Complex analytic Moduli space of Fano manifolds admitting Kähler-Ricci solitons

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Moduli problem

(M, ω): a C^∞-symplectic manifold.

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- (X, c) and (X', c') are biholomorphic if

$$\exists \phi : X \xrightarrow{bihol.} X' \text{ s.t. } c = \phi^* c'.$$

Complex structure on the moduli space

e.g. The moduli space of all K3 surfaces is a 20-dimensional complex manifold but for Hausdorffness. The moduli space of polarized K3 surafces is a 19-dimensional Hausdorff complex space.

As local semi-moduli (constructed by Kuranishi after Kodaira-Spencer) of deformation of complex manifolds are complex spaces, it is natural to expect: the global moduli space should also admit a natural structure of complex analytic space.

Problematic Fano

Main interest: $\pm\omega \in 2\pi c_1(M,\omega)$ ($c = \pm c_1(X)$) or $c_1(M,\omega) = 0$.

- $c_1(M,\omega) < 0$ general type: $\exists$ Hausdorff moduli.
- $c_1(M,\omega) = 0$ Calabi-Yau: $\exists$ Hausdorff moduli (under polarization).
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$[\mathcal{X}_0]$ cannot be separated from $[X]$!

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The following is a success story.

Theorem (Odaka '15, Li-Wang-Xu '15)

Kähler-Einstein Fano manifolds admit a natural Hausdorff, complex analytic moduli space.

Remark: YTD-correspondence (proved by Chen-Donaldson-Sun, Tian) on KE metrics and K-stability is used in the construction of the moduli.

Today's goal: enlargement

The goal of this talk:

Theorem (I. '17)

Let $\mathcal{KR}(M, \omega)$ be the set of Fano manifolds with Kähler-Ricci solitons of fixed symplectic type (M, ω). We can make this set $\mathcal{KR}(M, \omega)$ into a Hausdorff complex analytic space in a canonical way.

Kähler-Ricci soliton is a special metric on a Fano manifold, which generalizes Kähler-Einstein metric in view of Kähler-Ricci flow.

1. Kähler-Ricci soliton, moment map and K-optimal vector

Kähler-Ricci soliton

Definition (Kähler-Ricci soliton)

A Kähler-Ricci soliton on a Fano manifold X is a pair (g, ξ') of a Kähler metric g and a holomorphic vector field ξ' satisfying the following equation:

$$\mathrm{Ric}(g) - L_{\xi'} g = g.$$

Remember $\xi := \mathrm{Im}(\xi')$ generates a closed torus $T_\xi^{\mathbb{R}} = \overline{\exp \mathbb{R}\xi} \subset \mathrm{Aut}(X)$.

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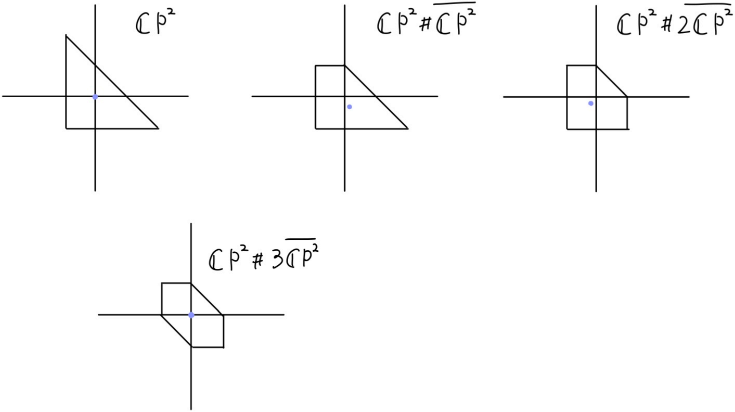
Theorem (Uniqueness, Tian-Zhu '02)

If (g_1, ξ_1') and (g_2, ξ_2') are two Kähler-Ricci solitons on a Fano manifold X, then there exists an element $\phi \in \mathrm{Aut}^0(X)$ such that

$$\phi^* \xi_1' = \xi_2', \quad \phi^* g_1 = g_2.$$

Example: All toric Fano manifolds admit Kähler-Ricci solitons.



Infinite dimensional symplectic manifold $\mathcal{J}_T$

Fix our notation.

- (M, ω): a simply connected C^∞-symplectic manifold.
- (T, ξ): a closed torus acting on (M, ω) and an element of $\mathrm{Lie}(T)$.
- θ_ξ: $d\theta_\xi = 2i_\xi\omega$ with the normalization $\int_M \theta_\xi \, e^{\theta_\xi}\omega^n = 0$.
- $\mathcal{J}_T$: the space of T-inv. almost complex srtuctures on (M, ω).
- Ham_T: the group of T-equiv. symplectic diffeomorphisms of (M, ω).

$\mathcal{J}_T$ admits a Ham_T-invariant smooth symplectic form Ω_ξ defined by

$$\Omega_{\xi,J}(A, B) := \int_M \mathrm{Tr}(JAB) \, e^{\theta_\xi}\omega^n$$

for $A, B \in T_J\mathcal{J}_T$.

Moment map $s_\xi : \mathcal{J}_T \to \mathfrak{ham}_T^*$

We identify the Lie algebra $\mathfrak{ham}_T := \mathrm{Lie}(\mathrm{Ham}_T)$ with

$$C^\infty_{T,\xi}(M) = \{f \in C^\infty_T(M) \mid \int_M f \; e^{\theta_\xi} \omega^n = 0\}.$$

Proposition (I. '17)

The map $S_\xi : \mathcal{J}_T \to \mathfrak{ham}_T^*$ defined by

$$J \mapsto \langle s_\xi(J), \bullet \rangle_\xi = \int_M \Big(s(g_J) + \bar{\Box}_{g_J} \theta_\xi - \bar{s} + \bar{\Box}_{g_J} \theta_\xi - \xi'_J \theta_\xi - \theta_\xi \Big) \bullet \; e^{\theta_\xi} \omega^n.$$

is a moment map of $(\mathcal{J}_T, \Omega_\xi) \curvearrowleft \mathrm{Ham}_T$.

If $c_1(M, \omega) > 0$, an integrable complex structure J satisfies $s_\xi(J) = 0$ iff g_J is a Kähler-Ricci solitons on the Fano manifold (M, J).

We have the following immediate cororally.

For $f \in \mathfrak{t} \subset \mathfrak{ham}_T = C^\infty_{T,\xi}(M)$, an easy caluculation shows that the modified Futaki invariant is equal to

$$\langle s_\xi, f \rangle_\xi = \int_M s_\xi f \ e^{\theta_\xi} \omega^n,$$

which shows the T-equivariant deformation invariance of the modified Futaki invariant restricted to $\mathfrak{t} \subset H^0(M, T_J M)$.

Because

$$\frac{d}{dt} \langle s_\xi(J_t), f \rangle_\xi = -\Omega_\xi(L_{X_f} J_t, \dot{J}_t) = 0.$$

Remark

If the set $s_\xi^{-1}(0)$ is not empty, then $\langle s_\xi, \bullet \rangle_\xi$ should vanish.

K-optimal vector

The following is a restatement of Tian-Zhu's result.

Proposition (Tian-Zhu '02)

For any $(M, \omega) \curvearrowleft T$, there exists a **unique** vector $\xi \in \mathrm{Lie}(T)$ with $\langle s_\xi, \bullet \rangle \equiv 0$.

Definition

- We call this unique vector ξ the K-optimal vector of $(M, \omega) \curvearrowleft T$.
- We call the action $(M, \omega) \curvearrowleft T$ is K-optimal if the K-optimal vector generates T.

The boundedness of Fano manifolds (Kollár-Miyaoka-Mori)
$\Longrightarrow$ Only finitely many non-equivalent K-optimal actions $(M, \omega) \curvearrowleft T_i$ with $s_{\xi_i}(J_i) = 0$ for some integrable $J_i \in \mathcal{J}_{T_i}$ and the K-optimal ξ_i.

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The uniqueness of KR-soliton $\Longrightarrow$ The topological space

$$K(M, \omega) := \coprod_i (s_{\xi_i}|_{\mathcal{J}_{T_i}^{\mathrm{int}}})^{-1}(0) \ / \ \mathrm{Ham}_{T_i}$$

can be naturally identified with the set of biholomorphism classes of Fano manifolds admitting KR-solitons of symplectic type (M, ω). The space $K(M, \omega)$ is Hausdorff as the action $\mathcal{J}_T \curvearrowleft \mathrm{Ham}_T$ is proper.

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Remark: The space $K(M, \omega)$ is naturally identified with the space

$$\{ J \in \mathcal{J}^{\mathrm{int}}(M, \omega) \mid (M, J) \text{ admits a Kähler-Ricci soliton } \} / \mathrm{bihol}.$$

as sets, but not as topological spaces. Actually, the latter topological space is not Hausdorff in general. The equivariant formulation is essential.

2. Structure of the moduli space

Charts

We must construct a structure of complex space on the space $K(M, \omega)$.

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1. A homeomorphism $\phi : U \to V \subset K(M,\omega)$ where V is an open neighbourhood of a point $X = (M,J) \in K(M,\omega)$ and U is a small open neighbourhood of $[0] \in H^1_{T_i}(X, TX) /\!/ \mathrm{Aut}_{T_i}(X)$.

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The map s_ξ is not holomorphic, $0 \in \mathfrak{ham}^*_T$ is not the regular value of s_ξ, the Fréchet Lie group Ham_T is not complex Lie group, etc.

:(

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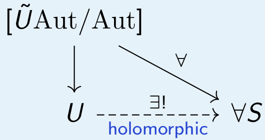
Generality on GIT quotient: a natural morphism $[\tilde{U}\mathrm{Aut}/\mathrm{Aut}] \to U$ enjoying the following universal property:



The story of holomorphic gluing

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$$[\tilde{U}_1\mathrm{Aut}_1/\mathrm{Aut}_1] \rightarrow \mathcal{K}(M,\omega) \leftarrow [\tilde{U}_2\mathrm{Aut}_2/\mathrm{Aut}_2]$$

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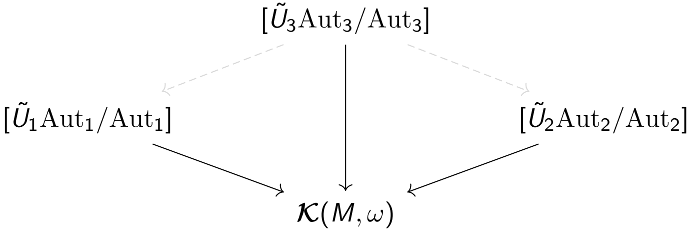

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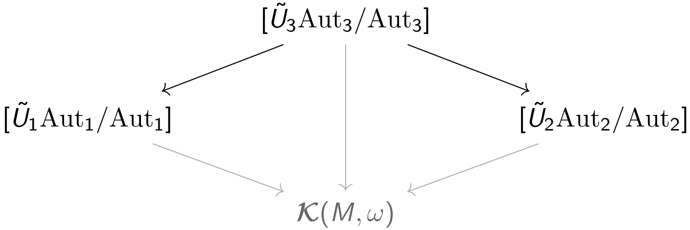

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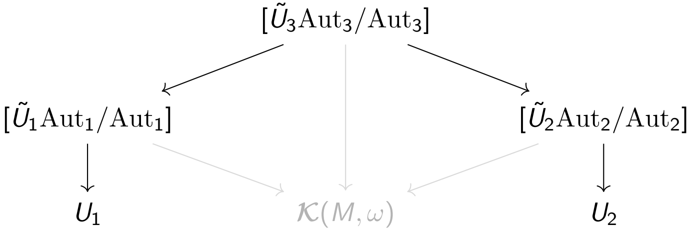

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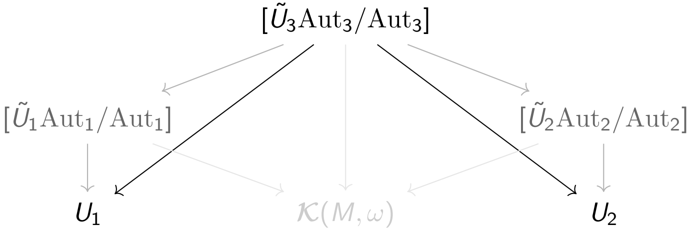

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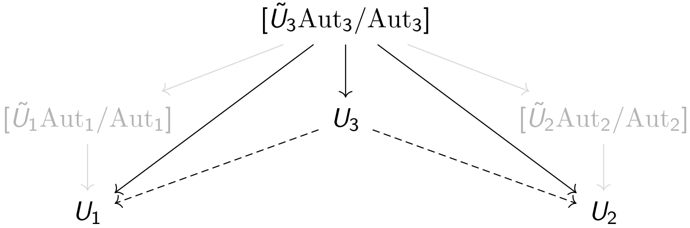

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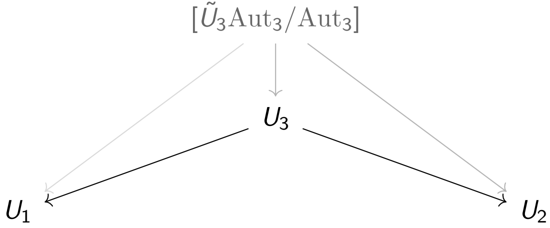

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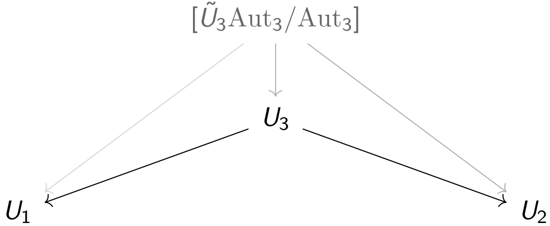

3. These morphisms are compatible with maps $\phi_i : U_i \to K(M, \omega)$ $(i = 1, 2, 3)$. $\Longrightarrow$ $U_3 \to U_i$ are biholomorphic onto its open image.

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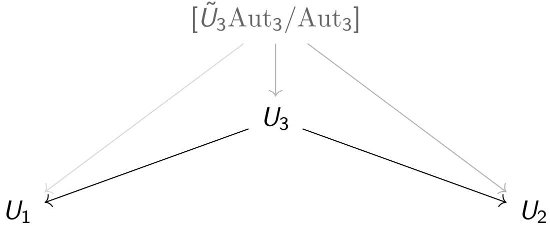

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4. The coordinate change $\phi_2^{-1} \circ \phi_1$ between U_1 and U_2 is holomorphic.

The universality of the moduli space

Theorem (I. '17)

The complex space $K(M,\omega)$ is uniquely characterized by the universality of a natural morphism $\mathcal{K}(M,\omega) \to K(M,\omega)$.

$$\begin{array}{ccc} \mathcal{K}(M,\omega) & & \\ \downarrow & \overset{\forall}{\searrow} & \\ K(M,\omega) & \overset{\exists!}{\dashrightarrow} & \forall S \end{array}$$

Gromov-Hausdorff topology

Proposition (I. '17)

The space

$$\mathcal{KR}_{GH}(M,\omega) = \{\text{Fano mfds of type } (M,\omega) \text{ with } \exists \text{ KR solitons}\}/\text{bihol.}$$

endowed with the Gromov-Hausdorff topology is naturally homeomorphic to $K(M,\omega)$.

Thank You !