Complex analytic Moduli space of Fano manifolds admitting Kähler-Ricci solitons

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Motivation

(M, ω): a C^∞-symplectic manifold.

Question: Can we construct a Hausdorff moduli space of bihol. classes of Kähler manifolds (X, L) of fixed symplectic type (M, ω).

We expect there should be a structure of complex analytic space on the moduli space, as the notion of complex spaces are designed to describe their own deformation theory.

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- $c_1(M, \omega) < 0$ general type: Yes, we can.
- $c_1(M, \omega) = 0$ Calabi-Yau: Yes, we can.
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- $c_1(M, \omega) > 0$ **Fano**: **Naively, it doesn't exist.**

Pathology: Naive moduli doesn't exist.

We can find many examples of Fano manifolds X and families of Fano manifolds $\pi : \mathcal{X} \to \Delta$ with the following property.

- Trivial away from the origin: $\pi^{-1}(\Delta^*) \cong \Delta^* \times X$
- The central fibre $\mathcal{X}_0 = \pi^{-1}(0)$ is a different Fano $\mathcal{X}_0 \not\cong X$.

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Metrics in general have good chemistry with Hausdorffness.

What about assuming the existence of some 'canonical metrics' on Fano manifolds in order to ensure separatedness?

Today's goal: rough statement without definitions

The goal of this talk:

Complex analytic moduli space, I. '17

Let $\mathcal{KR}_{GH}(n)$ be the set of n-dimensional Fano manifolds with Kähler-Ricci solitons. Then we can make this set $\mathcal{KR}_{GH}(n)$ into a Hausdorff complex analytic space in a **canonical way** in some sense.

Kähler-Ricci soliton is a special metric on a Fano manifold (with respect to the anti-canonical polarization $-K_X$), which generalizes Kähler-Einstein metric in view of Kähler-Ricci flow.

1. Differential geometric side - ‘real world’

Kähler-Ricci soliton

Definition (Kähler-Ricci soliton)

A **Kähler-Ricci soliton** on a Fano manifold X is a pair (g, ξ') of a Kähler metric g and a holomorphic vector field ξ' satisfying the following equation:

$$\mathrm{Ric}(g) - L_{\xi'} g = g.$$

Example: All toric Fano manifolds admit Kähler-Ricci solitons.

Remember $\xi := \mathrm{Im}(\xi')$ generates a closed torus $T_\xi^{\mathbb{R}} = \overline{\exp \mathbb{R}\xi} \subset \mathrm{Aut}(X)$.

Theorem (Uniqueness, Tian-Zhu '02)

If (g_1, ξ_1') and (g_2, ξ_2') are two Kähler-Ricci solitons on a Fano manifold X, then there exists an element $\phi \in \mathrm{Aut}^0(X)$ such that

$$\phi^* \xi_1' = \xi_2', \quad \phi^* g_1 = g_2.$$

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Theorem (Reductivity, Tian-Zhu '02)

If (g, ξ') is a Kähler-Ricci soliton on a Fano manifold X, then $\mathrm{Aut}(X, \xi')$ is a maximal reductive subgroup of $\mathrm{Aut}(X)$ and $\mathrm{Isom}(X, g)$ is its maximal compact subgroup.

Reductivity $\rightsquigarrow$ We can apply GIT (locally).

$\rightsquigarrow$ We should consider the pair (X, ξ') rather than mere X.
(in order to construct their moduli space.)

modified Futaki invariant

- (X, ξ') is a Fano manifold X with a holomorphic vector field ξ'.
- ω is a Kähler form in $2\pi c_1(X)$.
- $\theta_{\xi'}$ is a potential of ξ': $L_{\xi'}\omega = \sqrt{-1}\partial\bar{\partial}\theta_{\xi'}$ with $\int_X e^{\theta_{\xi'}}\omega^n = \int_X \omega^n$.
- h is a Ricci potential: $\sqrt{-1}\partial\bar{\partial}h = \mathrm{Ric}(\omega) - \omega$.

Define $\mathrm{Fut}_{\xi'} : \eta(X) \to \mathbb{C}$ by

$$\mathrm{Fut}_{\xi'}(v') := -\int_X v'(h - \theta_{\xi'})e^{\theta_{\xi'}}\omega^n.$$

- Independent of the choice of ω.
- If there exists a KR soliton (g, ξ'), $\mathrm{Fut}_{\xi'}$ should vanish.

K-optimal vector

If there exists a KR soliton (g, ξ'), $\mathrm{Fut}_{\xi'}$ should vanish.

- $X \curvearrowleft T^{\mathbb{C}} \cong (\mathbb{C}^*)^k$: torus action on a Fano manifold X.
- N: the 1-psg lattice of $T^{\mathbb{C}}$.
- $N_{\mathbb{R}} := N \otimes \mathbb{R} \subset \eta(X)$ by $\xi \mapsto \xi' := J\xi + \sqrt{-1}\xi$.

Proposition (Tian-Zhu '02)

For any $X \curvearrowleft T^{\mathbb{C}}$, where X does not necessarily admit any KR solitons, there exists a **unique** vector $\xi \in N_{\mathbb{R}}$ with $\mathrm{Fut}_{\xi'}|_{N_{\mathbb{R}}}$.

We call this unique vector **the K-optimal vector** of $X \curvearrowleft T^{\mathbb{C}}$.

Moment map picture

Fix our notation.

- (M, ω): a simply connected C^∞-symplectic manifold.
- (T, ξ): a closed torus acting on (M, ω) and an element of $\mathrm{Lie}(T)$.
- θ_ξ: $-d\theta_\xi = -2i_\xi\omega$ with the normalization $\int_M \theta_\xi\ e^{\theta_\xi}\omega^n = 0$.
- $\mathcal{J}_T$: the space of T-inv. almost complex srtuctures on (M, ω).
- Ham_T: the group of T-equiv. symplectic diffeomorphisms of (M, ω).

$\mathcal{J}_T$ admits a Ham_T-invariant smooth symplectic form Ω_ξ defined by

$$\Omega_{\xi,J}(A, B) := \int_M \mathrm{Tr}(JAB)e^{\theta_\xi}\omega^n$$

for $A, B \in T_J\mathcal{J}_T$.

Proposition (I. '17)

The map

$$s_\xi : \mathcal{J}_T \to C^\infty(M) : J \mapsto (s(g_J) + \bar{\square}\theta_\xi - n) + (\bar{\square}\theta_\xi - \xi'\theta_\xi - \theta_\xi)$$

defines a **moment map** of $(\mathcal{J}_T, \Omega_\xi) \curvearrowleft \mathrm{Ham}_T$. If $c_1(M, \omega) > 0$, then integrable complex structures J with $s_\xi(J) = 0$ precisely correspond to Kähler-Ricci solitons g_J on Fano manifolds (M, J).

We have the following immediate cororally.

For $f \in \mathfrak{t} \subset \mathrm{Lie}(\mathrm{Ham}_T) = C^\infty_T(M)/\mathbb{R}$, the modified Futaki invariant $\mathrm{Fut}_{\xi'}(X'_f)$ equals to

$$\langle s_\xi, f \rangle_\xi = \int_X s_\xi f \; e^{\theta_\xi} \omega^n,$$

which shows the T-equivariant deformation invariance of the modified Futaki invariant restricted to $\mathrm{Lie}(T) = N_\mathbb{R}$.

Summary of this section

In summary,

Input: a Fano $T^{\mathbb{C}}$-manifold X

$\Downarrow$

Intermediate: a C^∞-symplectic $T^{\mathbb{R}}$-manifold (M, ω) with $\omega \in 2\pi c_1(M, \omega)$

$\Downarrow$

Output: the **K-optimal vector** $\xi \in N_{\mathbb{R}}$ satisfying $\mathrm{Fut}_{\xi'}|_{N_{\mathbb{R}}} \equiv 0$.

The K-optimal vector $\xi \in N_{\mathbb{R}}$ is T-equivaraint deformation invariant!

From the uniqueness of KR-soliton, the topological space

$$(s_\xi|_{\mathcal{J}^{\mathrm{int}}_{T_\xi}})^{-1}(0) \; / \; \mathrm{Ham}_{T_\xi}$$

can be naturally identified with the set consisting of biholomorphism classes, not T_ξ-equivariant biholomorphism classes, of Fano manifolds admitting KR-solitons of symplectic type (M, ω, ξ).

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$$\Downarrow$$

Use K-stability !

2. Algebro-geometric side - ‘virtual world’

K-stability

From now on, the K-optimal vector ξ will be implicitly involved in our formulations, which is encoded in the torus action $X \curvearrowleft T^{\mathbb{C}} = T$.

A **special degeneration** for a $\mathbb{Q}$-Fano T-variety $(X, -K_X) \curvearrowleft T$ is a $T \times \mathbb{C}^*$-equivariant family of $\mathbb{Q}$-Fano varieties $\pi : \mathcal{X} \to \mathbb{C}$ (T acts on $\mathbb{C}$ trivially) endowed with a T-equivariant isomorphism $X \times \mathbb{C}^* \cong \pi^{-1}(\mathbb{C}^*)$ over $\mathbb{C}^* \subset \mathbb{C}$.

We assume $-K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier.

The **Donaldson-Futaki invariant** $DF_T(\pi)$ of a special degeneration $\pi : \mathcal{X} \to \mathbb{C}$ for $X \curvearrowleft T$ with respect to the K-optimal vector $\xi \in N_{\mathbb{R}}$ is given by

$$DF_T(\pi) := \mathrm{Fut}_{\xi'}(\frac{d}{dt}\lambda(t))$$

where $\lambda : \mathbb{C}^* \to \mathrm{Aut}_T(\mathcal{X}_0)$ is the 1-psg on the central fibre $\mathcal{X}_0$ generated by the $\mathbb{C}^*$-action on $\pi : \mathcal{X} \to \mathbb{C}$.

There is also an algebraic expression of DF_T.

A $\mathbb{Q}$-Fano T-variety X is said to be

- **K-semistable** if for any special degeneration $\pi : \mathcal{X} \to \mathbb{C}$ for $X \curvearrowleft T$, $DF_T(\pi) \geq 0$.
- **K-polystable** if it is K-semistable and $DF_T(\pi) = 0$ implies there is a T-equivariant isomorphism $\mathcal{X} \cong X \times \mathbb{C}$.
- **K-stable** if it is K-polystable and $\mathrm{Aut}^0_T(X) = T$.

Note here we use DF_T modified by the K-optimal vector ξ, which is different from the usual Donaldson-Futaki invariant when $\xi \neq 0$.

Theorem (Datar-Sékelyhidi '15, Chen-Sun-Wang '15 + BW '14)

A Fano T-manifold X admits a Kähler-Ricci soliton (g, ξ') with $\xi' \in \mathrm{Lie}(T) \subset \eta(X)$ if and only if $X \curvearrowleft T$ is K-polystable.

($\Rightarrow$ A Fano manifold X admits a KR soliton iff there exists some torus action $X \curvearrowleft T$ so that it is K-polystable.)

Complex analytic moduli

Equivariant formulation is essential

The existence of 'canonical metrics' is still not sufficient :(

There exists an isotrivial degeneration $\pi : \mathcal{X} \to \Delta$ of some KE-Fano manifold X with the following property.

- Trivial away from the origin $\pi^{-1}(\Delta^*) \cong \Delta^* \times X$.
- The central fibre $\mathcal{X}_0 \not\cong X$ admits Kähler-Ricci soliton with $\xi_0' \neq 0$.

This implies that, without our equivariant formulation, $\mathcal{X}_0$ will **not** be separated from X (in the "naive moduli space of Fano manifolds with Kähler-Ricci solitons", which algebraic geometers might naively imagine), though both of them admit 'canonical metrics'.

This is not $T_{\xi_0'}$-equivariant family.

We can show that our equivariant formulation will exclude such examples.

Definition (Moduli stack $\mathcal{K}_T(M,\omega)$)

The **moduli stack of K-semistable Fano T-manifolds**, denoted by $\mathcal{K}_T(M,\omega)$, is a category with

- **Object**: a T-equivariant family $\mathcal{X} \to S$ of K-semistable Fano T-manifolds over a complex space S where the underlying C^∞-symplectic structure of fibres are $(M,\omega) \curvearrowleft T^{\mathbb{R}}$,
- **Morphism**: $(f,\phi) : (S,\mathcal{X}) \to (S',\mathcal{X}')$ where ϕ is a fibrewise isomorphism over f.

We have a forgetful functor $\mathcal{K}_T(M,\omega) \to \mathbb{C}\mathbf{an} : (\mathcal{X} \to S) \mapsto S$, which makes $\mathcal{K}_T(M,\omega)$ into a stack over $\mathbb{C}\mathbf{an}$ and actually Artin analytic.

- A morphism from $\mathbb{C}\mathbf{an}_S$ to $\mathcal{K}_T(M,\omega)$ correspond to a T-equivariant family $\mathcal{X} \to S$ of K-semistable Fano T-manifolds.
- A morphism from $\mathcal{K}_T(M,\omega)$ to $\mathbb{C}\mathbf{an}_M$ gives a 'functorial' way to assign a morphism $S \to M$ to each family $\mathcal{X} \to S$ in $\mathcal{K}_T$.

Definition of moduli space in terms of the moduli stack

Definition

A complex space $\mathcal{K}$ with a morphism $\boldsymbol{\mathcal{K}}_T(M,\omega) \to \mathbb{C}\mathbf{an}_{\mathcal{K}}$ is called the **moduli space** of $\boldsymbol{\mathcal{K}}_T(M,\omega)$ if for any morphism $\boldsymbol{\mathcal{K}}_T(M,\omega) \to \mathbb{C}\mathbf{an}_S$ we have a unique holomorphic morphism $\mathcal{K} \to S$ completing the following commutative diagram.

$$\begin{array}{ccc} \boldsymbol{\mathcal{K}}_T(M,\omega) & & \\ \downarrow & \searrow & \\ \mathcal{K} & \dashrightarrow & S \end{array}$$

The moduli space is unique (up to bihol.) if it exists.

Main theorem

Theorem (I. '17)

The moduli space $\mathcal{K}_T(M,\omega) \to \mathcal{K}$ exists for any $(M,\omega) \curvearrowleft T$ with $\omega \in c_1(M,\omega)$ (possibly empty).

A morphism from a 'point' $\mathbb{C}\mathbf{an}_{\mathrm{pt}}$ to $\mathcal{K}_T(M,\omega)$ corresponds to a biholomorphism class of K-semistable Fano T-manifold.

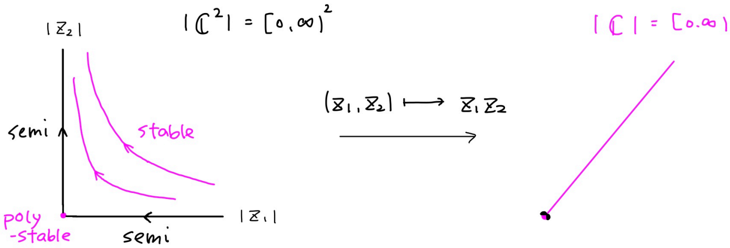
On the other hand, points of $\mathcal{K}$ corresponds to biholomorphism classes of K-polystable Fano T-manifolds (= Fano manifolds admitting KR-soliton).

Our moduli spaces look like GIT

Our moduli space $\mathcal{K}_T(M,\omega) \to \mathcal{K}$ is (étale) locally isomorphic to the GIT quotient $[H^1_T(X,\Theta)/\mathrm{Aut}_T(X)] \to H^1_T(X,\Theta) /\!\!/ \mathrm{Aut}_T(X)$.

Actually, we construct the moduli space $\mathcal{K}_T(M,\omega) \to \mathcal{K}$ by gluing them together where X runs all K-polystable Fano T-manifolds.

Here is an incomplete picture of $[\mathbb{C}^2/\mathbb{C}^*] \to \mathbb{C}^2 /\!\!/ \mathbb{C}^* = \mathbb{C}$ where $\mathbb{C}^*$ acts on $\mathbb{C}^2$ by $(z_1,z_2).t = (z_1.t, z_2.t^{-1})$.



Consistency

Proposition (I. '17)

The following are naturally identified as topological spaces.

- The moduli space $\mathcal{K}$ of $\boldsymbol{\mathcal{K}}_T(M, \omega)$.
- The symplectic reduction $s_{\xi,\mathrm{int}}^{-1}(0)/\mathrm{Ham}_T$ for the K-optimal vector ξ.
- The space $\mathcal{KR}_{GH,T}(M, \omega) = \{$ bihol. classes of Fano manifolds admitting KR solitons, T_ξ-equivariantly diffeo. to (M, ω, T) $\}$ endowed with the Gromov-Hausdorff topology.

To show this, we use the uniqueness of the moduli space $\mathcal{K}$ (funny! :)).

In the proof, we also show that the space $\mathcal{KR}_{GH}(n)$ consisting of n-dimensional Fano manifolds admitting KR solitons is a finite disjoint union of some $\mathcal{KR}_{GH,T_i}(M_i, \omega_i)$ with $\dim_{\mathbb{R}} M_i = 2n$.

Thank You !