μ-cscK metrics, μK-stability and a Lagrangian formalism

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Contents:

1. Heuristic picture: Lempert's Lagrangian formalism

Lagrangian system in physics

Lagrangian mechanics: coordinate free expression of Newtonian mechanics (convenient to deal with *holonomic constraint*)

$$\mathcal{H} = \mathbb{R}^3 \ni x, \quad T\mathcal{H} \ni (x, v).$$

In the Cartesian coordiante, the kinetic energy is $T(x, v) = \frac{1}{2} m|v|^2$. Consider a potential $V(x, v) = mg|x|$. The Lagrangian of this system is

$$\mathcal{L} = T - V : T\mathcal{H} \to \mathbb{R}.$$

The principle of least action says that the motion $\phi_t \in \mathcal{H}$ of a particle from $x_0 \in \mathcal{H}$ to $x_1 \in \mathcal{H}$ is characterized as the minimizer of the action functional

$$S(\phi) := \int_0^T \mathcal{L}(\phi_t, \dot{\phi}_t) dt.$$

The Euler–Lagrange equation is

$$\frac{\partial \mathcal{L}}{\partial x}(\phi_t, \dot{\phi}_t) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v}(\phi_t, \dot{\phi}_t) = 0.$$

Lempert's Lagrangian formalism

(X, L): a polarized manifold
$\mathcal{H}(X, L)$: the space of Kähler metrics in L

$$T\mathcal{H}(X, L) \cong \mathcal{H}(X, L) \times \mathbb{C}^\infty(X)/\mathbb{R}$$

Consider a functional

$$\mathcal{L} : T\mathcal{H} \to \mathbb{R}$$

which is invariant under the parallel translation (w.r.t. Mabuchi connection): for any smooth curve $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}u_t$, we have

$$\mathcal{L}(\omega_t, f \circ \varphi_t) = \mathcal{L}(\omega, f)$$

for $\varphi_t \in \mathrm{Diff}(X)$ generated by the time dep. vector field $(-1/2)\nabla_{\omega_t}\dot{u}_t$.

e.g. $\mathcal{L}(\omega, f) = \int_X |\hat{f}|^p \omega^n$

Lempert's Lagrangian formalism

Assume $\mathcal{L}$ is fibrewise convex.

Theorem (Lempert '20, principle of least action)

A weak geodesic (slightly regular) minimizes the action functional

$$S(\phi) := \int_a^b \mathcal{L}(\omega_t, \dot{\phi}_t) dt$$

among all path $\phi = \{\phi_t\}$ connecting given endpoints ω_a and ω_b.

Theorem (Lempert '20, Hadamard convexity)

Put

$$\mathscr{L}_T(\omega, \omega') = \inf\{S(\psi) \mid \psi : [0, T] \to \mathcal{H}(X, L) \text{ connecting } \omega, \omega'\}.$$

Then for weak geodesics ϕ, ϕ', $\mathscr{L}_T(\phi, \phi')$ is convex.

2. Extremal metric and Kähler–Ricci soliton

Extremal metric and Calabi functional

The Calabi functional $C : \mathcal{H}(X, L) \to \mathbb{R}$ is given by

$$C(\omega) := \frac{1}{2} \int_X \hat{s}(\omega)^2 \omega^n.$$

The critical points are extremal metrics: $\partial^\sharp s(\omega) = g^{i\bar{\jmath}} s_{\bar{\jmath}}$ is holomorphic.

Extremal metric and Calabi functional

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Donaldson–Futaki invariant:

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := (K_{\bar{\mathcal{X}}/\mathbb{P}^1}.\bar{\mathcal{L}}^{\cdot n}) - \frac{(K_X.L^{\cdot n-1})}{(n+1)(L^{\cdot n})}(\bar{\mathcal{L}}^{\cdot n+1})$$

Relative Donaldson–Futaki invariant:

$$\mathrm{DF}_\xi(\mathcal{X}, \mathcal{L}) := \mathrm{DF}(\mathcal{X}, \mathcal{L}) + \frac{1}{4\pi} \int_{\mathcal{X}_0} \hat{\theta}_\xi \theta_\eta \omega_0^n$$

If (X, L) admits an extremal metric with $\xi = \partial^\sharp s$, then we have $\mathrm{DF}_\xi(\mathcal{X}, \mathcal{L}) \geq 0$ (relatively K-semistable).

Extremal metric and Calabi functional

Donaldson's lower bound:

$$-\frac{4\pi \mathrm{DF}(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|} \leq (2C(\omega))^{1/2},$$

where we put

$$\|(\mathcal{X},\mathcal{L})\|^2 = \int_{\mathbb{R}} (t-b)^2 \mathrm{DH}(\mathcal{X},\mathcal{L})$$

with the barycenter $b := \int_{\mathbb{R}} t\mathrm{DH}(\mathcal{X},\mathcal{L})$.

Optimal destabilization conjecture

We have the equality

$$\sup_{(\mathcal{X},\mathcal{L})} -\frac{4\pi \mathrm{DF}(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|} = \inf_{\omega \in \mathcal{H}(X,L)} (2C(\omega))^{1/2}$$

and the supremum on the LHS is achieved by some test configuration.

Lagrangian formalism on Calabi functional

We consider the functional $W_{\mathrm{ext}} : T\mathcal{H}(X, L) \to \mathbb{R}$ given by

$$W_{\mathrm{ext}}(\omega, f) := -\frac{1}{2}\int_X (\hat{s}(\omega) - \hat{f})^2 \omega^n + \frac{1}{2}\int_X \hat{s}(\omega)^2 \omega^n.$$

Observations (written in my preprint in preparation)

- (Conservative) The Euler–Lagrange equation

$$\frac{\partial W_{\mathrm{ext}}}{\partial \omega}(\omega_t, \dot{\phi}_t) - \frac{d}{dt}\frac{\partial W_{\mathrm{ext}}}{\partial f}(\omega_t, \dot{\phi}_t) = 0$$

  for the action functional $S(\phi) := \int_a^b W_{\mathrm{ext}}(\omega_t, \dot{\phi}_t) dt$ is the geodesic equation.
- $W_{\mathrm{ext}}(\omega_t, \dot{\phi}_t)$ is monotonically decreasing along weak geodesic. (Convexity of the Mabuchi functional)

A new proof of Donaldson's lower bound

We note

$$C(\omega) = \sup_{f \in C^\infty(X)} W_{\mathrm{ext}}(\omega, f).$$

It is recently shown by C. Li that the slope of the Mabuchi functional along to the geodesic ray associated to a normal test configuration $(\mathcal{X}, \mathcal{L})$ is given by $M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \lim_{d \to \infty} d^{-1} \mathrm{DF}(\mathcal{X}_d, \mathcal{L}_d)$.
Using this, we obtain

$$\lim_{t \to \infty} W_{\mathrm{ext}}(\omega_t, \dot{\phi}_t) = -\frac{1}{2} \Big(4\pi M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) + \|(\mathcal{X}, \mathcal{L})\|^2 \Big).$$

We put for $\tau \geq 0$

$$C_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) := -\frac{1}{2} \Big(4\pi \tau M^{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) + \tau^2 \|(\mathcal{X}, \mathcal{L})\|^2 \Big).$$

By the monotonicity, we obtain

$$C_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = \lim_{t \to \infty} W_{\mathrm{ext}}(\omega_{\tau t}, \dot{\phi}_{\tau t}) \leq W_{\mathrm{ext}}(\omega, \dot{\phi}) \leq C(\omega)$$

A new proof of Donaldson's lower bound

When $\mathrm{DF}(\mathcal{X}, \mathcal{L}) \geq 0$, the maximum of $C_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau)$ is achieved at $\tau = 0$. When $\mathrm{DF}(\mathcal{X}, \mathcal{L}) < 0$, we have

$$C(\omega) \geq \sup_{\tau \geq 0} C_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = 2 \cdot 4\pi^2 \frac{\mathrm{DF}(\mathcal{X}, \mathcal{L})^2}{\|(\mathcal{X}, \mathcal{L})\|^2},$$

which shows Donaldson's lower bound on Calabi functional. □

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Theorem (Entropy maximization)

- If the supremum of C_{NA} is achieved by a product configuration, then (X, L) is relatively K-semistable. This is the case when (X, L) admits an extremal metric.
- If (X, L) is K-semistable, then C_{NA} is maximized at the trivial configuration.

Kähler–Ricci soliton and modified K-stability

Let X be a Fano manifold, i.e. $-K_X$ is ample. The normalized Kähler–Ricci flow on X is

$$\dot{\omega}_t = \mathrm{Ric}(\omega_t) - 2\pi\omega_t.$$

A Kähler–Ricci soliton is a self-similar solution: a pair of a Kähler metric ω and a holomorphic vector field ξ such that

$$L_\xi \omega = \mathrm{Ric}(\omega) - 2\pi\omega.$$

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We can define modified Futaki invariant $\mathrm{Fut}_\xi(\mathcal{X}, \mathcal{L})$ and hence modified K-stability of X w.r.t. ξ. (Tian–Zhu, Xiong, Berman, Datar–Székelyhidi)

There is a unique vector ξ such that $\mathrm{Fut}_\xi = 0$ for product configurations,

Problem But it is HARD to express the vector explicitly!
The invariant $\mathrm{Fut}_\bullet$ exists, but it is not explicitly given for X.

Nevertheless, we can check the modified K-stability in some case (toric, horospherical), without detecting ξ.

Kähler–Ricci soliton and H-functional

Let X be a Fano manifold. The H-functional $H : \mathcal{H}(X, -K_X) \to \mathbb{R}$ is given by

$$\frac{1}{2\pi} H(\omega) := \int_X h e^h \omega^n \Big/ \int_X e^h \omega^n - \log \int_X e^h \frac{\omega^n}{n!}.$$

The critical points are Kähler–Ricci solitons: $\partial^\sharp h$ is holomorphic.

Optimal destabilization along Kähler–Ricci flow

- (Chen–Sun–Wang) For a Fano manifold X, there exists a finitely generated filtration $\mathcal{F}$ of X such that the central fibre $\underline{X}$ is a modified K-semistable $\mathbb{Q}$-Fano variety, which moreover admits a special degeneration $\underline{\mathcal{X}}$ to a $\mathbb{Q}$-Fano variety $\underline{\mathcal{X}}_0$ with Kähler–Ricci soliton.
- (Han–Li) The filtration is uniquely characterized (modulo equivalence) as a maximizer of H-entropy.
- (Dervan–Székelyhidi) $\sup_{\mathcal{F}} H(\mathcal{F}) = \inf_{\omega \in \mathcal{H}(X,L)} H(\omega)$.

Lagrangian formalism on H-functional

Let X be a Fano manifold. We consider the functional $L : T\mathcal{H}(X, -K_X) \to \mathbb{R}$ given by

$$\frac{1}{2\pi} L(\omega, f) := -\int_X f e^h \omega^n \Big/ \int_X e^h \omega^n - \log \int_X e^{-f} \frac{\omega^n}{n!}.$$

Observations

- (Conservative) The Euler–Lagrange equation

$$\frac{\partial L}{\partial \omega}(\omega_t, \dot{\phi}_t) - \frac{d}{dt} \frac{\partial L}{\partial f}(\omega_t, \dot{\phi}_t) = 0$$

  for the action functional $S(\phi) := \int_a^b L(\omega_t, \dot{\phi}_t) dt$ is the geodesic equation.
- (Dervan–Székelyhidi) $L(\omega_t, \dot{\phi}_t)$ is monotonically decreasing along weak geodesic. (Convexity of the Ding functional) The limit along a geodesic ray gives the H-entropy.

3. A non-conservative Lagrangian formalism on μ-cscK metric

μ-cscK metric

For $\lambda \in \mathbb{R}$, we call a Kähler metric $\omega \in L$ μ^λ-cscK metric if

$$(s(\omega) + \bar{\Box}\theta_\xi) + (\bar{\Box}\theta_\xi - \xi\theta_\xi) - \lambda\theta_\xi = \mathrm{const}.$$

for some holomorphic vector field ξ with $\exists \theta_\xi \in C^\infty(X)$ s.t. $\sqrt{-1}\partial^\sharp\theta_\xi = \xi$.

Theorem (I. '19 + Lahdili '20)

- A Kähler–Ricci soliton $\omega_{\mathrm{KR}} \in -K_X$ is a $\mu^{2\pi}$-cscK metric.
- There is a Donaldson–Fujiki type moment map picture for μ^λ_ξ-cscK metric.
- For each (X, L), μ^λ-cscK metric is unique mod Aut for $\lambda \ll 0$.
- If there is an extremal metric $\omega_{\mathrm{ext}} \in L$, there is a family ω_λ of μ^λ-cscK metrics for $\lambda \ll 0$ such that ω_λ converges to ω_{ext}.

Perelman's W-functional and μ-cscK metrics

We consider the functional $\check{W}^\lambda : T\mathcal{H}(X, L) \to \mathbb{R}$ given by

$$\check{W}^\lambda(\omega, f) := -\frac{\int_X \big(s(\omega) + |\partial^\sharp f|^2 - \lambda(n - f)\big) e^{-f} \omega^n}{\int_X e^{-f} \omega^n} - \lambda \log \int_X e^{-f} \frac{\omega^n}{n!}.$$

Recall Perelman's W-functional is

$$W(g, f; \tau) = \frac{1}{(4\pi\tau)^{n/2}} \int_X \Big(\tau(R(g) + |\nabla f|^2) - (n - f)\Big) e^{-f} \mathrm{vol}_g$$

for a Riemannian metric g and $f \in C^\infty(X)$ with $\int_X e^{-f} \mathrm{vol}_g = 1$, usually considered for $\tau \geq 0$

Theorem (I. '20, to appear)

A state $(\omega, f) \in T\mathcal{H}$ is a critical point of $\check{W}^\lambda$ if and only if $\xi = \partial^\sharp f$ is holomorphic and ω is a μ^λ-cscK metric w.r.t. ξ.

W-functional as a non-conservative Lagrangian system

- (Non-conservative) The Euler–Lagrange equation $\frac{\partial \check{W}^\lambda}{\partial \omega}(\omega_t, \dot{\phi}_t) - \frac{d}{dt}\frac{\partial \check{W}^\lambda}{\partial f}(\omega_t, \dot{\phi}_t) = 0$ is NOT equivalent to the geodesic equation.
- The extremal path is geodesic iff $\partial^\sharp_{\omega_t} \dot{\phi}_t$ is holomorphic, which happens only when $\omega_t = \varphi_t^* \omega$ for $\varphi_t \in \mathrm{Aut}$ generated by a holomorphic vector field $\xi = \partial^\sharp f$.

Theorem (I. '20, to aapear)

- $\check{W}^\lambda$ is monotonically decreasing along smooth geodesics.
- For a smooth subgeodesic ray ϕ_t subordinated to a test configuration, the limit $\lim_{t\to\infty} \check{W}^\lambda(\omega_t, \dot{\phi}_t)$ is given by the non-archimedean μ-entropy.
- $\lim_{\lambda\to\pm\infty} \lambda\Big(\check{W}^\lambda(\omega, \lambda^{-1} f) - \check{W}^\lambda(\omega, 0)\Big) = W_{\mathrm{ext}}(\omega, f)$.

Archimedean μ-entropy

We define $\check{\boldsymbol{\mu}}^\lambda : \mathcal{H}(X, L) \to \mathbb{R}$ by

$$\check{\boldsymbol{\mu}}^\lambda(\omega) := \sup_{f \in C^\infty(X)} \check{W}^\lambda(\omega, f).$$

Theorem (I. '20, to appear)

1. For each $\lambda \leq 0$ and ω, there exists a unique maximizer $f \in C^\infty(X)$ of $\check{W}^\lambda(\omega, \cdot)$ modulo constant.
2. In this case, the functional $\check{\boldsymbol{\mu}}^\lambda : \mathcal{H}(X, L) \to \mathbb{R}$ is smooth.
3. Its critical points are precisely μ^λ-cscK metrics.
4. They are global minimizers of $\check{\boldsymbol{\mu}}^\lambda$ among all T-invariant Kähler metrics, where T is the center of a maximal compact.

The μ-entropy $\check{\boldsymbol{\mu}}^\lambda$ is an analogy of Calabi functional.

$\downarrow$

Question Is there an analogy of Donaldson's lower bound?

4. Volume minimization and μK-stability

Entropy maximization for product filtration

Suppose we have a holomorphic Hamiltonian action $(X, L) \curvearrowleft K$ by a compact Lie group. We define $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}} : \mathfrak{k} \to \mathbb{R}$ by

$$\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}(\xi) := \check{W}^\lambda(\omega, -\theta^\omega_\xi),$$

using a K-invariant metric $\omega \in L$ and $\theta_\xi \in C^\infty(X)$: $\sqrt{-1}\bar{\partial}\theta^\omega_\xi = i_\xi \omega$.

Theorem (I. '19 $+\alpha$)

- This is independent of the choice of ω and μ.
- If there exists a μ^λ_ξ-cscK metric, then ξ is a critical point of $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$. When $\lambda \leq 0$, ξ maximizes $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$ (among all vectors).
- There always exist a maximizer of $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$.
- (Phase transition) The value

  $$\lambda_{\mathrm{freeze}} := \sup\{\lambda \in \mathbb{R} \mid \forall \lambda' < \lambda \;\; \check{\boldsymbol{\mu}}^{\lambda'}_{\mathrm{NA}} \text{ admits a unique critical point}\}$$

  is not $\pm\infty$.

"Non-archimedean" μ-entropy of test configuration

Recall

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(-\frac{1}{2}\xi) = \check{W}^{\lambda}(\omega, \mu^{\omega}_{\xi}) = -\frac{\int_X (\mathrm{Ric}(\omega) + \bar{\Box}\mu_{\xi}) e^{\omega+\mu_{\xi}}}{\int_X e^{\omega+\mu_{\xi}}} + \lambda\Big(\frac{\int_X (\omega + \mu_{\xi}) e^{\omega+\mu_{\xi}}}{\int_X e^{\omega+\mu_{\xi}}} - \log \int_X e^{\omega+\mu_{\xi}}\Big)$$

Definition

For a test configuration $(\mathcal{X}, \mathcal{L})$ and $\tau \geq 0$, we put

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) := 2\pi \frac{(\kappa^{\mathbb{C}^\times}_{\mathcal{X}_0} . e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)}{(e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)} + \lambda\Big(\frac{(\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0 . e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)}{(e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)} - \log(e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)\Big)$$

"Non-archimedean" μ-entropy of test configuration

Proposition

By the localization on $\mathbb{P}^1$ and the equivariant Grothendieck–Riemann–Roch theorem for $\mathcal{X}_0/\{0\} \to \mathcal{X}/\mathbb{C}$, we get

$$(e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau) = (e^L) - \tau(e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \tau)$$
$$(\mathcal{L}_{\mathbb{C}^\times}|_{\mathcal{X}_0}.e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau) = (L.e^L) - \tau(\bar{\mathcal{L}}_{\mathbb{C}^\times}.e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \tau)$$
$$(\kappa^{\mathbb{C}^\times}_{\mathcal{X}_0}.e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau) = (K_X.e^L) - \tau(K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1}.e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \tau)$$

Replacing $K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1}$ with ${}^bK^{\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} = K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} + [\mathcal{X}_0^{\mathrm{red}}]^{\mathbb{C}^\times} - [\mathcal{X}_0]^{\mathbb{C}^\times}$, we obtain the following by the equivariant Stokes theorem.

Theorem (I. '20, to appear)

For a smooth subgeodesic ray ϕ_t subordinated to a test configuration $(\mathcal{X}, \mathcal{L})$, ${}^b\check{\boldsymbol{\mu}}^\lambda(\mathcal{X}, \mathcal{L}; \tau) = \lim_{t\to\infty} \check{W}^\lambda(\omega_{\tau t}, \dot{\phi}_{\tau t})$.

Toric expression

By the equivariant intersection formula, we obtain

Proposition (I. '20, to appear)

Let (X, L) be a toric variety and $P \subset \mathfrak{t}$ be the associated moment polytope. Take a defining convex piecewise affine function q of the moment polytope $Q = \{(\mu, t) \mid \mu \in P,\ q(\mu) \leq t \leq 0\}$ of a T-equivariant normal test configuration $(\mathcal{X}, \mathcal{L})$. Then

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = -2\pi \frac{\int_{\partial P} e^{\tau q} d\sigma}{\int_P e^{\tau q} d\mu} + \lambda \Big(\frac{\int_P (n + \tau q) e^{\tau q}}{\int_P e^{\tau q} d\mu} - \log \int_P e^{\tau q} d\mu \Big).$$

Question: Find an explicit example of a toric variety (X, L) and a non-product $(\mathcal{X}, \mathcal{L})$ (or filtration) maximizing $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$.

Entropy maximization for tc/filtration

The μ-entropy makes sense also for f.g. filtrations. By differentiating the μ-entropy at ξ to the direction of test configurations, we obtain

Theorem (I. '20, to appear)

A If there is a vector ξ such that

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\xi) \geq \check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau)$$

for every test configuration $(\mathcal{X}, \mathcal{L}; \tau)$, then (X, L) is μ^{λ}_{ξ}K-semistable.

B If for every test configuration $(\mathcal{X}, \mathcal{L}; \tau)$, there is a vector ξ such that

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\xi) \geq \check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau),$$

then (X, L) is μ^{λ}K-semistable.

5. Conjectural picture

Optimal destabilization

Conjecture (μK-stability is entropy maximization)

When $\lambda \leq 0$, the following are equivalent:

- (X, L) is μ_ξ^λK-semistable
- ξ is a maximizer of $\check{\boldsymbol{\mu}}_{\mathrm{NA}}^\lambda$ among all (f.g.) filtrations

Conjecture (Optimal destabilization conjecture for μ-entropy)

1. $$\sup_{\phi \in \mathcal{H}(X^{\mathrm{NA}}, L^{\mathrm{NA}})} \check{\boldsymbol{\mu}}_{\mathrm{NA}}^\lambda(\phi) = \inf_{\omega \in \mathcal{H}(X, L)} \check{\boldsymbol{\mu}}^\lambda(\omega).$$
2. When $\lambda \leq 0$, $\exists!$ a maximizer $\phi \in \mathcal{H}(X^{\mathrm{NA}}, L^{\mathrm{NA}})$ of $\check{\boldsymbol{\mu}}_{\mathrm{NA}}^\lambda$ modulo $\mathrm{Aut}(X, L)$.
3. "The central fibre" $\mathcal{X}_0$ of ϕ is mildly singular and μ_ξ^λK-semistable for the vector ξ generated by ϕ. (The NA Monge–Ampére measure $\mathrm{MA}^{\mathrm{NA}}(\phi)$ is a sum of dirac mass supporting on quasi-monomial valuations.)

Filtration

A (bounded) filtration $\mathcal{F}$ assigns a subspace $\mathcal{F}^\lambda R_m \subset R_m = H^0(X, L^{\otimes m})$ for each $\lambda \in \mathbb{R}$ and $m \geq 0$ which enjoys the following properties.

- $\mathcal{F}^\lambda R_m = \bigcap_{\lambda' < \lambda} \mathcal{F}^{\lambda'} R_m$.
- $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subset \mathcal{F}^{\lambda + \lambda'} R_{m+m'}$.
- $\exists C$ s.t. $\mathcal{F}^\lambda R_m = 0$ for $\lambda \geq Cm$ and $\mathcal{F}^\lambda R_m = R_m$ for $\lambda \leq -Cm$.

A filtration is finitely generated if there exists m such that for every k

$$\mathcal{F}^\lambda R_{mk} = \sum_{\substack{|I|=k \\ \sum_{i \in I} \lambda_i \geq \lambda}} \prod_{i \in I} \mathcal{F}^{\lambda_i} R_m.$$

Filtration and degeneration

- For $(X, L) \curvearrowleft T$ and $\xi \in \mathfrak{t}$, we put

$$\mathcal{F}_\xi^\lambda R_m := \bigoplus_{\langle \mu, \xi \rangle \geq \lambda} H^0(X, L^{\otimes m})_\mu$$

for $m \geq 1$ and $\mathcal{F}_\xi^\lambda R_0 = \mathbb{C}$ iff $\lambda \leq 0$. This is a f.g. filtration.

- Let $B_\sigma := \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap M] \curvearrowleft T$ be the affine toric variety associated to a strictly convex polyhedral cone $\sigma \subset N \otimes \mathbb{R}$ of full dimension. Let $(\mathcal{X}/B_\sigma, \mathcal{L})$ be a T-equivariant family of polarized schemes whose general fibre is isomorphic to (X, L). For $\mu \in M$ and $s \in H^0(X, L^{\otimes m})$, we define $\bar{s}e^{-\mu} \in H^0(X \times T, \mathcal{L}^{\otimes m})$ by $\bar{s}e^{-\mu}(x, t) = (s(x).t)\chi^{-\mu}(t)$. Put

$$\mathcal{F}_{(\mathcal{X}/B_\sigma, \mathcal{L})}^\mu R_m := \{ s \in H^0(X, L^{\otimes m}) \mid \bar{s}e^{-\mu} \text{ extends to a section over } \mathcal{X} \}$$

and for $\xi \in \sigma$,

$$\mathcal{F}_{(\mathcal{X}/B_\sigma, \mathcal{L}; \xi)}^\lambda R_m := \sum_{\langle \mu, \xi \rangle \geq \lambda} \mathcal{F}_{(\mathcal{X}/B_\sigma, \mathcal{L})}^\mu R_m.$$

Then $\mathcal{F}_{(\mathcal{X}/B_\sigma, \mathcal{L}; \xi)}$ is a f.g. filtration.

Filtration and non-archimedean metric

Boucksom–Jonsson, *A non-archimedean approach to K-satbility*

$$\{\text{filtrations}\}/\sim \xrightarrow{\mathrm{FS}} \mathrm{PSH}^{\uparrow}(X^{\mathrm{NA}}, L^{\mathrm{NA}}) \subset \mathrm{PSH}(X^{\mathrm{NA}}, L^{\mathrm{NA}}),$$

$$\{\text{f.g. filtrations}\}/\sim \xrightarrow{\sim} \mathcal{H}(X^{\mathrm{NA}}, L^{\mathrm{NA}}),$$

where

$$\mathcal{H}(X^{\mathrm{NA}}, L^{\mathrm{NA}}) = \Big\{ \frac{1}{m} \max_j (\log |s_j|_0 + \lambda_j) \Bigm| {}^{(s_j) \subset H^0(X, L^{\otimes m}),\ \bigcap_j s_j^{-1}(0) = \emptyset}_{\lambda_j \in \mathbb{R}} \Big\}.$$

Question Can we modify the definition of $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}$ so that it is independent of the choice of the filtration $\mathcal{F}$ with $\mathrm{FS}(\mathcal{F}) = \phi$?

1. Use a unique maximal filtration in the equivalent class of filtrations.
2. Express $\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\phi)$ as an integration on the Berkovich space X^{NA}, similarly as M^{NA}.

Towards non-archimedean formalism: non-archimedean moment map

For a $\mathbb{C}^\times$ action on (X, L) and a $U(1)$-invariant metric $\omega \in L$, we can associate a unique moment map $\mu : X \to \mathbb{R}$ normalized by $[\omega + \mu] = L_{\mathbb{C}^\times}$. The measure $\mathcal{D} = (\mathrm{id}_X \times \mu)_* \omega^n$ on $X \times \mathbb{R}$ determines μ and ω. For $w \in C^\infty(\mathbb{R})$, we have

$$\int_{X \times \mathbb{R}} w(t) \mathcal{D} = \int_X w(\mu) \omega^n.$$

For a test configuration $\phi = (\mathcal{X}, \mathcal{L})$, we consider the following measure $\mathcal{D}_\phi$ on $X^{\mathrm{NA}} \times \mathbb{R}$:

$$\mathcal{D}_\phi = \sum_E \mathrm{ord}_E(\mathcal{X}_0)(E.\mathcal{L}^{\dot{n}}).\delta_{v_E} \otimes \mathrm{DH}_{E, \mathcal{L}|_E}.$$

Then we have

$$\int_{X^{\mathrm{NA}} \times \mathbb{R}} w(-\tau t) \mathcal{D}_\phi = ([\mathcal{X}_0].w(\bar{\mathcal{L}}_{\mathbb{C}^\times}); \tau),$$

$$\int_{X^{\mathrm{NA}} \times \mathbb{R}} A_X(v) e^{-\tau t} \mathcal{D}_\phi = (K^{\log}_{\bar{\mathcal{X}}/\mathbb{P}^1} - K^{\log}_{X_{\mathbb{P}^1}/\mathbb{P}^1}.e^{\bar{\mathcal{L}}_{\mathbb{C}^\times}}; \tau).$$