Nonarchimedean μ-entropy and moment measure

@ Jussieu

(X, L) : a pol var / k char $k = 0$

X proper . L ample alg . cl

$\mathcal{F}$: a filtration on $R = \bigoplus_m H^0(X, mL)$

$\mathcal{F} : (\lambda \in \mathbb{R}, m \in \mathbb{N}) \longrightarrow \mathcal{F}_m^\lambda \subset R_m$

1. $\mathcal{F}_m^\lambda = \bigcap_{\lambda' < \lambda} \mathcal{F}_m^{\lambda'}$ $\rightsquigarrow$ (left conti decreasing

2. $\mathcal{F}_m^\lambda \cdot \mathcal{F}_{m'}^{\lambda'} \subset \mathcal{F}_{m+m'}^{\lambda+\lambda'}$

3 a $\mathcal{F}_m^\lambda = R_m$ $(\lambda \leq ma)$.

b $\mathcal{F}_m^\lambda = 0$ $(\lambda > ma')$

$s \in H^0(X, mL)$

$\mathcal{F}(s) := \underset{\max}{\sup} \{ \lambda \in \mathbb{R} \mid s \in \mathcal{F}_m^\lambda \}$

$s_1, \dots, s_N$: a basis of $H^0(X, mL)$

is <u>orthogonal</u> w.r.t. $\mathcal{F}$

if

$$\mathcal{F}\left(\sum a_i s_i\right) = \min_{a_i \neq 0} \mathcal{F}(s_i)$$

<u>Def</u> (s_i). orth w.r.t. $\mathcal{F}$

$$\nu(\mathcal{F}, m) := \sum_i \underline{\delta_{\frac{\mathcal{F}(s_i)}{m}}}$$

Dirac mass on $\frac{\mathcal{F}(s_i)}{m} \in \mathbb{R}$

<u>Fact</u> $\exists !\ DH_{\mathcal{F}}$

Radon measure on $\mathbb{R}$

total mass $(L^n)/n!$

$$\uparrow \text{weakly}$$

$$\frac{1}{m^n} \nu(\mathcal{F}, m)$$

equivariant Riemann - Roch

If $\mathrm{gr}\,\mathcal{F} = \bigoplus_m \bigoplus_\lambda \mathcal{F}_m^\lambda / \mathcal{F}_m^{\lambda+}$

$$\mathcal{F}_m^{\lambda+} = \sum_{\lambda < \lambda'} \mathcal{F}_m^{\lambda'}$$

is finitely generated,

$$\int_{\mathbb{R}} e^{-t}\, \nu(\mathcal{F}, m) = m^n \int_{\mathbb{R}} e^{-t}\, DH_{\mathcal{F}} + m^{n-1} R_{\mathcal{F}} + O(m^{n-2})$$

Fact

$$\mathcal{F} : \text{f.g.} \overset{\exists}{\rightsquigarrow} \hat{\mathcal{F}} : \text{f.g. w/ reduced } \mathrm{gr}\,\mathcal{F}.$$

$$\varphi_{\mathcal{F}} : \text{NA psh}$$

Def $\mathcal{F}$: f.g.

$$\mu_{NA}(\mathcal{F}) := 2 \frac{R_{\hat{\mathcal{F}}}}{\int_{\mathbb{R}} e^{-t}\, DH_{\hat{\mathcal{F}}}}$$

$T = 0$

$$(\rho \cdot \mathcal{F})_m^\lambda := \begin{cases} \mathcal{F}_m^{\rho^{-1}\lambda} & \rho > 0 \\ \mathcal{F}_{\mathrm{triv}, m}^\lambda & \rho = 0 \end{cases}$$

$$= \begin{cases} R_m & \lambda \leq 0 \\ 0 & \lambda > 0 \end{cases}$$

<u>Prop</u> $\mathcal{F}$: f.g.

$$\frac{d}{d\rho}\Big|_{\rho=0} \mu_{NA}(\rho \cdot \mathcal{F}) = -M_{NA}(\mathcal{F})$$

$$= -DF(\hat{\mathcal{F}})$$

Since $-M_{NA}(\rho \cdot \mathcal{F}) = -\rho M_{NA}(\mathcal{F})$,

we also have $\frac{d}{d\rho}\Big|_{\rho=0} (-M_{NA}(\rho \cdot \mathcal{F}))$

$$= -M_{NA}(\mathcal{F})$$

The crucial differece : boundedness

$-M_{NA}(\mathcal{F})$ is bounded from above

if and only if $M_{NA}(\mathcal{F}) \geqslant 0 \quad {}^{\forall}\mathcal{F}$

$\Leftrightarrow$ K-semistable

e.g. X = one pt blowing up of $\mathbb{P}^2$

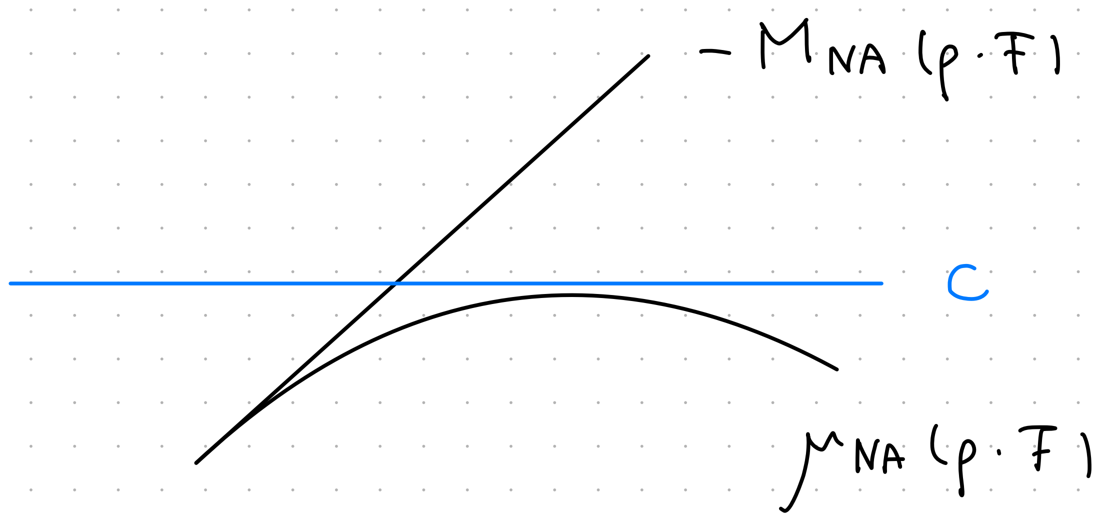
$L = -K_X$

NOT K-semistable.

Thm X : smooth

lower bound of Perelman μ-entropy

${}^{\exists}C \in \mathbb{R} \quad {}^{\forall}\mathcal{F}$: f.g. $\quad \mu_{NA}(\mathcal{F}) \leqslant C$



$$p_F^{exp} := \frac{e^{-t} DH_F}{\int_{\mathbb{R}} e^{-t} DH_F}$$

$$p_F := \frac{DH_F}{\int_{\mathbb{R}} DH_F}$$

$$a(F) := \int_{\mathbb{R}} \frac{dp_F^{exp}}{dp_F} \log \frac{dp_F^{exp}}{dp_F} \, dp_F$$

$$\mu_{NA}^{T}(F) := \mu_{NA}(F) - T\,a(F)$$

<u>Thm</u> (Chen-Sun-Wang, Dervan-Székelyhidi, Han-Li, Blum-Liu-Xu-Zhuang I)

If $TL = K_X$, X : klt

$\exists$ $\mathcal{F}$ f.g. maximizing μ^T_{NA}.

- $T \geq 0$ ... $\mathcal{F} = \mathcal{F}_{triv}$
- $T < 0$ ... $\mathcal{F}$ is constructed from Kähler-Ricci flow $g(t)$

$$\dot{g}(t) = -Ric(g(t)) + 2\pi T g(t)$$

$$\mathcal{F}^\lambda_m = \left\{ s \in H^0(X, mL) \;\middle|\; \frac{\log \|s\|_{g(t)}}{t} \leq \lambda \right\}$$

a priori depends on $g(0)$

$\hat{X} = \mathrm{Proj}\, gr\, \mathcal{F}$ is μK-semistable $\mathbb{Q}$-Fano.

Want to find maximizer of μ^T_{NA}

when $TL \neq K_X$

... Want to use compactness

... NA pluripotential theory

would be good place to

establish compactness.

<u>Thm</u> We can define DH_φ on $[-\infty, \infty)$

for $\varphi \in PSH(X^{an}, L^{an})$

$$\int_{[-\infty,\infty)} DH_\varphi = \frac{(L^n)}{n!}$$

Put $\mathcal{E}(X^{an}, L^{an})$

$$:= \{ \varphi \in PSH(X^{an}, L^{an}) \mid \int_{\{-\infty\}} DH_\varphi = 0 \}$$

$$E_{NA}(\varphi) = \int_{[-\infty,\infty)} t \, DH_\varphi$$

$$\rightsquigarrow \quad \mathcal{E}^1 \subset \mathcal{E}$$

$$E_{NA}^{exp}(\varphi) := -\int_{[-\infty,\infty)} e^{-t} \, DH_\varphi$$

$$\rightsquigarrow \quad \mathcal{E}^{exp,p} = \{ E_{NA}^{exp}(p \cdot \varphi) > -\infty \}$$

<u>Thm</u> We can define $\mu_{NA}^T(\varphi)$ for

$\varphi \in \mathcal{E}^{exp, 1+\varepsilon}$

If $\varphi_j \to \varphi$ weakly

$$\Rightarrow \quad E_{NA}^{exp}(p \cdot \varphi_j) \longrightarrow E_{NA}^{exp}(p \cdot \varphi)$$

$$\exists \, p > 2$$

then $\overline{\lim_{j \to \infty}} \, \mu_{NA}^T(\varphi_j) \leq \mu_{NA}^T(\varphi)$

We use a measure $\int e^{-t} D_\varphi$ on X^{an}

w/ total mass $\int_{\mathbb{R}} e^{-t} DH_\varphi$

$$\mu_{NA}(\varphi) = \frac{\int_{X^{an}} A_X \int e^{-t} D_\varphi}{\int_{\mathbb{R}} e^{-t} DH_\varphi}$$

$$+ \frac{\frac{1}{n!}\int_{\mathbb{R}} (K_X . 0)(L . \varphi\wedge\tau - \tau)^n e^{-\tau} d\tau}{\int_{\mathbb{R}} e^{-t} DH_\varphi}$$

Rem If $E^{\exp}_{NA}(\rho\cdot\varphi) < \infty$ $\exists \rho > 1$

$$E^{\exp}_{NA}(\varphi) = \frac{1}{(n+1)!}\int_{\mathbb{R}} (L . \varphi\wedge\tau - \tau)^{n+1} e^{-\tau} d\tau$$

For $\varphi = \varphi(x, z)$

$$\int e^{-t} D_\varphi = \sum_{E \subset \mathcal{X}_0} \operatorname{ord}_E \mathcal{X}_0 \int_{\mathbb{R}} e^{-t} DH_{(E, z|_E)} \, \delta_{v_E}$$

If $\varphi \in \mathcal{E}^{\exp, 2+\varepsilon}$. for $\psi \in \mathcal{E}^1$

$$\int_{X^{an}} \psi \int e^{-t} D_\varphi$$

$$= \int_{\mathbb{R}} d\tau \, e^{-\tau} \int_{X^{an}} (\psi - \sup \psi) \, MA(\varphi \cap \tau)$$

$$+ \sup \psi \int_{\mathbb{R}} e^{-\tau} DH_\varphi(\tau)$$

Thm (X, L) toric $\circlearrowleft \mathbb{T}$

$\exists\ \varphi_T \in \mathcal{E}^{\mathrm{exp}. \frac{n}{n-1}}_{\mathbb{T}}$

maximizes $\mu^T_{NA} \big|_{\mathcal{E}^{\exp}_{\mathbb{T}}}$.

- $T > 0$ maximizer is unique modulo const.
- $\exists!\ \varphi_0 = \lim_{T \nearrow 0} \varphi_T$
- $\dim X = 2$. φ_0 is continuous $\rightsquigarrow \mathcal{F}_{\varphi_0}$ filtration.