Perelman entropy in Kähler geometry

& Thermodynamical structure

Want to talk about.

1. Perelman μ^{λ}-entropy $\lambda \in \mathbb{R}$

   & Calabi energy (rescaled lim of $\lambda \to -\infty$

2. Non-archimedean μ^{λ}-entropy

   & L^2-Normalized DF inv

3. Donaldson type inequality

   and optimal degeneration.

4. Toric case

5. Optimal degeneration is equilibrium

   $-\lambda$ is absolute temperature.

The role of Perelman μ-entropy

in this talk

is analogous to Calabi fct'l

$$\mathrm{Cal}(\omega) = \frac{1}{2}\int_X \hat{S}(\omega)^2\,\omega^n$$

$$\geq 0$$

It is NOT something like

Mabuchi fct'l.

$$\delta M = -\int_X \hat{S}(\omega)\,\dot{\varphi}\,\omega^n$$

NOR

relative Mabuchi fct'l

$$\delta M_\xi = -\int_X (\hat{S}(\omega) - \hat{\theta}_\xi(\omega))\,\dot{\varphi}\,\omega^n$$

1. Perelman entropy $\mathcal{W}$ & μ

$(X, [\omega])$: cpt Kähler. $\lambda \in \mathbb{R}$

$$\mathcal{W}^\lambda = \mathcal{W} - \lambda \mathcal{S} \quad : TH(X, \omega) \to \mathbb{R}$$

$$\begin{array}{c} \| \\ (\omega_\phi, f) \in \mathcal{H}(X, \omega) \times C^\infty(X)/\mathbb{R} \end{array}$$

Normalize f : $\int_X e^f \underbrace{\mathrm{vol}_\phi}_{= \omega_\phi^n / n!} = 1$.

$\mathcal{W}(\omega_\phi, f)$

$$:= -\int_X \left(s(\omega_\phi) + \frac{1}{2} |\nabla_\phi f|_\phi^2 \right) e^f \mathrm{vol}_\phi$$

$\mathcal{S}(\omega_\phi, f) \longleftarrow$ relative entropy

$$:= -\int_X f e^f \mathrm{vol}_\phi$$

$e^f \mathrm{vol}_g$. vol_g

$$\mu^{\lambda}_{Per}(\omega_\phi) := \sup_f W^{\lambda}(\omega_\phi, f)$$

Thm Assume $\lambda \leq 0$.

Then μ^{λ}_{Per} is smooth fct'l

& its critical pts are
= minimizers

precisely $\mu\lambda$-cscK metrics

Proof

A tedious calculation

+ Rothaus's proof of log Sobolev ineq

+ implicit fct thm w/ linearization

$\Delta - \nabla f - \lambda$ ▨

Before going further.

let us observe. $\lambda \to -\infty$.

$$\beta := -\lambda^{-1} > 0 \quad (\lambda < 0)$$

$$\frac{W^{-\beta}(\omega_\phi, \beta f) - W^{-\beta}(\omega_\phi, 0)}{\beta}$$

$$\downarrow \; \beta \searrow 0 \quad (\lambda \searrow -\infty)$$

$$-\frac{1}{2\int_X \mathrm{vol}_\phi}\left[\int_X (\hat{s}(\omega_\phi) + \hat{f})^2 \mathrm{vol}_\phi - \int_X \hat{s}(\omega_\phi)^2 \mathrm{vol}_\phi\right]$$

$$=: W^{+}_{\mathrm{ext}}(\omega_\phi, f).$$

where
$$\begin{cases} \hat{s}(\omega_\phi) = s(\omega_\phi) - \dfrac{\int_X s(\omega_\phi)\,\mathrm{vol}_\phi}{\int_X \mathrm{vol}_\phi} \\[2ex] \hat{f} = f - \dfrac{\int_X f\,\mathrm{vol}_\phi}{\int_X \mathrm{vol}_\phi} \end{cases}$$

$$\sup_{f} W^{+}_{ext}(\omega_\phi . f)$$

$$= \frac{1}{2\int_X \mathrm{vol}_\phi} \int_X \hat{S}(\omega_\phi)^2 \, \mathrm{vol}_\phi$$

achieved at

$\hat{f} = -\hat{S}(\omega_\phi)$

... Calabi fct'l.

<u>Fact</u>

Crit pt of Cal

are precisely <u>extremal metrics.</u>

<u>Rem</u>

$$W^{+}_{ext}(\omega_\phi . f) = \frac{1}{\int_X \mathrm{vol}_\phi} \delta M(\omega_\phi . f)$$

$$- \frac{1}{2\int_X \mathrm{vol}_\phi} \int_X \hat{f}^2 \, \mathrm{vol}_\phi$$

nonlinear & equiv intersection

Def $\omega_\phi \in \mathcal{H}(X, \omega)$ is called

- extremal metric if $\exists f \in C^\infty(X)$

$\partial^\#_\phi f := g_\phi^{i\bar{j}} f_{,\bar{j}} \partial_i$

is holomorphic &

$S(\omega_\phi) = f$ modulo const.

- μ^λ-cscK metric if $\exists f \in C^\infty(X)$

$\partial^\#_\phi f$ is holomorphic &

$S(\omega_\phi) + \frac{1}{2}\Delta_\phi f$

$+ \frac{1}{2}(\Delta_\phi f - |\nabla_\phi f|^2) = \lambda f$.

module const.

<u>Thm</u>

$\exists$ extremal metric $\in [\omega]$

$\Rightarrow$ $\exists$ μ^λ-cscK metric $\in [\omega]$ for <u>$\lambda \ll 0$</u>

<u>Prop</u> When $T[\omega] = -c_1(X)$.

$\omega_\phi \in [\omega]$ is $\mu^{-2\pi T}$-cscK met.

iff it is a Kähler-Ricci soliton

i.e. $\exists f \in C^\infty(X)$

$\partial^\#_\phi f$ is holomorphic

& $Ric(\omega_\phi) - \mathcal{L}_{\partial^\# f}\, \omega_\phi = -2\pi T \omega_\phi$.

(limit / self similar solution

of KR flow)

Recall

$$\mathcal{M} : \mathcal{H}(X, \omega) \to \mathbb{R}$$

$$\delta \mathcal{M} = -\int (S(\omega_\varphi) - \bar{S})\, \dot{\varphi}\, \omega_\varphi^n$$

$\delta \mathcal{M}$: exact 1-form

$$\delta \mathcal{M} : T\mathcal{H}(X, \omega) \to \mathbb{R}$$

$$(\omega_\varphi, f) \longmapsto -\int (S(\omega_\varphi) - \bar{S})\, \underbrace{f}_{\text{linear}}\, \omega_\varphi^n$$

- the lim of slope of $\mathcal{M}$ along geod ray

$$= \lim_{t \to \infty} -\delta \mathcal{M}\, \underbrace{(\omega_{\phi_t}, -\dot{\phi}_t)}_{\text{curve in tangent bdl}}$$

- convexity

$$= \text{monotonicity of } \delta \mathcal{M}(\omega_{\phi_t}, -\dot{\phi}_t)$$

Recall also $W^{\lambda}(\omega_\phi . f)$. $W^{+}_{ext}(\omega_\phi . f)$

is well-def for $f \in C^{0,1}(X)$

2. <u>NA μ-entropy</u>

Consider a $C^{1,1}$-geod ray

$\{\phi_t\}_{t\in[0,\infty)}$ in $\mathcal{H}^{1,1}(X.\omega)$

<u>Fact</u> (Chu-Tosatti-Weinkove)

For a tc $(X.\mathcal{L})$,

we can assign a unique

$C^{1,1}$-geod ray $\{\phi_t^{(X.\mathcal{L})}\}_{t\in[0,\infty)}$

Thm $\rho \geq 0$

- $W^\lambda(\omega_{\phi_t}, -\rho\, \dot{\phi}_t)$. $W^+_{ext}(\omega_{\phi_t}, -\rho\, \dot{\phi}_t)$

$C^{0,1}$. $\int e^{-\dot{\phi}_t} \omega^n_{\phi_t} = 1$

are monotone decreasing along $C^{1,1}$-geod $\{\phi_t\}_{t\in[0,\infty)}$.

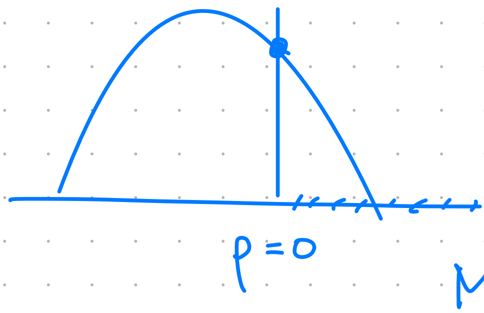
- (well-known) For a smooth tc $(\mathcal{X}, \mathcal{L})$,

$$\lim_{t\to\infty} W^+_{ext}(\omega_{\phi_t}, -\rho\, \dot{\phi}_t^{(\mathcal{X}, \mathcal{L})})$$

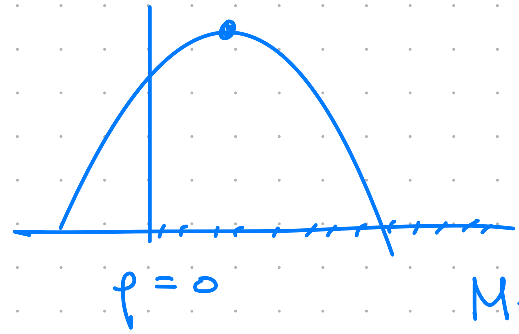
$$= -\frac{1}{2(L^n)}\Big(\rho\, 4\pi M_{NA}(\mathcal{X}, \mathcal{L}) + \underbrace{\rho^2}_{\text{NoT linear}} \underbrace{\int_{\mathbb{R}} (t-b)^2 DH_{(\mathcal{X}, \mathcal{L})}}_{= \|\mathcal{X}, \mathcal{L}\|^2}\Big)$$

$= C_{NA}(\mathcal{X}, \mathcal{L}; \rho)$

$$b = \frac{\int t\, DH_{(\mathcal{X}, \mathcal{L})}}{\int DH_{(\mathcal{X}, \mathcal{L})}}$$



$M_{NA} \geq 0$



$M_{NA} < 0$

$$\sup_{\rho \geq 0} C_{NA}(\mathcal{X}.\mathcal{L};\rho) = \begin{cases} 0 & M_{NA} \geq 0 \\ \dfrac{2\pi^2}{(L^n)} \dfrac{M_{NA}(\mathcal{X}.\mathcal{L})^2}{\|\mathcal{X}.\mathcal{L}\|^2} & M_{NA} < 0 \end{cases}$$

$$\begin{aligned} C_{NA}(\mathcal{X}.\mathcal{L};\rho) &= \lim W^+_{ext}(\omega_{\phi_t}. -\dot{\phi}_t) \\ &\leq W^+_{ext}(\omega_{\phi_0}. -\dot{\phi}_0) \\ &\leq Cal(\omega_{\phi_0}) \end{aligned}$$

Donaldson inequality on Calabi fct'l

Similarly, $\lim_{t\to\infty} W^\lambda(\omega_{\phi_t}, -\rho\, \dot{\phi}_t^{(\mathcal{X},\mathcal{L})})$

can be written by

$\mathbb{G}_m$-equivariant intersection of $K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}$

$$\mathbb{R}[\![\rho]\!] = \widehat{H}_{\mathbb{G}_m}(pt) \qquad \qquad K_{\bar{\mathcal{X}}/\mathbb{P}^1} - (\mathcal{X}_0 - \mathcal{X}_0^{red})$$

which we denote as $\mu^\lambda_{NA}(\mathcal{X}.\mathcal{L};\rho)$ ⌟ Thm.

When

$(X.L)$ & $(\mathcal{X}.\mathcal{L})$ is toric,

$$\mu^\lambda_{NA}(\mathcal{X}.\mathcal{L};\rho) = -2\pi \frac{\int_{\partial P} e^{\rho q}\, d\sigma}{\int_P e^{\rho q}\, d\mu}$$

$$+ \lambda \left[\frac{\int_P \rho q\, e^{\rho q}\, d\mu}{\int_P e^{\rho q}\, d\mu} - \log \int_P e^{\rho q}\, d\mu \right]$$

3. To sum up, we obtain

Thm (I. Donaldson)

$$\sup_{\substack{(\mathcal{X}.\mathcal{L})\\ \rho \geq 0}} \mu^{\lambda}_{NA}(\mathcal{X}.\mathcal{L};\rho) \leq \inf_{\omega_\phi \in [\omega]} \mu^{\lambda}_{Per}(\omega_\phi)$$

$$\sup_{(\mathcal{X}.\mathcal{L})} \frac{2\pi^2}{(L^n)} \frac{M_{NA}(\mathcal{X}.\mathcal{L})^2}{\|\mathcal{X}.\mathcal{L}\|^2} \leq \inf_{\omega_\phi \in [\omega]} Cal(\omega_\phi)$$

(Donaldson '05)

a little bit.

Cor (Non-trivial)

$\exists\ \mu^\lambda$-cscK $\Rightarrow$ μ^λ K-semistable
$\lambda \leq 0$

$\exists$ extremal $\Rightarrow$ relative K-semistable.

We can define μ^{λ}_{NA} for a suitable class of NA psh, including those coming from $\mathbb{R}$-filtrations

<u>Thm</u> If a filtration $\mathcal{F}$ maximizes μ^{λ}_{NA} and

$$gr\,\mathcal{F} = \bigoplus_m \bigoplus_\lambda \mathcal{F}^{\lambda}_m / \mathcal{F}^{\lambda+}_m$$

is finitely generated, then $\hat{X} = \mathrm{Proj}\, gr\,\mathcal{F}$ is $\mu^{\lambda}K$-ss w.r.t. ξ on $\hat{X}$ induced by $\mathcal{F}$.

$$\left(\mathcal{F}^{\lambda+}_m := \sum_{\lambda' > \lambda} \mathcal{F}^{\lambda'}_m = \bigcup_{\lambda' > \lambda} \mathcal{F}^{\lambda'}_m \right)$$

When $TL = K_X$ $(T \in \mathbb{R})$

$\exists$ KRs = $\mu^{-2\pi T}$-cscK met

$\Leftrightarrow$ $\mu^{-2\pi T}$ K ps (w.r.t. special tc)

(Berman - Witt Nyström
Datar - Szekelyhidi)

Thm When $TL = K_X$

$\mu^{-2\pi T}$ K ss w.r.t. ξ

$\Leftrightarrow$ $\mu^{-2\pi T}_{NA}$ maximized by $\mathcal{F}_\xi$

(When $T \geq 0$, this means
$\mathcal{F}_{triv}$ maximizes $\mu^{-2\pi T}_{NA}$)

Thm (Chen-Sun-Wang. Dervan-Székelyhidi

Han-Li. Blum-Liu-Xu-Zhuang.

I.)

If $TL = K_X$ $(T < 0)$

$\exists!$ maximizer $\mathcal{F}$ of μ^{λ}_{NA} or (H-entropy)

It is finitely generated, $\hat{X} = \mathrm{Proj}\, \mathrm{gr}\, \mathcal{F}$

is a $\mathbb{Q}$-Fano variety μ^{λ} K-ss

w.r.t. ξ generated by $\mathcal{F}$.

Moreover,

- $\exists$ ξ-equiv sp tc $\hat{\mathcal{X}}$ of $\hat{X}$ w/ $\hat{X}_0$ admits a KRS w.r.t. ξ

- NKRF $\dot{\omega}_t = -\mathrm{Ric}\,\omega_t + \lambda\omega_t$ on X converges to KRs on $\hat{X}_0$ in GH sense.

4. Toric case $\left(-\frac{\lambda}{2\pi} L \neq K_X\right)$

<u>Thm</u> There exists a $\overset{\text{lsc}}{\checkmark}$ convex fct

$g_\lambda : P \to (-\infty . \infty]$ which

maximizes μ_{NA}^{λ} among all

convex fcts w/ $\int_P e^{g'} d\mu < \infty$

- For $\lambda < 0$. g_λ is unique up to addition.
- $g_\lambda \underset{\lambda \nearrow 0}{\overset{\exists!}{\longrightarrow}} g_0$ up to addition.

  & g_0 maximizes μ_{NA}^{0}.
- $\dim P = 2$. g_0 is continuous.

Thermodynamical structure

Structure of (stochastic) thermodynamics

- Ω : system (a finite set)
- p : state (a probability measure on Ω)
- $S(p)$: entropy $\left(= -\sum_{i\in\Omega} p(i)\log p(i)\right)$
- $U_E(p)$: internal energy. $\left(= \sum_{i\in\Omega} E(i)p(i)\right)$

  $E : \Omega \to \mathbb{R}$

  Hamiltonian

physicists use this framework to understand "Maxwell demon"

… information theory.

Def (second law of thermodynamics)

p is equilibrium of internal energy U

if $U_E(p) = U$.

& $S_E(p) \geq S_E(p')$ for

p' w/ $U_E(p') = U$.

In stochastic thermodynamical setup.

we find p is equilibrium if

it is of the form.

$$p_\beta(j) = \frac{e^{-\beta E(j)}}{\sum_{i \in \Omega} e^{-\beta E(i)}}$$

(canonical distribution,

Gibbs distribution)

for some $\underset{\text{inverse temperature}}{\underline{\beta}}$ dep on U.

$$S^{eq}(U) := S(\text{equilibrium of internal energy } U)$$

$$= \sup \{ S(p) \mid U(p) = U \}$$

concave, increasing on

$$U \in [\min E, \sum_{i \in \Omega} E(i)]$$

$$\beta(U) := \frac{\partial S^{eq}}{\partial U} > 0.$$

$$T(U) := \frac{1}{\beta(U)}$$

Thermalization

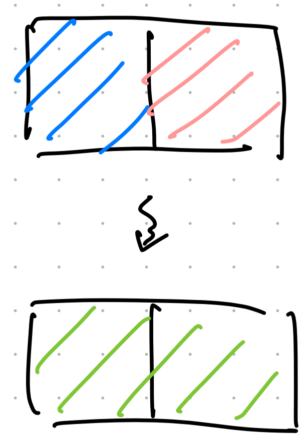
p_1 : equilibrium of Ω_1

p_2 : equilibrium of Ω_2

- $p_1 \times p_2$ is an equilibrium of $\Omega_1 \times \Omega_2$.

  if and only if

  $T(p_1) = T(p_2)$



- Otherwise. if $T(p_1) < T(p_2)$,

  we have

  $$T(p_1) < T(\hat{p}_1) = T(\hat{p}_2) < T(p_2)$$

  for equilibrium $\hat{p} = \hat{p}_1 \times \hat{p}_2$ of

internal energy $U_{E_1 \times E_2}(p_1 \times p_2) = U_{E_1}(p_1) + U_{E_2}(p_2)$

canonical dist p_β minimizes

Helmholtz free energy $F_\beta = U - \beta^{-1} S$

(Lagrange multiplier)

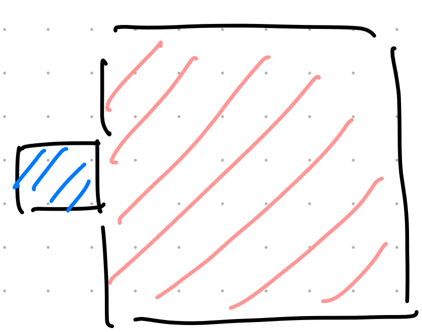
$-\beta F_\beta = S - \beta U$ (free entropy)

can be understood as

entropy of a composite system $\Omega \times \Omega_R^\infty$

heat bath

(heat resorvoir)



In our framework

system is a polytope P
(or (X, L))

state is a log convex fct^n on P

normalized as $\int_P u \, d\mu = 1$.

entropy $\quad S(u) = -\int_P u \log u \, d\mu$

internal energy $\quad U(u) = 2\pi \int_{\partial P} u \, d\sigma$

<u>Thm</u> equilibrium exists

and is of the form

$$u_\lambda = \frac{e^{g_\lambda}}{\int_P e^{g_\lambda} d\mu}$$ for the maximizer

g_λ of $\hat{\mu}^\lambda_{NA}$

(for some λ dep on U)

It turns out equilibrium of int energy

U is achieved by u_λ w/

$$\lambda = -2\pi \underbrace{\left(\frac{\partial S^{eq}}{\partial U}\right)^{-1}}_{\text{absolute temperature.}}$$

μ^λ_{NA} can be understood as "free energy", which we can understand as entropy of a composite system.

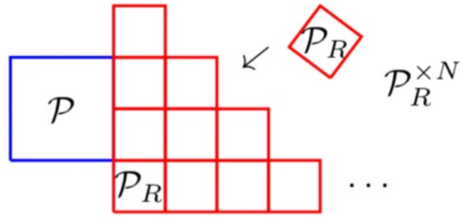


FIGURE 4. Realization of heat bath as the limit of infinitely many composition

Theorem 4.17 (Heat bath limit)**.** Let $\mathcal{P}$ be a K-unstable system and $\mathcal{P}_R$ be a mild K-unstable system. Fix $U \in \mathfrak{U}^*_{\mathcal{P}}$ and $T_R \in [0, \infty)$. For $N \in \mathbb{N}$, consider the composite system

$$\tilde{\mathcal{P}}_N = \mathcal{P} \times \mathcal{P}_R^{\times N}.$$

Let $\tilde{u}_N \in \mathcal{M}^{\exp,1}_{\mathrm{NA}}(\tilde{P}_N)$ be the equilibrium of internal μ-energy $U + NU^{\mathrm{can}}_{\mathcal{P}_R}(T_R)$. Then the following associated equilibrium on the subsystem P

$$u_N := \frac{1}{\int_{P_R^{\times N}} d\mu_{P_R}^{\times N}} \int_{P_R^{\times N}} \tilde{u}_N d\mu_{P_R}^{\times N} \in \mathcal{M}^{\exp,1}_{\mathrm{NA}}(P),$$

converges in L^p-topology ($p \in [1, \frac{n}{n-1})$) to the μ-canonical distribution $u_\infty \in \mathcal{M}^{\exp,1}_{\mathrm{NA}}(P)$ of temperature T_R, which is independent of the choice of $U \in \mathfrak{U}_{\mathcal{P}}$.

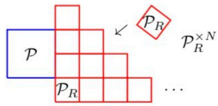
Let $u_R^{\times N} \in \mathcal{M}^{\exp,1}_{\mathrm{NA}}(P_R^{\times N})$ be the equilibrium of internal μ-energy $NU^{\mathrm{can}}_{\mathcal{P}_R}(T_R)$ on $\mathcal{P}_R^{\times N}$. Then for any $u \in \mathcal{M}^{\exp,1}_{\mathrm{NA}}(\mathcal{P}, U)$ of internal μ-energy U, the difference of composite entropy

$$\Delta S_{\tilde{\mathcal{P}}_N} := S_{\tilde{\mathcal{P}}_N}(\tilde{u}_N) - S_{\tilde{\mathcal{P}}_N}(u \times u_R^{\times N})$$

converges to

$$-\frac{1}{T_R}(F_{\mathcal{P}}(T_R, u_\infty) - F_{\mathcal{P}}(T_R, u))$$

as $N \to \infty$, which is independent of the choice of the mild system $\mathcal{P}_R$.



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converges in L^p-topology ($p \in [1, \frac{n}{n-1})$) to the μ-canonical distribution $u_\infty \in \mathcal{M}^{\exp,1}_{\mathrm{NA}}(P)$ of temperature T_R, which is independent of the choice of $U \in \mathfrak{U}_{\mathcal{P}}$.

Proof. We note $U + NU^{\mathrm{can}}_{\mathcal{P}_R}(T_R) \in \mathfrak{U}^*_{\tilde{\mathcal{P}}_N}$. Since $\mathcal{P}_R$ is mild, the composite system $\mathcal{P} \times \mathcal{P}_R^{\times N}$ is also mild. Let $T_N \in [0, \infty)$ be the element of the one point set

$$\mathbb{T}^{\mathrm{eq}}_{\tilde{\mathcal{P}}_N}(U + NU^{\mathrm{can}}_{\mathcal{P}_R}(T_R)).$$

Since $\tilde{u}_N$ on $\tilde{\mathcal{P}}_N$ is the μ-canonical distribution of temperature T_N, u_N on $\mathcal{P}$ is also the μ-canonical distribution of temperature T_N by Theorem 4.1.

Since

$$U^{\mathrm{can}}_{\mathcal{P}}(T_N) + NU^{\mathrm{can}}_{\mathcal{P}_R}(T_N) = U^{\mathrm{can}}_{\tilde{\mathcal{P}}_N}(T_N) = U + NU^{\mathrm{can}}_{\mathcal{P}_R}(T_R), \tag{4.21}$$

we compute

$$U^{\mathrm{can}}_{\mathcal{P}_R}(T_N) = \frac{1}{N}(U + NU^{\mathrm{can}}_{\mathcal{P}_R}(T_R) - U^{\mathrm{can}}_{\mathcal{P}}(T_N)) = U^{\mathrm{can}}_{\mathcal{P}_R}(T_R) + \frac{1}{N}(U - U^{\mathrm{can}}_{\mathcal{P}}(T_N)).$$

Since $U^{\mathrm{can}}_{\mathcal{P}}(T_N) \in \mathfrak{U}^*_{\mathcal{P}}$ is bounded, we get $U^{\mathrm{can}}_{\mathcal{P}_R}(T_N) \to U^{\mathrm{can}}_{\mathcal{P}_R}(T_R)$ as $N \to \infty$. It follows that $T_N \to T_R$. By the continuity we already proved, we conclude u_N converges to the μ-canonical distribution u_∞ of temperature T_R. $\square$

Now we obtain a characterization of free μ-energy in terms of equilibrium of composite system.

Theorem 4.18 (Free μ-energy as composite entropy)**.** Let $\mathcal{P}, \mathcal{P}_R, U, T_R$ and $\tilde{u}_N, u_\infty$ be the same as in the above theorem. Assume further the heat capacity $T_R \partial_T S^{\mathrm{can}}_{\mathcal{P}}(T_R)$ of $(\mathcal{P}_R, T_R)$ is positive. (See Remark 4.16.) Namely we assume $T_R > 0$, $S^{\mathrm{can}}_{\mathcal{P}_R}$ is differentiable at T_R and $\partial_T S^{\mathrm{can}}_{\mathcal{P}_R}(T_R) > 0$.

Let $u_R^{\times N} \in \mathcal{M}^{\exp,1}_{\mathrm{NA}}(P_R^{\times N})$ be the equilibrium of internal μ-energy $NU^{\mathrm{can}}_{\mathcal{P}_R}(T_R)$ on $\mathcal{P}_R^{\times N}$. Then for any $u \in \mathcal{M}^{\exp,1}_{\mathrm{NA}}(\mathcal{P}, U)$ of internal μ-energy U, the difference of composite entropy

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converges to

$$-\frac{1}{T_R}(F_{\mathcal{P}}(T_R, u_\infty) - F_{\mathcal{P}}(T_R, u))$$

as $N \to \infty$, which is independent of the choice of the mild system $\mathcal{P}_R$.