A sketch of the future of μ-cscK metrics and μK-stability

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Contents

1. Motivation

Dream

Want to construct a moduli theory of polarized varieties, generalizing that of Fano varieties, which encloses “unstable” objects.

What is the moduli theory of Fano varieties?, unstable case

What can we say about "unstable (= general)" Fano manifolds?

Let X be a $\mathbb{Q}$-Fano variety. In some way, we want to assign another "semistable" $\mathbb{Q}$-Fano variety X_0 for which we have a nice moduli theory.

What is the moduli theory of Fano varieties?, unstable case

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In this talk, a "degeneration" $(X, L) \overset{\mathcal{F}}{\rightsquigarrow} (X_0, L_0)$ of (X, L) means a f.g. filtration $\mathcal{F} = \{\mathcal{F}^\lambda\}_{\lambda \in \mathbb{R}}$ on the graded ring $R(X, L) = \bigoplus_m H^0(X, L^{\otimes m})$, which is an "irrational version" of test configuration (see [BHJ] and [CSW]). A prime divisor $E \subset Y \to X$ (over X) gives a (not necessarily f.g.) filtration $\mathcal{F}_E$.

For a "special degeneration" $X \overset{\mathcal{F}_E}{\rightsquigarrow} X_0$ of a $\mathbb{Q}$-Fano variety $(X, -K_X)$, we can define the H-entropy $\check{H}_{\mathrm{NA}}(\mathcal{F}) \in \mathbb{R}$ following [DS] and [HL]:

$$\check{H}_{\mathrm{NA}}(\mathcal{F}) := -2\pi((1 + \mathrm{ord}_E(K_Y/K_X)) + \log \int_{\mathbb{R}} e^{-t} \mathrm{DH}_{v_E}).$$

What is the moduli theory of Fano varieties?, unstable case

Theorem (CSW, DS, HL, BLXZ)

Let X be a $\mathbb{Q}$-Fano variety. Then there exists a unique "special degeneration" $X \overset{\mathcal{F}_E}{\rightsquigarrow} X_0$ which maximizes $\check{H}_{\mathrm{NA}}$. Moreover,

- the central fibre X_0 is a $\mathbb{Q}$-Fano variety with a vector field $\xi_{\mathcal{F}}$.
- X_0 admits a unique $\xi_{\mathcal{F}}$-equivariant test configuration whose central fibre is a $\mathbb{Q}$-Fano variety Z with Kähler–Ricci soliton w.r.t $\xi_{\mathcal{F}}$.
- When X is smooth, Z is the Gromov–Hausdorff limit of the normalized Kähler–Ricci flow.

What is the moduli theory of Fano varieties?, unstable case

Theorem (CSW, DS, HL, BLXZ)

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Remark: later we introduce another invariant $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$

- $\check{H}_{\mathrm{NA}}$ ... Ding type invariant ... makes sense for $L = -K_X$
- $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$ ... DF/Mabuchi type invariant ... makes sense in general

The same result is true also for $\check{\boldsymbol{\mu}}^{2\pi}_{\mathrm{NA}}$ instead of $\check{H}_{\mathrm{NA}}$.

What is the moduli theory of Fano varieties?, stable case

Kähler–Ricci soliton:

$$\mathrm{Ric}(\omega) - \mathscr{L}_{J\xi}\omega = 2\pi\omega.$$

Theorem (i + thesis)

There exists an algebraic moduli space of Fano manifolds with Kähler–Ricci solitons.

What is the moduli theory of Fano varieties?, stable case

Kähler–Ricci soliton:

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Theorem (i + thesis)

There exists an algebraic moduli space of Fano manifolds with Kähler–Ricci solitons.

Conjecture (Compact moduli space of KRs $\mathbb{Q}$-Fano varieties)

There exists a projective moduli space of $\mathbb{Q}$-Fano varieties with Kähler–Ricci solitons.

What is “stable”, then?

“Stability” in this line is

- modified Futaki invariant [BW] ... is defined for special test configuration
- $e^{\langle -, \xi \rangle}$-Ding invariant [HL] ... is a Ding type invariant defined for general test configuration
- $\mu_\xi^{2\pi}$-Futaki invariant [iii] ... is a DF/Mabuchi type invariant defined for general test configuration

These are a posteriori equivalent.

Dream, again

Want to construct a moduli theory of polarized varieties, generalizing that of Fano varieties.

||

Want to extend the theory of Kähler–Ricci soliton / modified K-stability to general polarized variety.

How?

⋮

K-stability is introduced based on moment map picture of cscK metric.
Observe the moment map picture of Kähler–Ricci soliton.

Dream, again

Want to construct a moduli theory of polarized varieties, generalizing that of Fano varieties.

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Want to extend the theory of Kähler–Ricci soliton / modified K-stability to general polarized variety.

How?

$\vdots$

K-stability is introduced based on moment map picture of cscK metric.

Observe the moment map picture of Kähler–Ricci soliton.

$\vdots$

There appears a new degree of freedom $\lambda \in \mathbb{R}$, which enriches the theory.

Tasks

Let's establish:

- Differential geometric theory on "stability": μ_ξ^λ-cscK metric
- Differential geometric theory on "instability": Perelman entropy

- Algebro-geometric theory on "stability": μ_ξ^λK-stability
- Algebro-geometric theory on "instability": μ^λK-optimal degeneration

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- Algebro-geometric theory on "stability": μ_ξ^λK-stability
- Algebro-geometric theory on "instability": μ^λK-optimal degeneration

We find two "phases" of the theory:

- Good behavior when $\lambda \leq 0$
- Phase transition when $\lambda \gg 0$

2. Theory on "stability": μ_ξ^λ-cscK metric and μ_ξ^λK-stability

μ_ξ^λ-cscK metric

The moment map picture on Kähler–Ricci soliton observed in [i] yields the following notion.

Definition

For $\lambda \in \mathbb{R}$, a Kähler metric $\omega \in L$ is called a μ^λ-cscK metric w.r.t ξ_f if there exists $f \in C^\infty(X)$ such that $\mathscr{L}_{\xi_f} J = 0$ for the Hamiltonian vector field ξ_f (i.e. $i_{\xi_f}\omega + df = 0$) and

$$s_f^\lambda(\omega) := (s(\omega) + \Box f) + (\Box f + |\partial^\sharp f|^2) - \lambda f$$

is constant.

Y. Nakagawa (2011) independently introduced the equation for

$$\lambda = -2\pi(K_X \cdot L^{\cdot n-1})/(L^{\cdot n})$$

(in view of K-energy on KRs) and studied Calabi ansatz example.

Basic property

- A Kähler–Ricci soliton $\mathrm{Ric}(\omega) - \mathscr{L}_{J\xi_f}\omega = 2\pi\beta\omega$ is a $\mu_f^{2\pi\beta}$-cscK metric, whose Kähler class L must satisfy $\beta L = -K_X$.
- Conversely, a $\mu^{2\pi\beta}$-cscK metric in the Kähler class L satisfying $\beta L = -K_X$ is a Kähler–Ricci soliton.
- A cscK metric $s(\omega) = \mathrm{const.}$ in any Kähler class L is a μ^λ-cscK metric with $f = 0$ and every $\lambda \in \mathbb{R}$.

Trivial construction:

- $\omega \in L$ is a μ^λ-cscK metric iff $c\omega \in cL$ is a $\mu^{\lambda/c}$-cscK metric $(c > 0)$.
- If (X, ω_X) and (Y, ω_Y) are μ^λ-cscK manifolds, then $(X \times Y, \omega_X \times \omega_Y)$ is a μ^λ-cscK manifold.

Perturbation: the invertibility of $\Delta - \nabla f - \lambda : C^\infty_{\xi_f}(X) \to C^\infty_{\xi_f}(X)$

- If $\omega \in L$ is a μ^λ-cscK metric for $\lambda \leq 0$, then there exists a μ^λ-cscK metric $\omega' \in L'$ in a small perturbation L' of L.
- If $\omega \in L$ is a μ^λ-cscK metric for $\lambda \leq 0$, then there exists a $\mu^{\tilde\lambda}$-cscK metric $\tilde\omega$ in L for a small perturbation $\tilde\lambda$ of λ.

DG theory on "stability"

For a given Kähler class L, we are interested in the existence and the uniqueness modulo $\mathrm{Aut}(X, L)$ of μ^λ-cscK metrics.

Proposition (Reductiveness, ii)

If a Kähler class $L = c_1(\mathfrak{L})$ admits a μ^λ-cscK metric ω w.r.t. ξ, then $\mathrm{Aut}^0_\xi(X, \mathfrak{L})$ is reductive with a maximal compact subgroup $\mathrm{Isom}^0_\xi(X, \omega)$. If $\lambda \leq 0$, $\mathrm{Aut}^0_\xi(X, \mathfrak{L})$ is a maximal reductive subgroup in $\mathrm{Aut}^0(X, \mathfrak{L})$.

Theorem (Uniqueness of μ_ξ^λ-cscK metric, Lahdili2)

Given a Kähler class L, $\lambda \in \mathbb{R}$ and ξ, μ^λ-cscK metrics w.r.t. ξ in L are unique modulo $\mathrm{Aut}_\xi(X, L)$.

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Theorem (Uniqueness of μ^λ_ξ-cscK metric, Lahdili2)

Given a Kähler class L, $\lambda \in \mathbb{R}$ and ξ, μ^λ-cscK metrics w.r.t. ξ in L are unique modulo $\mathrm{Aut}_\xi(X, L)$.

Conjecture (Uniqueness of phase)

Given a Kähler class L and $\lambda \leq 0$, μ^λ-cscK metrics in L are unique modulo $\mathrm{Aut}(X, L)$.

Moving $\lambda \in \mathbb{R}$: extremal limit and phase transition

Proposition (Extremal limit, ii)

Suppose there exists an extremal metric $\omega_{\mathrm{ext}} \in L$, then there exists a family of μ^λ-cscK metrics $\{\omega_\lambda\}$ in L for $\lambda \ll 0$ such that ω_λ converges smoothly to ω_{ext} as $\lambda \to -\infty$.

Proposition (Phase transition, ii)

Consider $L = c_1(\mathcal{O}(1))$ on $\mathbb{C}P^1$.

- For $\lambda \leq 2\pi$, every μ^λ-cscK metric has trivial $\xi = 0$.
- For $\lambda > 2\pi$, there exists a μ^λ-cscK metric w.r.t. non-trivial $\xi \neq 0$.

The latter metric is not the Fubini–Study metric and the isometry group reduces to $U(1) \subset SU(1)$.

Corollary (Non-uniqueness result)

Given a Kähler class L and $\lambda \gg 0$, μ^λ-cscK metrics in L are NOT unique modulo $\mathrm{Aut}(X, L)$, in general.

Examples by Calabi ansatz

Let Σ be a proper curve and $\mathcal{L}$ be a line bundle of degree $\ell \geq 1$. Consider the ruled surface $X := \mathbb{P}_\Sigma(\mathcal{L} \oplus \mathcal{O})$. In $H^2(X, \mathbb{R}) = [H^*(\Sigma)[B]/(B^2 - \ell B.F)]^{(2)} = \mathbb{R}B \oplus \mathbb{R}F$, the Kähler cone is given by

$$\mathcal{C}_X = \{xB + yF \mid x > 0, y > -\frac{\ell}{2}x\}$$

Proposition (Nakagawa, ii)

Every Kähler class in the subcone $\mathcal{C}'_X = \{xB + yF \mid x, y > 0\}$ admits a μ^λ-cscK metric w.r.t. non-trivial $\xi \neq 0$ for every $\lambda \geq 0$.

Remark: There exists $L \in \mathcal{C}'_X$ with no extremal metric.

Question

- What can we say about $L \in \mathcal{C}_X \setminus \mathcal{C}'_X$?
- Can we compute $\inf\{\lambda \in \mathbb{R} \mid \exists \mu^{\lambda'}\text{-cscK in } L \text{ for every } \lambda' > \lambda\}$?

Connecting KRs and extremal metric

Proposition (Connecting KRs and extremal metric, ii)

In the Kähler class $-K_X$ of $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, there exists a smooth family $\{\omega_\lambda\}_{\lambda \in [-\infty, 2\pi]}$ of μ^λ-cscK metrics, which gives an extremal metric at $\lambda = -\infty$ and a KRs at $\lambda = 2\pi$.

Question

What about the case $\lambda > 2\pi$? Observe phase transition in this example.

μ^λ_ξK-stability (in toric case, for brevity)

Since $\mathscr{L}_{\xi_f} g = 0$, $T_{\mathrm{cpt}} := \overline{\exp \mathbb{R}\xi_f} \subset \mathrm{Isom}(X, g)$ is a closed torus.

Keep this in mind, we consider an algebraic torus action T on (X, L). For $\lambda \in \mathbb{R}$, $\xi \in \mathfrak{t} := \mathrm{Hom}(T, \mathbb{C}^\times) \otimes \mathbb{R}$ and a T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$, we can define the μ^λ_ξ-Futaki invariant $\mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L})$.

When $(X, L) \circlearrowleft T$ is toric, a convex function q on the moment polytope $P := -\mu(X) \subset \mathfrak{t}^\vee$ encodes a T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$.

μ^0_ξ-Futaki invariant for toric test configuration

$$\frac{1}{2\pi}\mathrm{Fut}^0_\xi(q) = \frac{\int_{\partial P} q e^{-\langle x, \xi\rangle} d\sigma}{\int_P e^{-\langle x, \xi\rangle} d\mu} - \frac{\int_{\partial P} e^{-\langle x, \xi\rangle} d\sigma}{\int_P e^{-\langle x, \xi\rangle} d\mu} \frac{\int_P q e^{-\langle x, \xi\rangle} d\mu}{\int_P e^{-\langle x, \xi\rangle} d\mu}$$

$$= \frac{d}{dt}\Big|_{t=0} \frac{\int_{\partial P} e^{\langle x, -\xi\rangle + tq} d\sigma}{\int_P e^{\langle x, -\xi\rangle + tq} d\mu}$$

Conjectures based on moment map picture

Conjecture (Existence of μ^λ_ξ-cscK metric)

A Kähler class L admits a μ^λ-cscK metric w.r.t. ξ if and only if $\mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}) \geq 0$ (μ^λ_ξK-semistable) for every ξ-equivariant $(\mathcal{X}, \mathcal{L})$ and $\mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}) = 0$ only for product configurations (μ^λ_ξK-polystable).

Theorem (Lahdili1, iii, AJL)

$\Rightarrow$ is true (for smooth test configuration).

Conjecture (Moduli space of μ^λ_ξK-semistable polarized varieties)

For each $\lambda, \xi \in \mathfrak{t}$, there exists an algebraic moduli space of polarized varieties with μ^λ_ξ-cscK metrics, whose moduli stack consists of T-equivariant family of polarized varieties which are μ^λ_ξK-semistable.

3. Theory on “instability”: NA μ-entropy and Perelman entropy

AG theory on 'instability': (toric) NA μ-entropy

Recall

$$\frac{1}{2\pi}\mathrm{Fut}^0_\xi(q) = \frac{d}{dt}\Big|_{t=0} \frac{\int_{\partial P} e^{\langle x, -\xi\rangle + tq} d\sigma}{\int_P e^{\langle x, -\xi\rangle + tq} d\mu}.$$

Definition (Non-archimedean μ-entropy)

$$\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(q) := -2\pi \frac{\int_{\partial P} e^q d\sigma}{\int_P e^q d\mu} + \lambda\Big(n + \frac{\int_P q e^q d\mu}{\int_P e^q d\mu} - \log \int_P e^q d\mu\Big)$$

We can define $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{F})$ also for (not necessarily T-equiv.) degeneration $(X, L) \overset{\mathcal{F}}{\rightsquigarrow} (X_0, L_0)$ of general (X, L), but I don't explain it here.

NA μ-entropy and μK-stability

Theorem (Entropy maximization is optimal, v)

Suppose a "degeneration" $(X, L) \overset{\mathcal{F}}{\rightsquigarrow} (X_0, L_0)$ maximizes $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$, then (X_0, L_0) is μ^{λ}K-semistable w.r.t. $\xi_{\mathcal{F}}$.

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Conjecture (Optimal destabilization)

If X has only klt singularities, there exists a unique "degeneration" $(X, L) \overset{\mathcal{F}}{\rightsquigarrow} (X_0, L_0)$ which maximizes $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$.

Recall:

Theorem (CSW, DS, HL, BLXZ, v)

Let X be a $\mathbb{Q}$-Fano variety. Then there exists a unique "special degeneration" $X \overset{\mathcal{F}_E}{\rightsquigarrow} X_0$ which maximizes $\check{\boldsymbol{\mu}}^{2\pi}_{\mathrm{NA}}$, which is related to the Gromov–Hausdorff limit of the normalized Kähler–Ricci flow.

Restriction to affine functions (product tcs / vectors)

We can also write $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\langle \cdot, \zeta \rangle)$ as follows:

$$-\frac{\int_X s^{\lambda}_{-\mu_\zeta}(\omega) e^{-\mu_\zeta} \omega^n}{\int_X e^{-\mu_\zeta} \omega^n} + \lambda \Big(n - \log \int_X e^{-\mu_\zeta} \omega^n / n!\Big).$$

If $L = -K_X$ (Fano) and $\lambda = 2\pi$, the first term is n: Tian–Zhu functional.

Proposition (Volume minimization, ii)

If the exists a μ^{λ}-cscK metric w.r.t. $\xi \in \mathfrak{t}$, then ξ must be a critical point (actually, a maximizer) of the functional $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\langle \cdot, -\zeta \rangle)$ on $\mathfrak{t}$.

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Recall:

Conjecture (Uniqueness of phase)

Given a Kähler class L and $\lambda \leq 0$, μ^λ-cscK metrics in L are unique modulo $\mathrm{Aut}(X, L)$.

Since $\mathrm{Isom}_\xi(X, \omega)$ is maximally compact for $\lambda \leq 0$, it suffices to show the uniqueness of the critical points of $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\langle \cdot, -\zeta \rangle)$.

Phase transition, revival

In other words, the conjecture is equivalent to the following:

Conjecture (Uniqueness of phase, reduced)

Given a Kähler class L, $\lambda \leq 0$ and a maximal torus $T \subset \mathrm{Aut}(X, L)$, the critical points of $\check{\boldsymbol{\mu}}_{\mathrm{NA}}^{\lambda}(\langle \cdot, -\zeta \rangle)$ are unique.

A sophisticated version of Laplace principle shows the following:

Proposition (Phase transition, ii)

The functional $\check{\boldsymbol{\mu}}_{\mathrm{NA}}^{\lambda}(\langle \cdot, -\zeta \rangle)$ is proper and bounded from above, hence admits a critical point. Moreover,

$$\sup\{\lambda \in \mathbb{R} \mid \check{\boldsymbol{\mu}}_{\mathrm{NA}}^{\lambda'}(\langle \cdot, -\zeta \rangle) \text{ admits unique critical point for every } \lambda' < \lambda\}$$

is never $\pm\infty$. (The result $\neq +\infty$ was unexpected.)

Now we have shown the conjecture is true for non-positive $\lambda' < \sup\{\lambda \mid ...\}$.

DG theory on 'instability': Perelman's entropy

For $\omega \in \mathcal{H}(X, L)$ and $f \in C^\infty(X)$, we put

$$W^\lambda(\omega, f) := -\frac{\int_X s_f^\lambda(\omega) e^f \omega^n}{\int_X e^f \omega^n} + \lambda(n - \log \int_X e^f \omega^n / n!),$$

$$\boldsymbol{\mu}^\lambda_{\mathrm{Per}}(\omega) := \sup_f W^\lambda(\omega, f).$$

Theorem (Donaldson type inequality, iv)

$$\sup_{\mathcal{F}} \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{F}) \leq \inf_\omega \boldsymbol{\mu}^\lambda_{\mathrm{Per}}(\omega).$$

Conjecture (Minimax conjecture)

$$\sup_{\mathcal{F}} \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{F}) = \inf_\omega \boldsymbol{\mu}^\lambda_{\mathrm{Per}}(\omega).$$

Perelman's μ^λ-entropy and μ^λ-cscK metrics

Theorem (Perelman's μ^λ-entropy and μ^λ-cscK metrics, iv)

- Critical points of $W^\lambda : \mathcal{H}(X, L) \times C^\infty(X) \to \mathbb{R}$ are precisely the pairs (ω, f) of μ^λ-cscK metric ω w.r.t. ξ_f and the potential f.
- For $\lambda \leq 0$, $\boldsymbol{\mu}^\lambda_{\mathrm{Per}} : \mathcal{H}(X, L) \to \mathbb{R}$ is smooth and its critical points are precisely μ^λ-cscK metrics ω w.r.t. some ξ.
- For $\lambda \leq 0$, any μ^λ-cscK metric attains the minimum of $\boldsymbol{\mu}^\lambda_{\mathrm{Per}}$.
- For $\lambda \leq 0$, the vector ξ associated to a μ^λ-cscK metric gives a filtration $\mathcal{F}_\xi$ which attains the maximum of $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$ and we have the equality:

$$\sup_{\mathcal{F}} \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{F}) = \inf_{\omega} \boldsymbol{\mu}^\lambda_{\mathrm{Per}}(\omega).$$

Summary

Theorem (μ^λ-cscK metric is characterized by Perelman μ-entropy)

For $\lambda \leq 0$, TFAE:

- $\omega \in \mathcal{H}(X, L)$ is a μ^λ-cscK metric.
- ω minimizes $\boldsymbol{\mu}^\lambda_{\mathrm{Per}} : \mathcal{H}(X, L) \to \mathbb{R}$.

We put $\mathcal{H}^{\mathbb{R}}_{\mathrm{NA}}(X, L) := \{$ f.g. filtrations $\mathcal{F}\}/\sim$.

Theorem (Entropy maximization is optimal)

B implies A:

- A: The central fibre of $\mathcal{F} \in \mathcal{H}^{\mathbb{R}}_{\mathrm{NA}}(X, L)$ is μ^λK-semistable w.r.t. $\xi_{\mathcal{F}}$.
- B: $\mathcal{F}$ maximizes $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}} : \mathcal{H}^{\mathbb{R}}_{\mathrm{NA}}(X, L) \to \mathbb{R}$.

Conjecture

For $\lambda \leq 0$, A implies B.

Conjectures, I: YTD conjecture and moduli space

Conjecture (Compact moduli space of KRs $\mathbb{Q}$-Fano varieties)

There exists a projective moduli space of $\mathbb{Q}$-Fano varieties with Kähler–Ricci solitons.

Conjecture (Existence of μ^λ_ξ-cscK metric)

A Kähler class L admits a μ^λ-cscK metric w.r.t. ξ if and only if $\mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}) \geq 0$ (μ^λ_ξK-semistable) for every ξ-equivariant $(\mathcal{X}, \mathcal{L})$ and $\mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}) = 0$ only for product configurations (μ^λ_ξK-polystable).

Conjecture (Moduli space of μ^λ_ξK-semistable polarized varieties)

For each $\lambda, \xi \in \mathfrak{t}$, there exists an algebraic moduli space of polarized varieties with μ^λ_ξ-cscK metrics, whose moduli stack consists of T-equivariant family of polarized varieties which are μ^λ_ξK-semistable.

Conjectures, II: Existence and Uniqueness of optimal degeneration

Conjecture (Uniqueness of phase)

Given a Kähler class L and $\lambda \leq 0$, μ^λ-cscK metrics in L are unique modulo $\mathrm{Aut}(X, L)$.

Conjecture (Optimal destabilization)

$\exists!$ "degeneration" $(X, L) \overset{\mathcal{F}}{\rightsquigarrow} (X_0, L_0)$ which maximizes $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$.

Conjecture (Minimax conjecture)

$$\sup_{\mathcal{F}} \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{F}) = \inf_{\omega} \boldsymbol{\mu}^\lambda_{\mathrm{Per}}(\omega).$$

Approach: non-archimedean pluripotential theory

Conjecture (Optimal destabilization)

$\exists!$ "degeneration" $(X, L) \overset{\mathcal{F}}{\rightsquigarrow} (X_0, L_0)$ which maximizes $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$.

Want to consider a completion of the space of f.g. filtrations $\mathcal{H}^{\mathbb{R}}_{\mathrm{NA}}(X, L)$.

$\downarrow$

Use Boucksom–Jonsson's non-archimedean pluripotential theory

Theorem (v)

There exists

- a complete metric space $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$ consisting of functions on the Berkovich space X^{NA},
- a canonical embedding $\mathcal{H}^{\mathbb{R}}_{\mathrm{NA}}(X, L) \hookrightarrow \mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$ and
- an usc extension of $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}} : \mathcal{H}^{\mathbb{R}}_{\mathrm{NA}}(X, L) \to [-\infty, \infty)$ to $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L)$.

Toric case is verified

Theorem (New!)

There exists a lsc convex function q on P which maximizes toric $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$ for $\lambda \leq 0$ and $\int_P e^{\rho q} d\mu < \infty$ for $\rho \in [1, \frac{n}{n-1})$.

Conjecture

The maximizer q is piecewise affine.

References

- i: *The moduli space of Fano manifolds with KRs ...*
- ii: *Constant μ-scalar curvature ...*
- iii: *Equivariant calculus ...*
- iv: *Entropies in μ-framework ... , I*
- v: *Entropies in μ-framework ... , II*
- AJL: Apostolov, Jubert, Lahdili, *Weighted K-st. and coercivity ...*
- BHJ: Boucksom, Hisamoto, Jonsson, *Uniform K-stability, DH ...*
- BLXZ: Blum, Liu, Xu, Zhuang, *The existence of KRs degeneration*
- CSW: Chen, Sun, Wang, *Kähler–Ricci flow, KE metric, and ...*
- DS: Dervan, Székelyhidi, *The Kähler–Ricci flow and optimal ...*
- HL1: Han, Li, *On the YTD conjecture for generalized KRs ...*
- HL2: Han, Li, *Algebraic uniqueness of KRF limits ...*
- Lahdili1: *Kähler metrics with constant weighted scalar curvature ...*
- Lahdili2: *Convexity of weighted Mabuchi functional ...*
- Nakagawa: *On generalized Kähler–Ricci solitons*

Thank you!