Perelman's μ-entropy in Kähler geometry and μ-cscK metrics

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1. Canonical metrics in Kähler geometry

Setup

$X = (M, J)$: a compact Kähler manifold, $\dim_{\mathbb{C}} X = n$.

$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j = \sqrt{-1} \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j} dz^i \wedge d\bar{z}^j$: a Kähler metric (form)
$\rightsquigarrow L := [\omega] \in H^2(X, \mathbb{R})$

The set of Kähler classes

$$\mathcal{C}_X := \{ L \in H^2(X, \mathbb{R}) \mid \exists \omega \in L : \text{ Kähler } \}$$

gives an strictly convex open cone of $H^2(X, \mathbb{R})$. For $L \in \mathcal{C}_X$, we put

$$\mathcal{H}(X, L) := \{ \omega \in L \mid \omega : \text{ Kähler } \} \hookrightarrow C^\infty(X)/\mathbb{R}.$$

When L is the first Chern class of a line bundle $\boldsymbol{L}$, holomorphic sections of $\boldsymbol{L}^{\otimes k}$ gives an embedding $X \hookrightarrow \mathbb{C}P^{N_k}$ (Kodaira embedding).

A pair (X, L) for $L \in \mathcal{C}_X$ is called a polarized manifold.

What we are interested in?

Question For a given (X, L), can we find/define a "canonical metric" ω_{can} in L which characterizes the complex structure of (X, L)?

$C = (\Sigma, J)$: algebraic curve (Riemann surface),
$2\pi c_1(C) = \lambda L \in H^2(C, \mathbb{R}) = \mathbb{R}$.
$L \in \mathcal{C}_C$ iff

$$\begin{cases} \lambda > 0 & C = \mathbb{C}P^1, \\ \lambda = 0 & g(C) = 1, \\ \lambda < 0 & g(C) \geq 2. \end{cases}$$

For each $L \in \mathcal{C}_C$, $\exists\, \omega \in L$ Kähler–Einstein metric $\mathrm{Ric}(\omega) = \lambda\omega$, unique modulo $\mathrm{Aut}_{\mathrm{red}}(C)$.

$\rightsquigarrow$ applied to a construction of the moduli space of complex structures/isom, based on moment map picture. (Kempf–Ness, GIT, K-stability, Yau–Tian–Donaldson conjecture...)

Various notions of canonical metrics

Definition

Let (X, L) be a polarized manifold. A Kähler metric ω is called

- Kähler–Einstein metric if $\mathrm{Ric}(\omega) = \lambda\omega$.
- Kähler–Ricci soliton if $\mathrm{Ric}(\omega) - L_\xi \omega = \lambda\omega$ for a holo. vec. ξ.
  $\rightsquigarrow \lambda L = 2\pi c_1(X)$.
- CscK metric if $s(\omega) = \mathrm{const}$.
- Extremal metric if $\partial^\sharp s(\omega) := g^{i\bar{j}} s_{\bar{j}}.\partial/\partial z^i$ is holomorphic.

Remark

- (normalized) Kähler–Ricci flow: $\mathrm{Ric}(\omega_t) - \lambda\omega_t = \dot{\omega}_t$
  $\rightsquigarrow$ Kähler–Ricci soliton.
- Calabi functional: $C(\omega) := \int_X \hat{s}^2(\omega)\omega^n \rightsquigarrow$ Extremal metric.
- Unique modulo $\mathrm{Aut}_{\mathrm{red}}(X, L)$.

Examples: Kähler–Einstein metric/Kähler–Ricci soliton

- Fubini–Study metric on $\mathbb{C}P^n$: $H^2(\mathbb{C}P^n, \mathbb{R}) = \mathbb{R}$, $c_1(X) \in \mathcal{C}_X$,

$$\omega := \sqrt{-1}\partial\bar{\partial} \log \frac{1}{1+|z|^2} \in 2\pi c_1(X) \quad \Rightarrow \quad \mathrm{Ric}(\omega) = \omega.$$

- Aubin–Yau '77: $-c_1(X) \in \mathcal{C}_X$ (e.g. $g(C) \geq 2$)

  $\Rightarrow \exists! \omega \in -2\pi c_1(X)$ s.t. $\mathrm{Ric}(\omega) = -\omega$.
- (Calabi–)Yau '77: $c_1(X) = 0$

  $\Rightarrow \forall L \in \mathcal{C}_X\ \exists! \omega \in L$ s.t. $\mathrm{Ric}(\omega) = 0$.

Futaki '83 obstructions for $\exists$ KE met. on X w/ $c_1(X) \in \mathcal{C}_X$.

$\rightsquigarrow$ No Kähler–Einstein metric on $\mathbb{C}P^2 \# k\overline{\mathbb{C}P}^2$ for $k = 1, 2$.

($\rightsquigarrow$ K-stability: alg. criterion for $\exists$ KE met. $\rightsquigarrow$ YTD conjecture)

Still, $\exists$ Kähler–Ricci soliton

Indeed, every toric Fano manifold admits a KR soliton.

Examples: CscK metric/Extremal metric

When $\lambda L \neq 2\pi c_1(X)$, there is no KE metric/KR soliton in L.

$\rightsquigarrow \exists$ CscK metric/Extremal metric ?

- Gluing: $\exists$ cscK metric on (X, L) with $\#\mathrm{Aut} < \infty$
  $\Rightarrow \exists$ extremal metric on the blowup $(\tilde{X}, \pi^* L - \epsilon E)$ for small $\epsilon > 0$.
- Calabi ansatz: $X = \mathbb{P}_\Sigma(\mathcal{L} \oplus \mathcal{O})$, $\exists$ L admits an extremal metric.
- Toric surface: $\int_{\partial P} q d\sigma - \frac{(c_1(X).L^{\cdot n-1})}{(L^{\cdot n})} \int_P q d\mu > 0 \Rightarrow \exists$ cscK.
- Calabi dream manifold (Chen–Cheng '18): Let X be a Kähler surface with $-c_1(X) \in \mathcal{C}_X$ and no curve of negative self-intersection, then for every Kähler class L there exists a unique cscK metric in L.

Donaldson–Fujiki moment map picture

- $(M, \omega) \circlearrowleft T$: symplectic manifold
- $\mathcal{J} := \mathcal{J}_T(M, \omega)$: the space of T-inv alm cpx strs. comp. with ω
- Ω: a symplectic structure on $\mathcal{J}$
- $\mathrm{Ham} := \mathrm{Ham}_T(M, \omega)$: the group of T-equiv. Hamiltonian diffeomorphisms
- $\mathfrak{h} := \mathrm{Lie}(\mathrm{Ham}) = C^\infty_T(M)/\mathbb{R}$

Proposition (Donaldson–Fujiki '97, I. '18, Lahdili '19)

There are symplectic structures $\Omega_{\mathrm{cscK}} = \Omega_{\mathrm{ext}}$, Ω_{KRs} and moment maps

$$\mathscr{S}_{\mathrm{cscK}}, \mathscr{S}_{\mathrm{ext}}, \mathscr{S}_{\mathrm{KRs}} : \mathcal{J} \to \mathfrak{h}^\vee$$

with respect to $(\mathcal{J}, \Omega_\bullet) \circlearrowleft \mathrm{Ham}$ s.t. $\mathscr{S}_\bullet(J) = 0$ for integrable $J \in \mathcal{J}$ iff g_J is $\bullet = \mathrm{cscK}/\mathrm{ext}/\mathrm{KRs}$. As for KRs, we assume $[\omega] = c_1(X, \omega)$.

2. μ-cscK metrics

μ-cscK metric

For $\lambda \in \mathbb{R}$, we call a Kähler metric $\omega \in L$ μ^λ-cscK metric if

$$(s(\omega) + \bar{\square}\theta_\xi) + (\bar{\square}\theta_\xi - \xi\theta_\xi) - \lambda\theta_\xi = \mathrm{const}.$$

for some holomorphic vector field ξ with $\exists \theta_\xi \in C^\infty(X)$ s.t. $\partial^\sharp \theta_\xi = \xi$.

Theorem (I. '19 + Lahdili '20)

- A Kähler–Ricci soliton $\omega_{\mathrm{KRs}} \in c_1(X)$ is a $\mu^{2\pi}$-cscK metric.
- There is a Donaldson–Fujiki type moment map picture for μ^λ_ξ-cscK metric, which uses Ω_{KRs} on $\mathcal{J}(M, \omega)$.
- For each (X, L), μ^λ-cscK metric is unique mod Aut for $\lambda \ll 0$.
- If there is an extremal metric $\omega_{\mathrm{ext}} \in L$, there is a family ω_λ of μ^λ-cscK metrics for $\lambda \ll 0$ such that ω_λ converges to ω_{ext}.

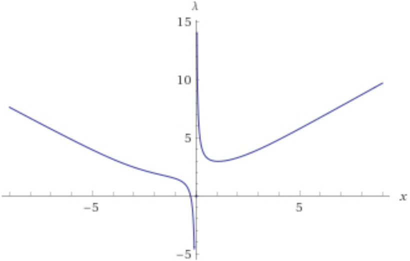
Phase transition phenomenon

- In general, μ_ξ^λ-cscK metric is unique mod Aut_ξ (Lahdili).
- When $\lambda \gg 0$, μ^λ-cscK metric is not unique due to the non-uniqueness of the candidates ξ mod Aut.
- For each $\lambda \leq 0$, the conjugate classes of the candidate vectors are at most finite.

For $\lambda \gg 0$, $\mathbb{C}P^1$ admits a μ^λ-cscK metric which is not a cscK metric.

Calabi ansatz on $\mathbb{C}P^2\#\overline{\mathbb{C}P^2} = \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O}(1)\oplus\mathcal{O})$

- The anti-canonical class $-K_X$ of $X = \mathbb{C}P^2\#\overline{\mathbb{C}P^2} = \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O}(1)\oplus\mathcal{O})$ admits both KRs and extremal metric (no cscK metrics).
- Calabi ansatz: $\exists\ \mu^\lambda$-cscK metrics for every $\lambda \in \mathbb{R}$ (with a negative $x_\lambda = \xi^\lambda/\eta = (6/11)\cdot\xi^\lambda/\xi_{\mathrm{ext}}$) connecting KRs and extremal metric.



(x from −9 to 9)

- We can see $2.9\times 2\pi < \lambda_{\mathrm{freeze}} < 3\times 2\pi$.

Calabi ansatz on $\mathbb{P}_\Sigma(L \oplus \mathcal{O})$

$X = \mathbb{P}_\Sigma(\mathcal{L} \oplus \mathcal{O})$: a ruled surface for a positive $\mathcal{L}$ on an algebraic curve Σ.

Let F denote a fibre and $B = \{(x, (0:1)) \mid x \in \Sigma\}$ denote the section at infinity. The Kähler cone is given by

$$\mathcal{C}_X = \{aF + bB \mid b > 0, \ \frac{a}{b} > -\frac{\deg \mathcal{L}}{2}\}.$$

Theorem (I. '20)

Every Kähler class L in the cone $\{aF + bB \mid a, b > 0\}$ admits a μ^λ-cscK metric for every $\lambda \geq 0$ (for some ξ).

On the other hand, for $g(\Sigma) \geq 2$ and small $\frac{a}{b}$, the Kähler class $aF + bB$ does not admit extremal metrics. (and no μ^λ-cscK metrics for $\lambda \ll 0$.)

3. Perelman's μ-entropy

Perelman's W-functional and μ-cscK metrics

Note

$$T\mathcal{H}(X,L) = \mathcal{H}(X,L) \times C^\infty(X)/\mathbb{R}.$$

We consider the functional $\check{W}^\lambda : T\mathcal{H}(X,L) \to \mathbb{R}$ given by

$$\check{W}^\lambda(\omega, f) := -\frac{\int_X \big(s(\omega) + |\partial^\sharp f|^2 - \lambda(n-f)\big) e^{-f}\omega^n}{\int_X e^{-f}\omega^n} - \lambda \log \int_X e^{-f}\frac{\omega^n}{n!}.$$

Recall Perelman's W-functional is

$$W(g,f;\tau) = \frac{1}{(4\pi\tau)^{n/2}} \int_X \Big(\tau(R(g) + |\nabla f|^2) - (n-f)\Big) e^{-f}\mathrm{vol}_g$$

for a Riemannian metric g and $f \in C^\infty(X)$ with $\int_X e^{-f}\mathrm{vol}_g = 1$, usually considered for $\tau \geq 0$

Theorem (I. '20, to appear)

A state $(\omega, f) \in T\mathcal{H}$ is a critical point of $\check{W}^\lambda$ if and only if $\xi = \partial^\sharp f$ is holomorphic and ω is a μ^λ-cscK metric w.r.t. ξ.

Archimedean μ-entropy

We define $\check{\boldsymbol{\mu}}^\lambda : \mathcal{H}(X, L) \to \mathbb{R}$ by

$$\check{\boldsymbol{\mu}}^\lambda(\omega) := \sup_{f \in C^\infty(X)} \check{W}^\lambda(\omega, f).$$

Theorem (I. '20, to appear)

1. For each $\lambda \leq 0$ and ω, there exists a unique maximizer $f \in C^\infty(X)$ of $\check{W}^\lambda(\omega, \cdot)$ modulo constant.
2. In this case, the functional $\check{\boldsymbol{\mu}}^\lambda : \mathcal{H}(X, L) \to \mathbb{R}$ is smooth.
3. Its critical points are precisely μ^λ-cscK metrics.
4. They are global minimizers of $\check{\boldsymbol{\mu}}^\lambda$ among all Kähler metrics.

The μ-entropy $\check{\boldsymbol{\mu}}^\lambda$ is an analogy of Calabi functional.

Entropy maximization

Suppose we have a holomorphic Hamiltonian action $(X, L) \circlearrowleft K$ by a compact Lie group. We define $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}} : \mathfrak{k} \to \mathbb{R}$ by

$$\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\xi) := \check{W}^\lambda(\omega, -\theta^\omega_\xi),$$

using a K-invariant metric $\omega \in L$ and $\theta_\xi \in C^\infty(X)$: $\partial^\sharp \theta^\omega_\xi = \xi$.

Theorem (I. '19 $+\alpha$)

- This is independent of the choice of ω and μ.
- If there exists a μ^λ_ξ-cscK metric, then ξ is a critical point of $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$. When $\lambda \leq 0$, ξ maximizes $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$ (among all vectors).
- There always exist a maximizer of $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$.
- (Phase transition) The value

  $$\lambda_{\mathrm{freeze}} := \sup\{\lambda \in \mathbb{R} \mid \forall \lambda' < \lambda \;\; \check{\boldsymbol{\mu}}^{\lambda'}_{\mathrm{NA}} \text{ admits a unique critical point}\}$$

  is not $\pm\infty$.

W-functional along geodesics

A path $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$ in $\mathcal{H}(X, L)$ is called a geodesic if

$$\ddot{\phi}_t - |\bar{\partial}\dot{\phi}_t|^2 = 0.$$

Theorem (I. '20, to aapear)

- $\check{W}^\lambda$ is monotonically decreasing along smooth geodesics.
- For a smooth subgeodesic ray ϕ_t subordinated to a test configuration, the limit $\lim_{t\to\infty} \check{W}^\lambda(\omega_t, \dot{\phi}_t)$ is given by the non-archimedean μ-entropy.

4. Non-archimedean μ-entropy

"Non-archimedean" μ-entropy of test configuration

Recall

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(-\frac{1}{2}\xi) = \check{W}^{\lambda}(\omega, \mu^{\omega}_{\xi}) = -\frac{\int_X (\mathrm{Ric}(\omega) + \bar{\Box}\mu_\xi) e^{\omega+\mu_\xi}}{\int_X e^{\omega+\mu_\xi}} + \lambda\Big(\frac{\int_X (\omega+\mu_\xi) e^{\omega+\mu_\xi}}{\int_X e^{\omega+\mu_\xi}} - \log \int_X e^{\omega+\mu_\xi}\Big)$$

Definition

For a test configuration $(\mathcal{X}, \mathcal{L})$ and $\tau \geq 0$, we put

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) := 2\pi \frac{(\kappa^{\mathbb{C}^\times}_{\mathcal{X}_0} . e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)}{(e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)} + \lambda\Big(\frac{(\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0 . e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)}{(e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)} - \log(e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)\Big)$$

"Non-archimedean" μ-entropy of test configuration

Proposition

By the localization on $\mathbb{P}^1$ and the equivariant Grothendieck–Riemann–Roch theorem for $\mathcal{X}_0/\{0\} \to \mathcal{X}/\mathbb{C}$, we get

$$(e^{\mathcal{L}_{\mathbb{C}^\times}|_{\mathcal{X}_0}}; \tau) = (e^L) - \tau(e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \tau)$$

$$(\mathcal{L}_{\mathbb{C}^\times}|_{\mathcal{X}_0}.e^{\mathcal{L}_{\mathbb{C}^\times}|_{\mathcal{X}_0}}; \tau) = (L.e^L) - \tau(\bar{\mathcal{L}}_{\mathbb{C}^\times}.e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \tau)$$

$$(\kappa^{\mathbb{C}^\times}_{\mathcal{X}_0}.e^{\mathcal{L}_{\mathbb{C}^\times}|_{\mathcal{X}_0}}; \tau) = (K_X.e^L) - \tau(K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1}.e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \tau)$$

Replacing $K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1}$ with ${}^bK^{\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} = K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} + [\mathcal{X}_0^{\mathrm{red}}]^{\mathbb{C}^\times} - [\mathcal{X}_0]^{\mathbb{C}^\times}$, we obtain the following by the equivariant Stokes theorem.

Theorem (I. '20, to appear)

For a smooth subgeodesic ray ϕ_t subordinated to a test configuration $(\mathcal{X}, \mathcal{L})$, ${}^b\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = \lim_{t\to\infty} \check{W}^\lambda(\omega_{\tau t}, \dot{\phi}_{\tau t})$.

Toric expression

By the equivariant intersection formula, we obtain

Proposition (I. '20, to appear)

Let (X, L) be a toric variety and $P \subset \mathfrak{t}$ be the associated moment polytope. Take a defining convex piecewise affine function q of the moment polytope $Q = \{(\mu, t) \mid \mu \in P,\ q(\mu) \leq t \leq 0\}$ of a T-equivariant normal test configuration $(\mathcal{X}, \mathcal{L})$. Then

$$ {}^{b}\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = -2\pi \frac{\int_{\partial P} e^{\tau q} d\sigma}{\int_P e^{\tau q} d\mu} + \lambda \Big(\frac{\int_P (n + \tau q) e^{\tau q}}{\int_P e^{\tau q} d\mu} - \log \int_P e^{\tau q} d\mu \Big). $$

Question: Find an explicit example of a toric variety (X, L) and a non-product $(\mathcal{X}, \mathcal{L})$ (or filtration) maximizing $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$.

Entropy maximization for tc/filtration

The μ-entropy makes sense also for f.g. filtrations. By differentiating the μ-entropy at ξ to the direction of test configurations, we obtain

Theorem (I. '20, to appear)

A If there is a vector ξ such that

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\xi) \geq \check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau)$$

for every test configuration $(\mathcal{X}, \mathcal{L}; \tau)$, then (X, L) is μ^{λ}_{ξ}K-semistable.

B If for every test configuration $(\mathcal{X}, \mathcal{L}; \tau)$, there is a vector ξ such that

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\xi) \geq \check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau),$$

then (X, L) is μ^{λ}K-semistable.

Optimal destabilization conjecture

Conjecture (μK-stability is entropy maximization)

When $\lambda \leq 0$, the following are equivalent:

- (X, L) is μ^λ_ξK-semistable
- ξ is a maximizer of $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$ among all (f.g.) filtrations

Conjecture (Optimal destabilization conjecture for μ-entropy)

1. $$\sup_{\phi \in \mathcal{H}(X^{\mathrm{NA}}, L^{\mathrm{NA}})} \check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\phi) = \inf_{\omega \in \mathcal{H}(X, L)} \check{\boldsymbol{\mu}}^\lambda(\omega).$$
2. When $\lambda \leq 0$, $\exists!$ a maximizer $\phi \in \mathcal{H}(X^{\mathrm{NA}}, L^{\mathrm{NA}})$ of $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$ modulo $\mathrm{Aut}(X, L)$.
3. "The central fibre" $\mathcal{X}_0$ of ϕ is mildly singular and μ^λ_ξK-semistable for the vector ξ generated by ϕ. (The NA Monge–Ampére measure $\mathrm{MA}^{\mathrm{NA}}(\phi)$ is a sum of dirac mass supporting on quasi-monomial valuations.)