Perelman's entropy in Kähler geometry

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23, July, 2021, Hayama

Perelman's W-entropy

X: a compact Kähler manifold
L: a Kähler class on X; $L = c_1(\boldsymbol{L})$

Perelman's W-entropy

For a Kähler metric $\omega \in \mathcal{H}(X, L)$ and a smooth function $f \in C^\infty(X, \mathbb{R})$ normalized as $\int_X e^f \omega^n/n! = 1$, we put

$$\check{W}(\omega, f) := -\int_X (s(\omega) + |\partial^\sharp f|^2) e^f \omega^n/n!,$$

$$\check{S}(\omega, f) := \int_X (n + f) e^f \omega^n/n!,$$

$$\check{W}^\lambda(\omega, f) := \check{W}(\omega, f) + \lambda \check{S}(\omega, f).$$

$$\check{W}^\lambda : T\mathcal{H}(X, L) = \mathcal{H}(X, L) \times C^\infty(X)/\mathbb{R} \to \mathbb{R}.$$

μ-cscK metric

Theorem (I. '21)

A state $(\omega, f) \in T\mathcal{H}(X, L)$ is a critical point of $\check{W}^\lambda$ if and only if $\xi = \partial^\sharp f$ is holomorphic and ω satisfies the equation

$$(s(\omega) + \bar{\Box} f) + (\bar{\Box} f + |\partial^\sharp f|^2) = \lambda f - \mathrm{const}.$$

We call a Kähler metric ω *μ^λ-cscK metric* if it satisfies the above equation for some f with holomorphic $\partial^\sharp f$. We put

$$s_f(g) := (s(g) + \bar{\Box} f) + (\bar{\Box} f + |\partial^\sharp f|^2).$$

- $\mu_\xi^{2\pi}$-cscK metric in $c_1(X) \Leftrightarrow$ KRs $\mathrm{Ric}(\omega) - L_\xi \omega = \omega$.
- Extremal metric $\Rightarrow$ μ^λ-cscK metric for $\lambda \ll 0$.

μ-entropy

We put

$$\boldsymbol{\mu}^\lambda(\omega) := \sup_{\int_X e^f \omega^n/n! = 1} \check{W}^\lambda(\omega, f).$$

Theorem (Rothaus '81, I. '21)

For each λ and ω, there exists f attaining the maximum of $\check{W}^\lambda(\omega, \cdot)$. When $\lambda \leq 0$, such f is unique. Consequently, $\boldsymbol{\mu}^\lambda$ is smooth on $\mathcal{H}(X, L)$ for $\lambda \leq 0$. The critical points are precisely μ^λ-cscK metrics in this case.

The functional is analogous to Calabi functional for extremal metric.

Moment map picture

$(M, \omega) \curvearrowleft T$: Hamiltonian torus action. For $\xi \in \mathfrak{t}$ and $\lambda \in \mathbb{R}$, the map

$$\mathcal{S}_\xi^\lambda : \mathcal{J}_T \to \mathfrak{ham}_T^\vee : J \mapsto (s_{\mu_\xi}(g_J) - \lambda \mu_\xi) e^{\mu_\xi} \omega^n$$

gives a moment map for a symplectic manifold $(\mathcal{J}_T, \Omega_\xi) \curvearrowleft \mathrm{Ham}_T$.

μ_ξ^λ-Futaki invariant

$$\mathrm{Fut}_\xi^\lambda(\zeta) := -\int_X \mu_\zeta (s_{\mu_\xi}(\omega) - \lambda \mu_\xi) e^{\mu_\xi} \omega^n.$$

$\rightsquigarrow$ generalized to test configuration via equivariant intersection formula

$\Rightarrow$ μ_ξ^λK-stability $\supset$ modified K-stability

Theorem (Lahdili '19, I. '20, Apostolov–Jubert–Lahdili '21)

If (X, L) admits a $\check{\mu}_\xi^\lambda$-cscK metric, then it is $\check{\mu}_\xi^\lambda$K-semistable.

Volume mimimization: μ-Futaki invariant vanishing

Proposition (A generlaization of Tian–Zhu's result. I. '19)

$(X, \omega) \curvearrowleft K$: Hamiltonian action. The functional

$$\begin{aligned} \boldsymbol{\mu}^{\lambda}_{\mathrm{NA}} : \ \mathfrak{k} &\longrightarrow \mathbb{R} \\ \xi &\mapsto \check{W}^{\lambda}(\omega, \mu^{\omega}_{\xi}) \end{aligned}$$

is independent of the choice of $\omega \in \mathcal{H}(X, L)$.

1. Its derivative at $\xi \in \mathfrak{k}$ is $\mathrm{Fut}^{\lambda}_{\xi}$.
2. It is proper and bounded from above, hence attains a maximum, which implies the existence of ξ with $\mathrm{Fut}^{\lambda}_{\xi} = 0$.
3. The value

   $$\lambda_{\mathrm{ice}}(X, L) := \sup\{\lambda \in \mathbb{R} \mid \boldsymbol{\mu}^{\lambda'}_{\mathrm{NA}} \text{ has a unique crit. pt. for } \forall \lambda' < \lambda\}$$

   is finite (never $\pm\infty$).

Non-archimedean μ-entropy

We can extend $\boldsymbol{\mu}^\lambda_{\mathrm{NA}}$ to test configurations $(\mathcal{X}, \mathcal{L})$.

Example: Non-archimedean μ-entropy for toric configuration

Let P be the moment polytope of a toric variety (X, L). For a convex function q normalized as $\int_P e^q d\mu = 1$, we put

$$\boldsymbol{\mu}^\lambda_{\mathrm{NA}}(q) := \int_{\partial P} e^q d\sigma - \lambda \int_P (n+q) e^q d\mu$$

We have $\boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(q_{(\mathcal{X}, \mathcal{L})})$.

Theorem (I. '21)

$$\sup_{(\mathcal{X}, \mathcal{L})} \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) \leq \inf_{\omega \in \mathcal{H}(X, L)} \boldsymbol{\mu}^\lambda(\omega)$$

Sketch of proof

- $\check{W}^\lambda(\omega_{\phi_t}, -\dot{\phi}_t)$ is monotonically decreasing.
  Consider the action functional $\mathcal{A}(t) :=$ "$-\int_0^t \check{W}^\lambda(\omega_{\phi_s}, -\dot{\phi}_s) ds$" and show its convexity by a tensor calculus on equivariant differential forms and Berman–Berndtsson's subharmonicity argument. (cf. the convexity of Mabuchi functional)
- $\lim_{t\to\infty} \check{W}^\lambda(\omega_{\phi_t}, -\dot{\phi}_t) = \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L})$
  The equivariant Stokes theorem and the regularity of geodesic ray across the central fibre (Chu–Tosatti–Weinkove).

$$\check{\boldsymbol{\mu}}^\lambda(\omega) \geq \check{W}^\lambda(\omega_{\phi_{\tau.t}}, -\dot{\phi}_{\tau.t}) \to \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau)$$

Cororally

For $\lambda \leq 0$, all the critical points of $\boldsymbol{\mu}^\lambda$ are global minimizers, which are precisely μ^λ-cscK metrics. If there is a $\check{\mu}^\lambda_\xi$-cscK metric ω, we have

$$\sup_{(\mathcal{X},\mathcal{L};\tau)} \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\phi_\xi) = \boldsymbol{\mu}^\lambda(\omega) = \inf_{\omega_\varphi \in \mathcal{H}(X,L)} \boldsymbol{\mu}^\lambda(\omega_\varphi).$$

Thank you for listening!