathrm{Aut}^\circ(X)$ if it exists.
- $\mathrm{Aut}(X, \xi')$ is reductive.

  This is important in order to apply local GIT in the construction of our coming moduli space.

Kähler-Ricci soliton

$$\boxed{\mathrm{Ric}\, g - L_{\xi'}\, g = g}$$

. $\mathrm{Im}\,\xi'$ is related to the K-optimal vector.

This equation naturally arises in the limit of the normalized Kähler-Ricci flow : $\frac{\partial}{\partial t} g(t) = -\mathrm{Ric}(g(t)) + g(t)$.

A DIGRESSION ...

Chen-Sun-Wang's surprising result arXiv '15

nKR flow on ANY Fano mfds X converges to some KR soliton on some $\mathbb{Q}$-Fano var X_0 (in the Gromov-Hausdorff sense)

If X admits a KR soliton, then $X_0 = X$.

$X \rightsquigarrow X_0'$ two step degeneration $\rightsquigarrow X_0$

Their result can be thought as an analogy of

Harder-Narasimhan filtration	& **Jordan-Hölder filtration**
$\forall$ coh shvs $\rightsquigarrow$ semistable coh shvs	semistable coh shvs $\rightsquigarrow$ polystable coh shvs (This is GIT)

$\underline{Q.}$ How about the uniqueness?

Anyway, there should be large amount of examples ...!

III. Examples

- ANY toric Fano mfd admits KR solitons [Wang-Zhu '04]

  $\Rightarrow$ ANY toric Fano mfd is mK-stable w.r.t. max torus action

  possibly (often) mK-polystable w.r.t. a smaller torus.

  eg. $\left\{\begin{array}{l} \cdot\ \mathbb{CP}^2 \text{ is mK-ps w.r.t. } T = \{1\}. \\ \cdot\ 2\text{pt blow up of } \mathbb{CP}^2 \text{ is mK-ps w.r.t. some } T = \mathbb{C}^* \end{array}\right.$

We call $X \circlearrowleft T$ is K-optimal if the K-op vec $\xi \in N_{\mathbb{R}}$ is not contained in any smaller $N'_{\mathbb{R}} \subsetneq N_{\mathbb{R}}$ w/ $N' \subsetneq N$: sublattice.

Rem.

If $X \circlearrowleft T_1, T_2$ are K-optimal actions so that $X \circlearrowleft T_1, T_2$ are both mK-ss. Then T_1 and T_2 are conjugate to each other.

· ANY horospherical Fano mfd is mK-polystable w.r.t. some K-optimal action

$\cdot\shortparallel\cdot$

cptf'tn of toric bdl / homogeneous sp.

[Delcroix '18]

In this class. an important example X_0 w/ non Einstein KR soliton

[Pasquier-Perrin '10] + [Delcroix '18]

$\left\{\begin{array}{l} \text{the Kuranishi family of } X_0 \text{ is a 1-dim isotrivial family} \\ \begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathbb{C} \end{array} \quad \text{whose general fibre } X_t \cong X_1 \text{ is the orth} \\ \text{Grass } Gr_g(2.7) \text{, which has KE met. (esp. KR sol)} \end{array}\right.$

Though both X_0, X_1 admit KR solitons, they cannot be separated

in the family $\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathbb{C} \end{array}$ ...

$\boxed{\begin{array}{l} \text{It is NOT } T\text{-equiv deformation !} \\ (T = \text{the K-op action on } X_0) \end{array}}$

Our observations and examples illustrate

$$\left\{ \text{family of Fano mfds } \begin{matrix} \mathcal{X} \\ \downarrow \\ S \end{matrix} \text{ whose each fibre admits KR sol's} \right\} \Big/ S\text{-isom}$$

(⇔ each fibre is mK-ps for some T-action)

$$\uparrow$$

$\underline{\text{not full}}$ (nor faithful ?)

$$\left\{ \begin{array}{l} \text{T-equiv family of Fano T-mfds } \begin{matrix} \mathcal{X} \circlearrowleft T \\ \downarrow \\ S \end{matrix} \text{ whose each fibre} \\ \text{is mK-ps K-op} \end{array} \right\} \Big/ \begin{array}{l} \text{T-equiv} \\ S\text{-isom} \end{array},$$

while

$$\left\{ \text{Fano mfd } X \text{ which admits KR solitons} \right\} / \text{isom}$$

$$\updownarrow \quad 1 : 1$$

$$\left\{ \text{mK-ps K-op Fano T-mfd } X \circlearrowleft T \right\} / \text{T-equiv isom}.$$

IV. Moduli space - the main theorem

SUPEREXPESS INTRODUCTION to STACKS (cf. [I. Appendix A])

A stack over $\mathrm{Alg}_{\acute{e}t}/\mathrm{Cpx}$ is a category over $\mathrm{Alg}_{\acute{e}t}/\mathrm{Cpx}$,

i.e. a functor $\mathcal{C} \to \mathrm{Alg}_{\acute{e}t}/\mathrm{Cpx}$

enjoying some geometric assumptions (such as $\exists$ p.b., can glue obj's)

A typical example is $\mathrm{Alg}_X \to \mathrm{Alg}_{\acute{e}t}$ for $X \in \mathrm{Alg}$.

$$(S \to X) \mapsto S$$

2-Yoneda : $\mathrm{Alg} \hookrightarrow \mathrm{Stacks}/_{\mathrm{Alg}_{\acute{e}t}}$: $X \longmapsto (\mathrm{Alg}_X \to \mathrm{Alg}_{\acute{e}t})$

"fully faithful"

Let $\mathcal{K}$ be a stack over $\mathrm{Alg}_{\acute{e}t}$ $(\mathrm{Sch}_{\acute{e}t})$ $/\ \mathbb{C}px$ consisting of T-equivariant families of mK-ss K-op Fano T-mfds.

Namely, $\mathcal{K}$ is a category whose object is a T-equivariant family $\begin{matrix} \mathcal{X} \circlearrowleft T \\ \downarrow \\ S \end{matrix}$ whose each fiber is a mK-ss K-op Fano mfd.

Morphism is a pair of a base morphism $f: S \to S'$ and a T-equiv Cartesian morphism $/_f$ $\phi: \mathcal{X} \to \mathcal{X}'$.

$\mathcal{K} \to \mathrm{Alg}_{\acute{e}t}/\mathbb{C}px$: $\begin{matrix} \mathcal{X} \circlearrowleft T \\ \downarrow \\ S \end{matrix} \longmapsto S$ makes $\mathcal{K}$ into a stack.

<u>Prop.</u>(I) $\mathcal{K}$ is Artin over $\mathbb{C}px$. (Artin over $\mathrm{Alg}_{\acute{e}t}$? : NOT KNOWN)

3/IV

Def. the moduli space of ${}^{\text{a stack}}\ \mathcal{C}/\mathrm{Cpx}$ is a complex analytic space M with a morphism $\mathcal{C} \to M$ enjoying the following universal property.

$$\begin{tikzcd} \mathcal{C} \arrow[d] \arrow[dr, "\forall"] & \\ M \arrow[r, dashed, "\exists !"'] & X \end{tikzcd} \qquad \begin{array}{c} \mathrm{Stacks}/\mathrm{Cpx} \\ \cup \\ \mathrm{Cpx} \end{array}$$

The moduli space is unique up to isom if it exists.

e.g. $[\mathbb{C}^2/\mathbb{C}^*] \to pt = \mathbb{C}^2/\!/\mathbb{C}^*$ (good in the sense of Alper)

$[\mathbb{C}^2 \setminus \{(0,0)\} / \mathbb{C}^*] \to \mathbb{CP}^1$ (actually, this is isom)

$[\mathbb{CP}^1/\mathbb{C}^*] \to pt$ (not good in the sense of Alper)

$0, \infty \in \mathbb{CP}^1$: distinct closed orbits goes to the same pt.

$$|\mathcal{X}| := \{ \underset{\mathrm{Spec}\,\mathbb{C}}{\overset{\shortparallel}{pt}} \to \mathcal{X} / \sim \} = \{ \text{mK-ss K-op Fano T-mfd } X \}$$

Thm (I.) For any $X \in |\mathcal{X}|$, there exists an $\overset{\text{T-equiv}}{\text{isotrivial}}$ degeneration $\begin{matrix} \mathcal{X} \\ \downarrow \\ \mathbb{C} \end{matrix}$ of X whose central fiber $\mathcal{X}_0$ is a mK-ps K-op $\mathbb{Q}$-Fano T-var.

Moreover, the central fiber is uniquely determined up to T-equiv isom for any such degenerations of X.

We define an equivalence relation $\overset{K}{\sim}$ on $|\mathcal{X}|$ by

$X \overset{K}{\sim} Y$ if the central fibers in the above thm coincides.

From the thm, for a K-equivalence class $[X]_K$, if there exists an mK-ps smooth $X_0 \in [X]$, then it is unique up to isom.

Let $\mathcal{K}^\circ \subset \mathcal{K}$ be a substack of $\mathcal{K}$ consisting of T-equiv families of mK-ss K-op Fano T-mfds which are K-equivalent ($\overset{K}{\sim}$) to some smooth mK-ps K-op Fano T-mfds (i.e. $\exists\, X_0 \in [X]$: mK-ps smooth)

<u>Main thm</u> (I.)

There exists the moduli space $\mathcal{K}^\circ \xrightarrow{\phi} M$ of $\mathcal{K}^\circ$

Moreover, ϕ induces a bijective map $|\mathcal{K}^\circ| / \overset{K}{\sim} \longrightarrow M$.

$$|\mathcal{K}^\circ| / \overset{K}{\sim} = \{\text{mK-ps K-op Fano T-mfd}\} = \{\text{Fano mfd which admits KR sol's}\}$$

~~Proof~~ gluing "$H^1_T(X.TX) /\!/ \mathrm{Aut}_T(X)$" together. ... $\blacksquare$

cf. Slide B in Hayama symposium

Some remarks.

Prop (I.) The topology on $M \mathrel{\hat{=}} \{$ Fano with $\exists$ KR solitons $\}$ coincides with the Gromov-Hausdorff topology w.r.t. KR solitons.

There is a moment map picture suitable in our setting, which I do not explain here.

Actually, I used the moment map to construct the moduli space.

V. Why modified K-stability?

- conjectural pictures

Harder-Narasimhan filtration for $\mathbb{Q}$-Fano var X

"Conj" There exists a sequence of sp dgs $\begin{matrix}\mathcal{X}_1\\ \downarrow\\ \mathbb{C}\end{matrix}$, $\begin{matrix}\mathcal{X}_2\\ \downarrow\\ \mathbb{C}\end{matrix}$, ..., $\begin{matrix}\mathcal{X}_s\\ \downarrow\\ \mathbb{C}\end{matrix}$

where $\begin{matrix}\mathcal{X}_1\\ \downarrow\\ \mathbb{C}\end{matrix}$ is a T_0-equiv sp dg of $X \curvearrowleft \underline{T_0}$

maximal K-op action
i.e. if $X \curvearrowleft T_0'$ is K-op, $T_0' \supset T_0$ then $T_0' = T_0$

with minimal $c_1 := {}^{w}\mathrm{DF}/\mathrm{norm} \leq 0$

Set $T_i := T_{i-1} \times \mathbb{C}^*$ and X_i : the central fibre of $\begin{matrix}\mathcal{X}_i\\ \downarrow\\ \mathbb{C}\end{matrix} \curvearrowleft T_{i-1} \times \mathbb{C}^*$.

Then $\begin{matrix}\mathcal{X}_{i+1}\\ \downarrow\\ \mathbb{C}\end{matrix}$ is a T_i-equiv sp dg of $X_i \curvearrowleft T_i$ with minimal $c_{i+1} := {}^{w}\mathrm{DF}/\mathrm{norm}$.

such that $c_{i+1} > c_i$ & $c_s = 0$, & $X_s \curvearrowleft T_s$ is mK-ss.

(toric Fano $\doteq$ lim bdl $\subset$ coh shv ?)