μ-cscK metrics, μK-stability and a Lagrangian formalism

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1. Heuristic picture: Lempert’s Lagrangian formalism
Lagrangian system in physics

**Lagrangian mechanics**: coordinate free expression of Newtonian mechanics (convenient to deal with *holonomic constraint*)

\[ \mathcal{H} = \mathbb{R}^3 \ni x, \quad T\mathcal{H} \ni (x, v). \]

In the Cartesian coordinate, the kinetic energy is \( T(x, v) = \frac{1}{2} m |v|^2 \).

Consider a potential \( V(x, v) = mg|x| \). The Lagrangian of this system is

\[ \mathcal{L} = T - V : T\mathcal{H} \to \mathbb{R}. \]

The **principle of least action** says that the motion \( \phi_t \in \mathcal{H} \) of a particle from \( x_0 \in \mathcal{H} \) to \( x_1 \in \mathcal{H} \) is characterized as the minimizer of the *action functional*

\[ S(\phi) := \int_0^T \mathcal{L}(\phi_t, \dot{\phi}_t) dt. \]

The **Euler–Lagrange equation** is

\[ \frac{\partial \mathcal{L}}{\partial x}(\phi_t, \dot{\phi}_t) - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial v}(\phi_t, \dot{\phi}_t) = 0. \]
(X, L): a polarized manifold
\( \mathcal{H}(X, L) \): the space of Kähler metrics in \( L \)

\[ T\mathcal{H}(X, L) \cong \mathcal{H}(X, L) \times \mathbb{C}^\infty(X)/\mathbb{R} \]

Consider a functional

\[ \mathcal{L} : T\mathcal{H} \to \mathbb{R} \]

which is invariant under the parallel translation (w.r.t. Mabuchi connection): for any smooth curve \( \omega_t = \omega + \sqrt{-1} \partial \bar{\partial} u_t \), we have

\[ \mathcal{L}(\omega_t, f \circ \varphi_t) = \mathcal{L}(\omega, f) \]

for \( \varphi_t \in \text{Diff}(X) \) generated by the time dep. vector field \((-1/2) \nabla_{\omega_t} \dot{u}_t \).

e.g. \( \mathcal{L}(\omega, f) = \int_X |\hat{f}|^p \omega^n \)
Lempert’s Lagrangian formalism

Assume $\mathcal{L}$ is fibrewise convex.

**Theorem (Lempert ’20, principle of least action)**

A weak geodesic (slightly regular) minimizes the action functional

$$S(\phi) := \int_a^b \mathcal{L}(\omega_t, \dot{\phi}_t)dt$$

among all path $\phi = \{\phi_t\}$ connecting given endpoints $\omega_a$ and $\omega_b$.

**Theorem (Lempert ’20, Hadamard convexity)**

Put

$$\mathcal{L}_T(\omega, \omega') = \inf \{S(\psi) \mid \psi : [0, T] \rightarrow \mathcal{H}(X, L) \text{ connecting } \omega, \omega'\}.$$ 

Then for weak geodesics $\phi, \phi', \mathcal{L}_T(\phi, \phi')$ is convex.
2. Extremal metric and Kähler–Ricci soliton
Extremal metric and Calabi functional

The **Calabi functional** $C : \mathcal{H}(X, L) \to \mathbb{R}$ is given by

$$C(\omega) := \frac{1}{2} \int_X \hat{s}(\omega)^2 \omega^n.$$

The critical points are **extremal metrics**: $\partial^\# s(\omega) = g^{i\overline{j}} s_{\overline{j}}$ is holomorphic.
Extremal metric and Calabi functional

The Calabi functional $C : \mathcal{H}(X, L) \to \mathbb{R}$ is given by

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Donaldson–Futaki invariant:

$$DF(X, L) := (K_{\bar{\mathcal{X}}/\mathbb{P}^1} \cdot \bar{\mathcal{L}}^n) - \frac{(K_X \cdot L^{n-1})}{(n+1)(L^n)} (\bar{\mathcal{L}}^{n+1})$$

Relative Donaldson–Futaki invariant:

$$DF_\xi(X, L) := DF(X, L) + \frac{1}{4\pi} \int_{X_0} \hat{\theta}_\xi \theta_\eta \omega_0^n$$

If $(X, L)$ admits an extremal metric with $\xi = \partial^# s$, then we have $DF_\xi(X, L) \geq 0$ (relatively K-semistable).
Extremal metric and Calabi functional

Donaldson’s lower bound:

\[- \frac{4\pi \text{DF}(\mathcal{X}, \mathcal{L})}{\| (\mathcal{X}, \mathcal{L}) \|} \leq (2C(\omega))^{1/2},\]

where we put

\[\| (\mathcal{X}, \mathcal{L}) \|^2 = \int_{\mathbb{R}} (t - b)^2 \text{DH}(\mathcal{X}, \mathcal{L})\]

with the barycenter \( b := \int_{\mathbb{R}} t \text{DH}(\mathcal{X}, \mathcal{L}) \).

Optimal destabilization conjecture

We have the equality

\[\sup_{(\mathcal{X}, \mathcal{L})} - \frac{4\pi \text{DF}(\mathcal{X}, \mathcal{L})}{\| (\mathcal{X}, \mathcal{L}) \|} = \inf_{\omega \in \mathcal{H}(\mathcal{X}, \mathcal{L})} (2C(\omega))^{1/2}\]

and the supremum on the LHS is achieved by some test configuration.
Lagrangian formalism on Calabi functional

We consider the functional $W_{\text{ext}} : T\mathcal{H}(X, L) \to \mathbb{R}$ given by

$$W_{\text{ext}}(\omega, f) := -\frac{1}{2} \int_X (\hat{s}(\omega) - \hat{f})^2 \omega^n + \frac{1}{2} \int_X \hat{s}(\omega)^2 \omega^n.$$ 

Observations (written in my preprint in preparation)

- (Conservative) The Euler–Lagrange equation

$$\frac{\partial W_{\text{ext}}}{\partial \omega}(\omega_t, \phi_t) - \frac{d}{dt} \frac{\partial W_{\text{ext}}}{\partial f}(\omega_t, \phi_t) = 0$$

for the action functional $S(\phi) := \int_a^b W_{\text{ext}}(\omega_t, \phi_t) dt$ is the geodesic equation.

- $W_{\text{ext}}(\omega_t, \phi_t)$ is monotonically decreasing along weak geodesic.

(Convexity of the Mabuchi functional)
A new proof of Donaldson’s lower bound

We note

\[ C(\omega) = \sup_{f \in C^\infty(X)} W_{\text{ext}}(\omega, f). \]

It is recently shown by C. Li that the slope of the Mabuchi functional along to the geodesic ray associated to a normal test configuration \((X, L)\) is given by \(M_N^A(X, L) = \lim_{d \to \infty} d^{-1} \text{DF}(X_d, L_d)\).

Using this, we obtain

\[ \lim_{t \to \infty} W_{\text{ext}}(\omega_t, \phi_t) = -\frac{1}{2} \left( 4\pi M_N^A(X, L) + \| (X, L) \|^2 \right). \]

We put for \(\tau \geq 0\)

\[ C_{N\text{A}}(X, L; \tau) := -\frac{1}{2} \left( 4\pi \tau M_N^A(X, L) + \tau^2 \| (X, L) \|^2 \right). \]

By the monotonicity, we obtain

\[ C_{N\text{A}}(X, L; \tau) = \lim_{t \to \infty} W_{\text{ext}}(\omega_{\tau t}, \phi_{\tau t}) \leq W_{\text{ext}}(\omega, \phi) \leq C(\omega). \]
A new proof of Donaldson’s lower bound

When $DF(\mathcal{X}, \mathcal{L}) \geq 0$, the maximum of $C_{NA}(\mathcal{X}, \mathcal{L}; \tau)$ is achieved at $\tau = 0$. When $DF(\mathcal{X}, \mathcal{L}) < 0$, we have

$$C(\omega) \geq \sup_{\tau \geq 0} C_{NA}(\mathcal{X}, \mathcal{L}; \tau) = 2 \cdot 4\pi^2 \frac{DF(\mathcal{X}, \mathcal{L})^2}{\| (\mathcal{X}, \mathcal{L}) \|^2},$$

which shows Donaldson’s lower bound on Calabi functional. $\square$
A new proof of Donaldson’s lower bound

When $\text{DF}(\mathcal{X}, \mathcal{L}) \geq 0$, the maximum of $C_{\text{NA}}(\mathcal{X}, \mathcal{L}; \tau)$ is achieved at $\tau = 0$. When $\text{DF}(\mathcal{X}, \mathcal{L}) < 0$, we have

$$C(\omega) \geq \sup_{\tau \geq 0} C_{\text{NA}}(\mathcal{X}, \mathcal{L}; \tau) = 2 \cdot 4\pi^2 \frac{\text{DF}(\mathcal{X}, \mathcal{L})^2}{\| (\mathcal{X}, \mathcal{L}) \|^2},$$

which shows Donaldson’s lower bound on Calabi functional. □

Theorem (Entropy maximization)

- If the supremum of $C_{\text{NA}}$ is achieved by a product configuration, then $(\mathcal{X}, L)$ is relatively $K$-semistable. This is the case when $(\mathcal{X}, L)$ admits an extremal metric.
- If $(\mathcal{X}, L)$ is $K$-semistable, then $C_{\text{NA}}$ is maximized at the trivial configuration.
Kähler–Ricci soliton and modified K-stability

Let $X$ be a Fano manifold, i.e. $-K_X$ is ample. The normalized Kähler–Ricci flow on $X$ is

$$\dot{\omega}_t = \text{Ric}(\omega_t) - 2\pi \omega_t.$$ 

A Kähler–Ricci soliton is a self-similar solution: a pair of a Kähler metric $\omega$ and a holomorphic vector field $\xi$ such that

$$L_\xi \omega = \text{Ric}(\omega) - 2\pi \omega.$$
Kähler–Ricci soliton and modified K-stability

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We can define modified Futaki invariant $\text{Fut}_\xi(X, L)$ and hence modified K-stability of $X$ w.r.t. $\xi$. (Tian–Zhu, Xiong, Berman, Datar–Székelyhidi)

There is a unique vector $\xi$ such that $\text{Fut}_\xi = 0$ for product configurations, but it is HARD to express the vector explicitly! The invariant $\text{Fut}_\bullet$ exists, but it is not explicitly given for $X$.

Nevertheless, we can check the modified K-stability in some case (toric, horospherical), without detecting $\xi$. 
Kähler–Ricci soliton and $H$-functional

Let $X$ be a Fano manifold. The $H$-functional $H : \mathcal{H}(X, -K_X) \to \mathbb{R}$ is given by

$$
\frac{1}{2\pi} H(\omega) := \int_X e^{h_\omega} \omega^n / \int_X e^h \omega^n - \log \int_X e^h \omega^n \frac{n!}{n!}.
$$

The critical points are Kähler–Ricci solitons: $\partial^\# h$ is holomorphic.

Optimal destabilization along Kähler–Ricci flow

- **(Chen–Sun–Wang)** For a Fano manifold $X$, there exists a finitely generated filtration $\mathcal{F}$ of $X$ such that the central fibre $X$ is a modified K-semistable $\mathbb{Q}$-Fano variety, which moreover admits a special degeneration $\mathcal{X}$ to a $\mathbb{Q}$-Fano variety $\mathcal{X}_0$ with Kähler–Ricci soliton.

- **(Han–Li)** The filtration is uniquely characterized (modulo equivalence) as a maximizer of $H$-entropy.

- **(Dervan–Székelyhidi)** $\sup_{\mathcal{F}} H(\mathcal{F}) = \inf_{\omega \in \mathcal{H}(X, L)} H(\omega)$. 
Lagrangian formalism on $H$-functional

Let $X$ be a Fano manifold. We consider the functional $L : T\mathcal{H}(X, -K_X) \to \mathbb{R}$ given by

$$
\frac{1}{2\pi} L(\omega, f) := -\int_X f e^h \omega^n \bigg/ \int_X e^h \omega^n - \log \int_X e^{-f} \frac{\omega^n}{n!}.
$$

Observations

- (Conservative) The Euler–Lagrange equation

$$
\frac{\partial L}{\partial \omega} (\omega_t, \dot{\phi}_t) - \frac{d}{dt} \frac{\partial L}{\partial f} (\omega_t, \dot{\phi}_t) = 0
$$

for the action functional $S(\phi) := \int_a^b L(\omega_t, \dot{\phi}_t) dt$ is the geodesic equation.

- (Dervan–Székelyhidi) $L(\omega_t, \dot{\phi}_t)$ is monotonically decreasing along weak geodesic. (Convexity of the Ding functional) The limit along a geodesic ray gives the $H$-entropy.
3. A non-conservative Lagrangian formalism on $\mu$-cscK metric
-cscK metrics, $\mu$-K-stability and a Lagrangian formalism (Eiji Inoue)

**$\mu$-cscK metric**

For $\lambda \in \mathbb{R}$, we call a Kähler metric $\omega \in L$ a $\mu^\lambda$-cscK metric if

$$
(s(\omega) + \bar{\Box} \theta_\xi) + (\bar{\Box} \theta_\xi - \xi \theta_\xi) - \lambda \theta_\xi = \text{const.}
$$

for some holomorphic vector field $\xi$ with $\exists \theta_\xi \in C^\infty(X)$ s.t.

$$\sqrt{-1} \partial^\# \theta_\xi = \xi.$$

**Theorem (I. ’19 + Lahdili ’20)**

- A Kähler–Ricci soliton $\omega_{KR} \in -K_X$ is a $\mu^{2\pi}$-cscK metric.
- There is a Donaldson–Fujiki type moment map picture for $\mu^\lambda_\xi$-cscK metric.
- For each $(X, L)$, $\mu^\lambda$-cscK metric is unique mod $\text{Aut}$ for $\lambda \ll 0$.
- If there is an extremal metric $\omega_{\text{ext}} \in L$, there is a family $\omega_\lambda$ of $\mu^\lambda$-cscK metrics for $\lambda \ll 0$ such that $\omega_\lambda$ converges to $\omega_{\text{ext}}$. 
We consider the functional $\tilde{W}^\lambda : \mathcal{T}\mathcal{H}(X, L) \to \mathbb{R}$ given by

$$
\tilde{W}^\lambda(\omega, f) := -\int_X \left( s(\omega) + |\partial^\# f|^2 - \lambda(n-f) \right) e^{-f} \omega^n \int_X e^{-f} \omega^n - \lambda \log \int_X e^{-f} \omega^n \frac{n!}{n}.
$$

Recall Perelman’s $W$-functional is

$$
W(g, f; \tau) = \frac{1}{(4\pi \tau)^{n/2}} \int_X \left( \tau(R(g) + |\nabla f|^2) - (n-f) \right) e^{-f} \text{vol}_g
$$

for a Riemannian metric $g$ and $f \in C^\infty(X)$ with $\int_X e^{-f} \text{vol}_g = 1$, usually considered for $\tau \geq 0$.

**Theorem (I. ’20, to appear)**

A state $(\omega, f) \in \mathcal{T}\mathcal{H}$ is a critical point of $\tilde{W}^\lambda$ if and only if $\xi = \partial^\# f$ is holomorphic and $\omega$ is a $\mu^\lambda$-cscK metric w.r.t. $\xi$. 

The $W$-functional as a non-conservative Lagrangian system

- **(Non-conservative)** The Euler–Lagrange equation
  \[ \frac{\partial W^\lambda}{\partial \omega} (\omega_t, \dot{\phi}_t) - \frac{d}{dt} \frac{\partial W^\lambda}{\partial f} (\omega_t, \dot{\phi}_t) = 0 \]
  is NOT equivalent to the geodesic equation.

- The extremal path is geodesic iff $\partial^\sharp_{\omega_t} \dot{\phi}_t$ is holomorphic, which happens only when $\omega_t = \varphi_t^* \omega$ for $\varphi_t \in \text{Aut}$ generated by a holomorphic vector field $\xi = \partial^\sharp f$.

**Theorem (I. '20, to appear)**

- $\tilde{W}^\lambda$ is monotonically decreasing along smooth geodesics.
- For a smooth subgeodesic ray $\phi_t$ subordinated to a test configuration, the limit $\lim_{t \to \infty} \tilde{W}^\lambda(\omega_t, \dot{\phi}_t)$ is given by the non-archimedean $\mu$-entropy.
- $\lim_{\lambda \to \pm \infty} \lambda \left( \tilde{W}^\lambda(\omega, \lambda^{-1} f) - \tilde{W}^\lambda(\omega, 0) \right) = W_{\text{ext}}(\omega, f)$. 
Archimedean $\mu$-entropy

We define $\tilde{\mu}^\lambda : \mathcal{H}(X, L) \to \mathbb{R}$ by

$$\tilde{\mu}^\lambda(\omega) := \sup_{f \in C^\infty(X)} W^\lambda(\omega, f).$$

**Theorem (I. ’20, to appear)**

1. For each $\lambda \leq 0$ and $\omega$, there exists a unique maximizer $f \in C^\infty(X)$ of $W^\lambda(\omega, \cdot)$ modulo constant.
2. In this case, the functional $\tilde{\mu}^\lambda : \mathcal{H}(X, L) \to \mathbb{R}$ is smooth.
3. Its critical points are precisely $\mu^\lambda$-cscK metrics.
4. They are global minimizers of $\tilde{\mu}^\lambda$ among all $T$-invariant Kähler metrics, where $T$ is the center of a maximal compact.

The $\mu$-entropy $\tilde{\mu}^\lambda$ is an analogy of Calabi functional.

**Question** Is there an analogy of Donaldson’s lower bound?
4. Volume minimization and $\mu$K-stability
Entropy maximization for product filtration

Suppose we have a holomorphic Hamiltonian action \((X, L) \subset K\) by a compact Lie group. We define \(\tilde{\mu}_\text{NA}^\lambda : \mathfrak{k} \to \mathbb{R}\) by

\[
\tilde{\mu}_\text{NA}^\lambda(\xi) := \tilde{W}^\lambda(\omega, -\theta_\xi),
\]

using a \(K\)-invariant metric \(\omega \in L\) and \(\theta_\xi \in C^\infty(X)\): \(\sqrt{-1}\bar{\partial}\theta_\xi = i_\xi \omega\).

**Theorem (I. ’19 +α)**

- This is independent of the choice of \(\omega\) and \(\mu\).
- If there exists a \(\mu_\xi^\lambda\)-cscK metric, then \(\xi\) is a critical point of \(\tilde{\mu}_\text{NA}^\lambda\).
  When \(\lambda \leq 0\), \(\xi\) maximizes \(\tilde{\mu}_\text{NA}^\lambda\) (among all vectors).
- There always exist a maximizer of \(\tilde{\mu}_\text{NA}^\lambda\).
- (Phase transition) The value

\[
\lambda_{\text{freeze}} := \sup\{\lambda \in \mathbb{R} \mid \forall \lambda' < \lambda \quad \tilde{\mu}_\text{NA}^{\lambda'} \text{ admits a unique critical point}\}
\]

is not \(±\infty\).
"Non-archimedean" $\mu$-entropy of test configuration

Recall

$$\tilde{\mu}_{\text{NA}}^{\lambda}(-\frac{1}{2}\xi) = \tilde{W}^{\lambda}(\omega, \mu_{\xi}^\omega) = -\frac{\int_X (\text{Ric}(\omega) + \Box\mu_{\xi})e^{\omega+\mu_{\xi}}}{\int_X e^{\omega+\mu_{\xi}}}$$

$$+ \lambda\left(\frac{\int_X (\omega + \mu_{\xi})e^{\omega+\mu_{\xi}}}{\int_X e^{\omega+\mu_{\xi}}} - \log\int_X e^{\omega+\mu_{\xi}}\right)$$

**Definition**

For a test configuration $(\mathcal{X}, \mathcal{L})$ and $\tau \geq 0$, we put

$$\tilde{\mu}_{\text{NA}}^{\lambda}(\mathcal{X}, \mathcal{L}; \tau) := 2\pi \frac{(\kappa_{\mathcal{X}_0}^\mathcal{X} \cdot \mathcal{L}_{\mathcal{C} \times |x_0}, \tau)}{(e^\mathcal{L}_{\mathcal{C} \times |x_0}, \tau)}$$

$$+ \lambda\left(\frac{(\mathcal{L}_{\mathcal{C} \times |x_0} \cdot e^\mathcal{L}_{\mathcal{C} \times |x_0}, \tau)}{(e^\mathcal{L}_{\mathcal{C} \times |x_0}, \tau)} - \log(e^\mathcal{L}_{\mathcal{C} \times |x_0}, \tau)\right)$$
Proposition

By the localization on $\mathbb{P}^1$ and the equivariant Grothendieck–Riemann–Roch theorem for $\mathcal{X}_0/\{0\} \to \mathcal{X}/\mathbb{C}$, we get

$$\left(e^{\mathcal{L}_{\mathbb{C}^\times}|_{\mathcal{X}_0}; \tau}\right) = \left(e^L\right) - \tau\left(e^{\tilde{\mathcal{L}}_{\mathbb{C}^\times}}; \tau\right)$$

$$\left(\mathcal{L}_{\mathbb{C}^\times}|_{\mathcal{X}_0} \cdot e^{\mathcal{L}_{\mathbb{C}^\times}|_{\mathcal{X}_0}; \tau}\right) = \left(L \cdot e^L\right) - \tau\left(\tilde{\mathcal{L}}_{\mathbb{C}^\times} \cdot e^{\tilde{\mathcal{L}}_{\mathbb{C}^\times}}; \tau\right)$$

$$\left(K_{\mathcal{X}_0}^{\mathbb{C}^\times} \cdot e^{\mathcal{L}_{\mathbb{C}^\times}|_{\mathcal{X}_0}; \tau}\right) = \left(K_{\mathcal{X}} \cdot e^L\right) - \tau\left(K_{\mathcal{X}/\mathbb{P}^1}^{\mathbb{C}^\times} \cdot e^{\tilde{\mathcal{L}}_{\mathbb{C}^\times}}; \tau\right)$$

Replacing $K_{\mathcal{X}/\mathbb{P}^1}^{\mathbb{C}^\times}$ with $b K_{\mathcal{X}/\mathbb{P}^1}^{\mathbb{C}^\times} = K_{\mathcal{X}/\mathbb{P}^1}^{\mathbb{C}^\times} + [\mathcal{X}_0^{\text{red}}]^{\mathbb{C}^\times} - [\mathcal{X}_0]^{\mathbb{C}^\times}$, we obtain the following by the equivariant Stokes theorem.

Theorem (I. ’20, to appear)

For a smooth subgeodesic ray $\phi_t$ subordinated to a test configuration $(\mathcal{X}, \mathcal{L})$, $b \tilde{\mu}^\lambda(\mathcal{X}, \mathcal{L}; \tau) = \lim_{t \to \infty} \tilde{\mathcal{W}}^\lambda(\omega_{\tau t}, \phi_{\tau t})$. 
Toric expression

By the equivariant intersection formula, we obtain

**Proposition (I. ’20, to appear)**

Let $(X, L)$ be a toric variety and $P \subset t$ be the associated moment polytope. Take a defining convex piecewise affine function $q$ of the moment polytope $Q = \{(\mu, t) \mid \mu \in P, \ q(\mu) \leq t \leq 0\}$ of a $T$-equivariant normal test configuration $(X, L)$. Then

$$\tilde{\mu}_{NA}^{\lambda}(X, L; \tau) = -2\pi \frac{\int_{\partial P} e^{\tau q} d\sigma}{\int_P e^{\tau q} d\mu} + \lambda \left( \frac{\int_P (n + \tau q) e^{\tau q}}{\int_P e^{\tau q} d\mu} - \log \int_P e^{\tau q} d\mu \right).$$

**Question** Find an explicit example of a toric variety $(X, L)$ and a non-product $(X, L)$ (or filtration) maximizing $\tilde{\mu}_{NA}^{\lambda}$. 
Entropy maximization for tc/filtration

The $\mu$-entropy makes sense also for f.g. filtrations. By differentiating the $\mu$-entropy at $\xi$ to the direction of test configurations, we obtain

**Theorem (I. ’20, to appear)**

**A** If there is a vector $\xi$ such that

$$\tilde{\mu}_{\text{NA}}^\lambda(\xi) \geq \tilde{\mu}_{\text{NA}}^\lambda(\mathcal{X}, \mathcal{L}; \tau)$$

for every test configuration $(\mathcal{X}, \mathcal{L}; \tau)$, then $(X, L)$ is $\mu^\lambda K$-semistable.

**B** If for every test configuration $(\mathcal{X}, \mathcal{L}; \tau)$, there is a vector $\xi$ such that

$$\tilde{\mu}_{\text{NA}}^\lambda(\xi) \geq \tilde{\mu}_{\text{NA}}^\lambda(\mathcal{X}, \mathcal{L}; \tau),$$

then $(X, L)$ is $\mu^\lambda K$-semistable.
5. Conjectural picture
Optimal destabilization

**Conjecture (μK-stability is entropy maximization)**

When $\lambda \leq 0$, the following are equivalent:

- $(X, L)$ is $\mu_\xi^\lambda$K-semistable
- $\xi$ is a maximizer of $\tilde{\mu}_{\text{NA}}^\lambda$ among all (f.g.) filtrations

**Conjecture (Optimal destabilization conjecture for μ-entropy)**

1. 
   \[
   \sup_{\phi \in \mathcal{H}(X^{\text{NA}}, L^{\text{NA}})} \tilde{\mu}_{\text{NA}}^\lambda(\phi) = \inf_{\omega \in \mathcal{H}(X, L)} \tilde{\mu}_{\text{NA}}^\lambda(\omega).
   \]

2. When $\lambda \leq 0$, $\exists!$ a maximizer $\phi \in \mathcal{H}(X^{\text{NA}}, L^{\text{NA}})$ of $\tilde{\mu}_{\text{NA}}^\lambda$ modulo $\text{Aut}(X, L)$.

3. “The central fibre” $\mathcal{X}_0$ of $\phi$ is mildly singular and $\mu_\xi^\lambda$K-semistable for the vector $\xi$ generated by $\phi$. (The NA Monge–Ampère measure $\text{MA}^{\text{NA}}(\phi)$ is a sum of dirac mass supporting on quasi-monomial valuations.)
Filtration

A (bounded) filtration $\mathcal{F}$ assigns a subspace $\mathcal{F}^\lambda R_m \subset R_m = H^0(X, L^{\otimes m})$ for each $\lambda \in \mathbb{R}$ and $m \geq 0$ which enjoys the following properties.

- $\mathcal{F}^\lambda R_m = \bigcap_{\lambda' < \lambda} \mathcal{F}^{\lambda'} R_m$.
- $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subset \mathcal{F}^{\lambda + \lambda'} R_{m+m'}$.
- $\exists C$ s.t. $\mathcal{F}^\lambda R_m = 0$ for $\lambda \geq Cm$ and $\mathcal{F}^\lambda R_m = R_m$ for $\lambda \leq -Cm$.

A filtration is finitely generated if there exists $m$ such that for every $k$

$$\mathcal{F}^\lambda R_{mk} = \sum_{|I|=k} \prod_{i \in I} \mathcal{F}^{\lambda_i} R_m.$$
Filtration and degeneration

- For \((X, L) \curvearrowleft T\) and \(\xi \in t\), we put

\[
\mathcal{F}_\xi^\lambda R_m := \bigoplus_{\langle \mu, \xi \rangle \geq \lambda} H^0(X, L^{\otimes m}_\mu)
\]

for \(m \geq 1\) and \(\mathcal{F}_\xi^\lambda R_0 = \mathbb{C}\) iff \(\lambda \leq 0\). This is a f.g. filtration.

- Let \(B_\sigma := \text{Spec} \mathbb{C}[\sigma^\vee \cap M] \curvearrowleft T\) be the affine toric variety associated to a strictly convex polyhedral cone \(\sigma \subset N \otimes \mathbb{R}\) of full dimension. Let \((\mathcal{X}/B_\sigma, L)\) be a \(T\)-equivariant family of polarized schemes whose general fibre is isomorphic to \((X, L)\). For \(\mu \in M\) and \(s \in H^0(X, L^{\otimes m}_\mu)\), we define \(\bar{s}e^{-\mu} \in H^0(X \times T, L^{\otimes m})\) by \(\bar{s}e^{-\mu}(x, t) = (s(x).t)\chi^{-\mu}(t)\). Put

\[
\mathcal{F}_{(\mathcal{X}/B_\sigma, L)}^{\mu} R_m := \{s \in H^0(X, L^{\otimes m}) \mid \bar{s}e^{-\mu} \text{ extends to a section over } \mathcal{X}\}
\]

and for \(\xi \in \sigma\),

\[
\mathcal{F}_{(\mathcal{X}/B_\sigma, L; \xi)}^{\lambda} R_m := \sum_{\langle \mu, \xi \rangle \geq \lambda} \mathcal{F}_{(\mathcal{X}/B_\sigma, L)}^{\mu} R_m.
\]

Then \(\mathcal{F}_{(\mathcal{X}/B_\sigma, L; \xi)}\) is a f.g. filtration.
Filtration and non-archimedean metric

Boucksom–Jonsson, *A non-archimedean approach to K-stability*

\[ \{\text{filtrations}\} / \sim \overset{\text{FS}}{\longrightarrow} \text{PSH}^\dagger(X^{\text{NA}}, L^{\text{NA}}) \subset \text{PSH}(X^{\text{NA}}, L^{\text{NA}}), \]

\[ \{\text{f.g. filtrations}\} / \sim \overset{\sim}{\longrightarrow} \mathcal{H}(X^{\text{NA}}, L^{\text{NA}}), \]

where

\[ \mathcal{H}(X^{\text{NA}}, L^{\text{NA}}) = \left\{ \frac{1}{m} \max_j (\log |s_j|_0 + \lambda_j) \ \middle| \ (s_j) \subset H^0(X, L^{\otimes m}), \ \bigcap_j s_j^{-1}(0) = \emptyset, \ \lambda_j \in \mathbb{R} \right\}. \]

**Question** Can we modify the definition of \( \tilde{\mu}^\lambda_{\text{NA}} \) so that it is independent of the choice of the filtration \( \mathcal{F} \) with \( \text{FS}(\mathcal{F}) = \phi \)?

1. Use a unique maximal filtration in the equivalent class of filtrations.
2. Express \( \tilde{\mu}^\lambda_{\text{NA}}(\phi) \) as an integration on the Berkovich space \( X^{\text{NA}} \), similarly as \( M^{\text{NA}} \).
Towards non-archimedean formalism: non-archimedean moment map

For a $\mathbb{C}^\times$ action on $(X, L)$ and a $U(1)$-invariant metric $\omega \in L$, we can associate a unique moment map $\mu : X \to \mathbb{R}$ normalized by $[\omega + \mu] = L_{\mathbb{C}^\times}$. The measure $\mathcal{D} = (\text{id}_X \times \mu)_* \omega^n$ on $X \times \mathbb{R}$ determines $\mu$ and $\omega$. For $w \in C^\infty(\mathbb{R})$, we have

\[ \int_{X \times \mathbb{R}} w(t) \mathcal{D} = \int_X w(\mu) \omega^n. \]

For a test configuration $\phi = (\mathcal{X}, \mathcal{L})$, we consider the following measure $\mathcal{D}_\phi$ on $X^{\text{NA}} \times \mathbb{R}$:

\[ \mathcal{D}_\phi = \sum_E \text{ord}_E(\mathcal{X}_0)(E.\mathcal{L}^\nabla).\delta_v \otimes \text{DH}_{E,\mathcal{L}|_E}. \]

Then we have

\[ \int_{X^{\text{NA}} \times \mathbb{R}} \mathcal{A}_X(v)e^{-\tau t} \mathcal{D}_\phi = (K^\log_{\mathcal{X}/\mathbb{P}^1} - K^\log_{\mathbb{P}^1/\mathbb{P}^1}.e_{\mathbb{C}^\times}; \tau). \]