μ-cscK metric (on arXiv:1902:00664)

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$X \circlearrowleft T \cong (U(1))^{\times k}$: holomorphic action on a complex (Kähler) manifold

μ-scalar curvature

For $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{t}$ and a T-equivariant Kähler metric $\omega + \mu$, we put

$$s_\xi^\lambda(\omega) := (s(\omega) - \Delta\mu_\xi) - (\Delta\mu_\xi + 2|\nabla\mu_\xi|^2) + 2\lambda\mu_\xi.$$

- $s(\omega) - \Delta\mu_\xi = \mathrm{tr}_\omega(\mathrm{Ric}(\omega) + L_{\nabla\mu_\xi}\omega)$
- $L_{\nabla\mu_\xi}(e^{-2\mu_\xi}\omega^n) = (\Delta\mu_\xi + 2|\nabla\mu_\xi|^2)\ e^{-2\mu_\xi}\omega^n$

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Definition

A Kähler metric ω is a μ_ξ^λ-cscK metric if $s_\xi^\lambda(\omega)$ is constant.

- Independent of the choice of the moment map μ for ω.
- μ_0^λ-cscK metric $\iff$ cscK metric.

μ-scalar curvature: Kähler-Ricci soliton

X: Fano manifold, i.e. $c_1(X) > 0$.

Kähler-Ricci soliton

A (ξ-invariant) Kähler metric $\omega \in 2\pi c_1(X) = [\mathrm{Ric}(\omega)]$ is called a Kähler-Ricci soliton if

$$\mathrm{Ric}(\omega) + L_{\nabla \mu_\xi} \omega = \omega.$$

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- $s(\omega) - \Delta \mu_\xi = \mathrm{tr}_\omega (\mathrm{Ric}(\omega) + L_{\nabla \mu_\xi} \omega) = n$.

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$$\begin{aligned} L_{\nabla \mu_\xi} (\mathrm{Ric}(\omega) + L_{\nabla \mu_\xi} \omega) &= \sqrt{-1}\partial\bar{\partial}(\Delta \mu_\xi + 2|\nabla \mu_\xi|^2) \\ &= L_{\nabla \mu_\xi} \omega = \sqrt{-1}\partial\bar{\partial}(2\mu_\xi). \end{aligned}$$

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Proposition (I. '18)

Fix $\xi \in \mathfrak{t}$. $\omega \in 2\pi c_1(X)$ is KRS for $\xi \iff \mu_\xi^1$-cscK metric.

Proof of "KRS $\Leftarrow$ μ^1-cscK"

Proof.

- $h : \mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial} h$,
- $\theta_\xi := -2\mu_\xi$,
- $\xi^J = J\xi + \sqrt{-1}\xi = \bar{\partial}^\sharp \theta_\xi$.

$$L_{\xi^J}(\mathrm{Ric}(\omega) - \omega) = \sqrt{-1}\partial\bar{\partial}(\bar{\square}\theta_\xi - \theta_\xi)$$
$$= L_{\xi^J}\sqrt{-1}\partial\bar{\partial} h = \sqrt{-1}\partial\bar{\partial}\xi^J h$$

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$$= L_{\xi^J}\sqrt{-1}\partial\bar{\partial}h = \sqrt{-1}\partial\bar{\partial}\xi^J h$$

$$\Rightarrow \bar{\square}\theta_\xi - \theta_\xi - \xi^J h = \mathrm{const.}$$

$$\Rightarrow s^1_\xi(\omega) = -(\bar{\square} - \xi^J)(h - \theta_\xi) + \mathrm{const.}$$

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$$L_{\xi^J}(\mathrm{Ric}(\omega) - \omega) = \sqrt{-1}\partial\bar{\partial}(\bar{\Box}\theta_\xi - \theta_\xi)$$
$$= L_{\xi^J}\sqrt{-1}\partial\bar{\partial} h = \sqrt{-1}\partial\bar{\partial}\xi^J h$$
$$\Rightarrow \bar{\Box}\theta_\xi - \theta_\xi - \xi^J h = \text{const.}$$
$$\Rightarrow s^1_\xi(\omega) = -(\bar{\Box} - \xi^J)(h - \theta_\xi) + \text{const.}$$

Therefore, $s^1_\xi(\omega) = \text{const.} \Rightarrow (\bar{\Box} - \xi^J)(h - \theta_\xi) = \text{const.}$
$\Rightarrow h - \theta_\xi = \text{const.} \Rightarrow \mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}\theta_\xi$: KRS □

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- $\theta_\xi := -2\mu_\xi$,
- $\xi^J = J\xi + \sqrt{-1}\xi = \bar{\partial}^\sharp \theta_\xi$.

$$
\begin{aligned}
L_{\xi^J}(\mathrm{Ric}(\omega) - \omega) &= \sqrt{-1}\partial\bar{\partial}(\bar{\Box}\theta_\xi - \theta_\xi) \\
&= L_{\xi^J}\sqrt{-1}\partial\bar{\partial} h = \sqrt{-1}\partial\bar{\partial}\xi^J h
\end{aligned}
$$

$$\Rightarrow \bar{\Box}\theta_\xi - \theta_\xi - \xi^J h = \text{const.}$$

$$\Rightarrow s^1_\xi(\omega) = -(\bar{\Box} - \xi^J)(h - \theta_\xi) + \text{const.}$$

Therefore, $s^1_\xi(\omega) = \text{const.} \Rightarrow (\bar{\Box} - \xi^J)(h - \theta_\xi) = \text{const.}$

$$\Rightarrow h - \theta_\xi = \text{const.} \Rightarrow \mathrm{Ric}(\omega) - \omega = \sqrt{-1}\partial\bar{\partial}\theta_\xi\text{: KRS}$$

□

Summary: Similarly as cscK generalizes KE, μ-cscK generalizes KRS for general Kähler class.

μ-scalar curvature: "naturality" of the concept

Recall

Donaldson-Fujiki moment map picture

(M, ω): C^∞-symplectic manifold. Scalar curvature gives a moment map on $\mathcal{J}(M, \omega)$. Namely, the map $\mathcal{S} : \mathcal{J}(M, \omega) \to \mathrm{Lie}(\mathrm{Ham}(M, \omega))^\vee$ given by

$$\langle \mathcal{S}(J), f \rangle = \int_M (s(g_J) - \bar{s}) f \omega^n$$

is a moment map for the symplectic structure Ω on $\mathcal{J}(M, \omega)$:

$$\Omega_J(A, B) = \int_M (JA, B)_{g_J} \omega^n.$$

μ-scalar curvature: "naturality" of the concept

Put

$$\bar{s}^\lambda_\xi := \int_M s^\lambda_\xi(g_J)\ e^{\theta_\xi}\omega^n \Big/ \int_M e^{\theta_\xi}\omega^n.$$

Proposition (Moment map picture for μ-cscK, I. '18, Lahdili '18)

$(M,\omega) \circlearrowleft T$: C^∞-symplectic manifold. μ-scalar curvature gives a moment map on $\mathcal{J}_T(M,\omega)$. Namely, the map $\mathcal{S}^\lambda_\xi : \mathcal{J}_T(M,\omega) \to \mathrm{Lie}(\mathrm{Ham}_T(M,\omega))^\vee$ given by

$$\langle \mathcal{S}^\lambda_\xi(J), f\rangle = \int_M (s^\lambda_\xi(g_J) - \bar{s}^\lambda_\xi) f\ e^{\theta_\xi}\omega^n$$

is a moment map for the symplectic structure Ω_ξ on $\mathcal{J}_T(M,\omega)$:

$$\Omega_{\xi,J}(A,B) = \int_M (JA,B)_{g_J}\ e^{\theta_\xi}\omega^n.$$

2. Expected results: Futaki invariant, Reductiveness...

μ-Futaki invariant

Put

$$\mathfrak{h}_0(X) := \{\zeta' \in H^0(X, TX) \mid \exists \theta_{\zeta'} \in C^\infty_{\mathbb{C}}(X) \text{ s.t. } i_{\zeta'}\omega = \sqrt{-1}\bar{\partial}\theta_\zeta\}.$$

Proposition (Lahdili '18, I. '19)

For $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{t}$, we define $\mathrm{Fut}^\lambda_\xi : \mathfrak{h}_0(X) \to \mathbb{C}$ by

$$\mathrm{Fut}^\lambda_\xi(\zeta') := \int_X (s^\lambda_\xi(\omega) - \bar{s}^\lambda_\xi)\theta_{\zeta'}\ e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n,$$

taking a ξ-invariant metric ω. Then this only depends on $(X, [\omega], \xi, \lambda)$.

- Fut^λ_ξ vanishes if there exists a μ^λ_ξ-cscK metric in $[\omega]$.
- Fut^λ_ξ is well-defined on $\mathfrak{h}_0$ not only on $\mathfrak{h}_{0,\xi} := \{\zeta' \in \mathfrak{h}_0 \mid [\xi^J, \zeta'] = 0\}$.
- $\mathrm{Fut}^\lambda_\xi|_{\mathfrak{t}}$ is T-equivariant deformation invariant.

Reductiveness

Proposition (Lahdili '18, I. '19)

Suppose $b^1(X) = 0$ for simplicity. If there exists a μ^λ_ξ-cscK metric for some $\lambda \in \mathbb{R}$, the identity component $\mathrm{Aut}^0_\xi(X)$ of the group of biholomorphisms compatible with ξ is a reductive linear algebraic group. When $\lambda \leq 0$, $\mathrm{Aut}^0_\xi(X)$ is maximal reductive in $\mathrm{Aut}^0(X)$.

Combined with Luna's étale slice theorem and GIT, reductiveness is applied as follows in the case $[\omega] = 2\pi c_1(X)$:

- Chen-Donaldson-Sun: K-polystable $\Rightarrow \exists$ KE
- Construction of the moduli of Fano manifolds with KE/KRS metrics

Non-emptiness: perturbation of Kähler class

Proposition (I. '19)

Suppose a Kähler manifold $(X, [\omega])$ admits a μ_ξ^λ-cscK metric ω_λ satisfying $\lambda \leq \lambda_1$ for the first eigenvalue λ_1 of $\Delta - \nabla\theta_\xi$. Then there exists a positive number $\epsilon > 0$ and a neighbourhood U of $[\omega]$ in the Kähler cone of X such that for every $[\tilde{\omega}] \in U$ and $\tilde{\lambda} \in (\lambda - \epsilon, \lambda + \epsilon)$ there exists a $\mu_{\tilde{\xi}}^{\tilde{\lambda}}$-cscK metric $\tilde{\omega}_{\tilde{\lambda}}$ in $[\tilde{\omega}]$ for some $\tilde{\xi} \in \mathrm{Lie}(T_{\max})$.

Example

- For every cscK manifold (X, ω), a perturbation of $[\omega]$ admits a μ^λ-cscK metric for every $\lambda \leq 0$.
- For every toric Fano manifold X, a perturbation of $[\omega] = 2\pi c_1(X)$ admits a $\mu^{1\pm\epsilon}$-cscK metric.

3. Volume minimization

Tian-Zhu's volume minimization for KRS

Let X be a Fano manifold and ω be a Kähler metric in $2\pi c_1(X)$. Let $X \circlearrowleft K$ be a holomorphic action of a compact group which is Hamiltonian with respect to ω.
Take for each $\xi \in \mathfrak{k} = \mathrm{Lie}(K)$, take a moment map μ_ξ and normalize $\theta_\xi := -2\mu_\xi$ so that $\bar{\Box}\theta_\xi - \theta_\xi - \xi^J h = 0$.

Proposition (Tian-Zhu '02)

The modified Futaki invariant Fut^1_ξ is given as the derivative of the following proper convex functional at $\xi \in \mathfrak{k}$ (up to positive multiple):

$$f(\xi) := \int_X e^{\theta_\xi} \omega^n.$$

Cororally

There exists a unique vector $\xi \in \mathfrak{k}$ such that $\mathrm{Fut}^1_\xi \equiv 0$.

μ-volume functional

Let $(X, [\omega]) \circlearrowleft K$ be a general compact Kähler manifold. Put

$$\mathrm{Vol}^\lambda(\xi) := e^{\bar{s}^\lambda_\xi} \Big(\int_X e^{\theta_\xi} \omega^n \Big)^\lambda.$$

Proposition (I. '19)

$$d_\xi \mathrm{Vol}^\lambda = \mathrm{Vol}^\lambda(\xi) \cdot \mathrm{Fut}^\lambda_\xi$$

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1. Properness of Vol^λ?
2. Uniqueness of the critical points of Vol^λ?

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1. Properness of Vol^λ? - TRUE
2. Uniqueness of the critical points of Vol^λ? - TRUE for $\lambda \ll 0$.

However, NOT TURE for $\lambda \gg 0$!

Properness of Vol^λ

Proposition (I. '19)

$\log \mathrm{Vol}^\lambda(\xi) = \bar{s}^\lambda_\xi + \lambda \log \int_X e^{\theta_\xi} \omega^n$ goes to $+\infty$ as $|\xi| \to \infty$.

Proof: Replacing θ_ξ by $\theta_\xi +$ const. doesn't affect the value of Vol^λ. Normalize θ_ξ so that $\max \theta_\xi = 0$ and put $\Sigma := \{x \in X \mid \theta_\xi(x) = 0\}$.

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Normalize θ_ξ so that $\max \theta_\xi = 0$ and put $\Sigma := \{x \in X \mid \theta_\xi(x) = 0\}$.
As θ_ξ is a perfect Morse-Bott function, we can write as

$$\theta_\xi = -(x_1^2 + \cdots + x_{2k}^2)$$

near Σ, where $2k = \dim_{\mathbb{R}} \Sigma$ and x_i are normal to Σ.

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Applying Gaussian integral near Σ, we can show for $p = 0, 1, \ldots$ a convergence of measures $(-1)^p t^{k+p} \theta_\xi^p e^{t\theta_\xi} \omega^n \to dm_\infty$ as $t \to \infty$ to some strictly positive measure dm_∞ supported on Σ.

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$$\bar{s}^0_{t\xi}/t = \int_X (s(\omega)/t) e^{t\theta_\xi} \omega^n \Big/ \int_X e^{t\theta_\xi} \omega^n + \int_X \bar{\Box} \theta_\xi e^{t\theta_\xi} \omega^n \Big/ \int_X e^{t\theta_\xi} \omega^n$$

$$\to 0 + \int_\Sigma \bar{\Box} \theta_\xi \, dm_\infty \Big/ \int_\Sigma dm_\infty > 0 \quad (\Leftarrow (\bar{\Box} \theta_\xi)|_\Sigma > 0)$$

Continuation: proof of the properness of Vol^λ

As for the rest terms:

$$\begin{aligned}\log \mathrm{Vol}^\lambda(t\xi)/t - \bar{s}^0_{t\xi}/t &= \lambda\Big(-\int_X \theta_\xi e^{t\theta_\xi}\omega^n \Big/ \int_X e^{t\theta_\xi}\omega^n + t^{-1}\log \int_X e^{t\theta_\xi}\omega^n\Big) \\ &= O(t^{-1}) + O(t^{-1}\log t) \to 0\end{aligned}$$

It follows that $\log \mathrm{Vol}^\lambda(t\xi) \approx C_\xi t \nearrow \infty$ as $C_\xi = \liminf_{t\to\infty} \bar{s}^0_{t\xi}/t > 0$. □

Cororally

For each λ, there exists a minimizer ξ of Vol^λ. In particular, there exists a vector $\xi \in \mathfrak{k}$ such that $\mathrm{Fut}^\lambda_\xi \equiv 0$.

Non-Uniqueness of critical points of Vol^λ when $\lambda \gg 0$

The second derivative is given as follows:

$$\left(\frac{d}{dt}\right)^2\Big|_{t=0} \log \mathrm{Vol}^\lambda(\xi + t\zeta)$$
$$= -2\frac{\int_X \theta_\zeta e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n}\mathrm{Fut}^\lambda_\xi(\zeta) + \frac{\int_X (\hat{s}^\lambda_\xi \theta_\zeta^2 + 4|\zeta|^2_{g_J})e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n}$$
$$-\lambda\left(\frac{\int_X \theta_\zeta^2 e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} - \left(\frac{\int_X \theta_\zeta e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n}\right)^2\right).$$

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$$- \lambda\left(\frac{\int_X \theta_\zeta^2 e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} - \left(\frac{\int_X \theta_\zeta e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n}\right)^2\right).$$

Suppose $\mathrm{Fut} \equiv 0$. Then $0 \in \mathfrak{k}$ is a critical point of Vol^λ for every λ as $\mathrm{Fut} = \mathrm{Fut}^\lambda_0$. We should have

$$\lambda \leq \frac{\int_X (s(\omega)\theta_\zeta^2 + 4|\zeta|^2_{g_J})\omega^n}{\int_X \omega^n} \Big/ \left(\frac{\int_X \theta_\zeta^2\omega^n}{\int_X \omega^n} - \left(\frac{\int_X \theta_\zeta\omega^n}{\int_X \omega^n}\right)^2\right)$$

in order that $0 \in \mathfrak{k}$ is a local minimizer of Vol^λ. This fails for $\lambda \gg 0$.

4. Go to the limit $\lambda \searrow -\infty$

Extremal metric

Let $(X, [\omega])$ be a (compact) Kähler manifold. A Kähler metric $\omega \in [\omega]$ is called an extremal metric if it is a critical point of the Calabi functional $C(\omega) := \int_X (s(\omega) - \bar{s})^2 \omega^n$.

$\iff$ $s(\omega) - \theta_\xi$ is constant for some ξ generating a (rank 1) torus.

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$\iff$ $s(\omega) - \theta_\xi$ is constant for some ξ generating a (rank 1) torus.

Recall

$$s_\xi^\lambda(\omega) = (s(\omega) + \bar{\square}\theta_\xi) + (\bar{\square}\theta_\xi - 2|\xi|^2) - \lambda\theta_\xi.$$

If we have a sequence of triples $\{(\xi_i, \lambda_i, \omega_i) \in \mathfrak{t} \times \mathbb{R} \times \mathcal{H}_{[\omega]}\}_{i=1}^\infty$ such that $\xi_i \to 0$ and $\lambda_i \xi_i \to \xi_\infty$ for some $\xi_\infty \in \mathfrak{t}$ and $\omega_i \to \omega_\infty$ for some Kähler metric $\omega_\infty \in [\omega]$, then $s_{\xi_i}^{\lambda_i}(\omega_i)$ converges to $s(\omega_\infty) - \theta_{\xi_\infty}$.

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- If ω_i are $\mu_{\xi_i}^{\lambda_i}$-cscK metrics, then ω_∞ is an extremal metric.
- Taking $\omega_i = \omega$ shows that $\mathrm{Fut}_{\xi_i}^{\lambda_i}$ converges to $\mathrm{Fut}_{\xi_\infty}^{\mathrm{ext}} = \int_X (\hat{s}(\omega) - \hat{\theta}_{\xi_\infty}) \bullet \omega^n / \int_X \omega^n$.

Claim of this section

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If we take a sequence $\{(\xi_i, \lambda_i)\}$ so that $\lambda_i \to -\infty$ and $\mathrm{Fut}_{\xi_i}^{\lambda_i} \equiv 0$, then $\xi_i \to 0$ and $\lambda_i \xi_i \to \xi_{\mathrm{ext}}$, where ξ_{ext} is the (extremal) vector uniquely characterized by $\mathrm{Fut}_{\xi_{\mathrm{ext}}}^{\mathrm{ext}} \equiv 0$.

Remark: such a sequence always exists.

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Remark: such a sequence always exists.

Note

$$\mathrm{Fut}^{\lambda}_{\xi}(\xi) = \mathrm{Fut}^{0}_{\xi}(\xi) - \lambda \left(\frac{\int_X \theta_\xi^2 e^{\theta_\xi} \omega^n}{\int_X e^{\theta_\xi} \omega^n} - \left(\frac{\int_X \theta_\xi e^{\theta_\xi} \omega^n}{\int_X e^{\theta_\xi} \omega^n} \right)^2 \right).$$

Put

$$\lambda_\xi := \mathrm{Fut}^{0}_{\xi}(\xi) \cdot \left(\frac{\int_X \theta_\xi^2 e^{\theta_\xi} \omega^n}{\int_X e^{\theta_\xi} \omega^n} - \left(\frac{\int_X \theta_\xi e^{\theta_\xi} \omega^n}{\int_X e^{\theta_\xi} \omega^n} \right)^2 \right)^{-1} \quad \text{for } \xi \neq 0,$$

then $\mathrm{Fut}^{\lambda}_{\xi}(\xi) = 0 \iff \lambda = \lambda_\xi$ for $\xi \neq 0$.

λ as a function of ξ

λ is a continuous function on $\mathfrak{k} \setminus \{0\}$. So if a sequence of pairs (ξ_i, λ_i) satisfies $\mathrm{Fut}_{\xi_i}^{\lambda_i} \equiv 0$ and $\lambda_i \to -\infty$, then $\lambda_i = \lambda_{\xi_i}$ and $|\xi_i|$ should satisfy either $|\xi_i| \to 0$ or $|\xi_i| \to \infty$.

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The set $\{\xi \in \mathfrak{k} \mid \lambda_\xi \leq 0\}$ is bounded. In particular, we have $|\xi_i| \to 0$.

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Lemma

A normalized function $\lambda_\xi |\xi|$ on $\mathfrak{k}^\times = \mathfrak{k} \setminus \{0\}$ continuously extends to the real bowing-up $\hat{\mathfrak{k}} = \mathfrak{k}^\times \cup \{|\xi| = 1\} \to \mathfrak{k}$. In particular, $\lambda_i \xi_i$ is bounded.

When $t \to 0$, $\quad |t\xi|^{-1} \mathrm{Fut}^0_{t\xi}(t\xi) \to \int_X (s(\omega) - \bar{s}) \theta_{\xi/|\xi|} \omega^n / \int_X \omega^n$ and

$$|t\xi|^{-2} \Big(\frac{\int_X \theta_{t\xi}^2 e^{\theta_{t\xi}} \omega^n}{\int_X e^{\theta_{t\xi}} \omega^n} - \Big(\frac{\int_X \theta_{t\xi} e^{\theta_{t\xi}} \omega^n}{\int_X e^{\theta_{t\xi}} \omega^n} \Big)^2 \Big) \to \Big(\frac{\int_X \theta_{\xi/|\xi|}^2 \omega^n}{\int_X \omega^n} - \Big(\frac{\int_X \theta_{\xi/|\xi|} \omega^n}{\int_X \omega^n} \Big)^2 \Big) \ \square$$

Continuation: proof of the claim

Thus we can find a subsequence of $\{\lambda_i \xi_i\}$ converging to a vector ξ_∞. As $\mathrm{Fut}^{\lambda_i}_{\xi_i} \to \mathrm{Fut}^{\mathrm{ext}}_{\xi_\infty}$ and $\mathrm{Fut}^{\lambda_i}_{\xi_i} \equiv 0$, we have $\mathrm{Fut}^{\mathrm{ext}}_{\xi_\infty} \equiv 0$. Such a vector ξ_∞ is the unique critical point of the following proper convex functional:

$$W^{\mathrm{ext}}(\xi) := \int_X (s - \bar{s} - (\theta_\xi - \bar{\theta}_\xi))^2 \omega^n.$$

It follows that ξ_∞ is independent of the choice of the subsequence, so that the original sequence $\lambda_i \xi_i$ converges to ξ_{ext}.

From extremal metric to μ-cscK, recent updates

Proposition (I. '19)

Suppose a Kähler manifold $(X, [\omega])$ admits an extremal metric ω_{ext}, then for every $\lambda \ll 0$ there exists a μ_ξ^λ-cscK metric ω_λ for some $\xi \neq 0$. Moreover, ω_λ converges to ω_{ext} as $\lambda \to -\infty$.

Proposition (I. '19)

The functional Vol^λ has unique critical points for $\lambda \ll 0$.

Questions and ...

1. Can we expect the uniqueness of the critical points of Vol^λ for every $\lambda \leq 0$?
2. When $[\omega] = 2\pi c_1(X)$, what about for $\lambda \leq 1$?
3. Prove the uniqueness of $\mu^\lambda_{(\xi)}$-cscK metric for each $\lambda \leq 0$ (mod Aut).
4. Relate the existence of μ^λ_ξ-cscK (resp. μ^λ-cscK) metric to μK-stability of $(X, [\omega], \xi, \lambda)$ (resp. μK-stability of $(X, [\omega], \lambda) \circlearrowleft T$) (in progress).
5. Construct moduli spaces of Kähler manifolds with μ^λ_ξ-cscK metric.
6. Wall-crossing: When $[\omega] = 2\pi c_1(X)$, observe what happens on the moduli spaces when λ changes from 1 (KRS) to $-\infty$ (extremal).

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多謝！ (Thank you !)