# Theory on Kähler metrics with constant exponentially weighted scalar curvature and exponentially weighted K-stability including Kähler–Ricci solitons

# （ケーラー・リッチ・ソリトンを包括する指数偏スカラー曲率一定のケーラー計量と指数偏K安定性の理論）

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# Preface

This thesis consists of the following two parts:

- In part I, we establish a framework unifying both the frameworks on *cscK metrics & K-stability* and *Kähler–Ricci solitons & modified K-stability*, which we call the framework on *$\mu$-cscK metrics & $\mu$K-stability* of polarized manifolds.

- In part II, we study moduli problems on Fano manifolds admitting Kähler–Ricci solitons.

Part I is a reorganization of the articles [Ino2], [Ino3] and that in part II is a reorganization of the articles [Ino1], [Ino3], while these articles appear in the order of [Ino1], [Ino2], [Ino3].

The main aim of part I is to formulate a proper framework on $\mu$-cscK metrics and $\mu$K-stability. Another aim, which is especially important in the case of Kähler–Ricci solitons, is to refine the definition of modified K-stability related to the existence of Kähler–Ricci solitons. Though modified Futaki invariant is defined only for special degenerations, it is desirable in view of application to moduli problem that such invariants are defined also for general test configurations. The framework on $\mu$-cscK helps to distinguish the role of the polarization $\mathcal{L}$ and the anti-canonical 'polarization' $-K_{\mathcal{X}/\mathbb{C}}$ of test configurations, which are indistinguishable for special degenerations.

In chapter 1 of part I, we introduce the notion of $\mu$-scalar curvature for Kähler metric, motivated by a Donaldson–Fujiki type moment map picture on Kähler–Ricci solitons proved in chapter 3 (cf. [Ino1]). The concept fits into a Lahdili's more general framework on weighted cscK metrics, where we use an exponential weight. We study constraints for the existence of Kähler metrics with constant $\mu$-scalar curvature, which we call *$\mu$-cscK metrics* for short. Some compactness results on a generalization of Tian–Zhu's functional in this $\mu$-cscK setup shows that not only the concept unifies the framework of cscK metrics and Kähler–Ricci solitons, but also it has an intriguing connection with extremal metrics, which cannot expected from moment map picture and is special aspect of $\mu$-cscK metrics among weighted cscK metrics. This chapter corresponds to the article [Ino2].

In chapter 2 of part I, we introduce the notion of $\mu$K-stability of polarized manifolds as a framework unifying both K-stability and modified K-stability. We show that the existence of $\mu$-cscK metrics in the first Chern class of the polarization implies the $\mu$K-semistability of the polarized manifold. We also construct a characteristic class which generalizes Paul–Tian's CM line bundle to the context of $\mu$K-stability. We make use of this product in chapter 4 in part II. This chapter is based on the article [Ino3].

The aim of part II is to unveil a proper formulation of moduli problem on Fano manifolds with Kähler–Ricci solitons and to construct its separated complex analytic/algebraic moduli space.

In chapter 3 of part II, we construct a complex analytic moduli space of Fano manifolds admitting Kähler–Ricci solitons. It is observed that the moduli problem must be formulated as a construction of a complex analytic space enjoying a universal property with respect to an Artin stack of families of adequate Fano manifolds *equivariant with respect to the torus action* generated by a vector field $\xi$ with vanishing modified Futaki invariant $\mathrm{Fut}_\xi \equiv 0$. We directly make use of the moment map picture on Kähler–Ricci solitons to construct local charts on the moduli space and then show the holomorphy of the coordinate changes by establishing a uniqueness result on degenerations of a Fano manifold to Fano manifolds with Kähler–Ricci solitons, using Donaldson–Sun's argument on Gromov–Hausdorff limit of algebraic varieties. This chapter corresponds to the article [Ino1] and is written independently from the results in part I.

In chapter 4 of part II, we show applications of Theorem G in chapter 2 of part I to the algebraic moduli problem. We firstly show that the moduli space constructed in chapter 3 is indeed algebraic as predicted. Theorem G is applied to show Zariski openness of the subset consisting of Fano manifolds which degenerate to some Fano manifolds with Kähler–Ricci solitons. We then propose an approach to the compactification problem of the moduli space. This chapter is based on the article [Ino3].

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# Part I

# Constant $\mu$-scalar curvature Kähler metrics and $\mu$K-stability

# Introduction for Part I

In this part I, we introduce the notions on $\mu$-cscK metrics and $\mu$K-stability of polarized manifolds and establish a foundation on these concepts. The framework unifies the frameworks on cscK metrics and Kähler–Ricci solitons and propose a refinement of modified K-stability related to the existence of Kähler–Ricci solitons.

The $\mu$-scalar curvature is firstly introduced as a family of functions associated to a Kähler metric on a Kähler manifold with an action by a torus $T$ which is parametrized by the parameters $\xi \in \mathfrak{t}$ and $\lambda \in \mathbb{R}$. We call $\mu_\xi^\lambda$-scalar curvature the function corresponding to the parameter $(\xi, \lambda)$. When $\xi = 0$, $\mu_\xi^\lambda$-scalar curvature is nothing but the usual scalar curvature. On the other hand, when $\xi \neq 0$, Kähler–Ricci solitons give typical examples of $\mu$-cscK metrics for $\xi \neq 0$: for Kähler metrics in the cohomology class $c_1(X)$, the $\mu_\xi^{2\pi}$-scalar curvature is constant if and only if the metric is a Kähler–Ricci soliton with respect to the vector field $\xi$: $\mathrm{Ric}(\omega) - L_{J\xi}\omega = \omega$.

Our starting point on $\mu$-scalar curvature is a Donaldson–Fujiki type moment map picture. For fixed $\xi \in \mathfrak{t}$, the $\mu_\xi^\lambda$-scalar curvature has an interpretation as a moment map on the space $\mathscr{J}_T(M, \omega)$ of $T$-invariant almost complex structures on a symplectic manifold $(M, \omega)$ with respect to a symplectic structure $\Omega_\xi$ on $\mathscr{J}_T(M, \omega)$ associated to the measure $e^{-2\mu_\xi}\omega^n$ for a moment map $\mu : M \to \mathfrak{t}^\vee$ of $\omega$. This moment map picture formally predicts that the existence of $\mu_\xi^\lambda$-cscK metrics must be characterized by the positivity of some numerical invariant associated to degenerations of the given polarized manifold. The invariant will be introduced and studied in chapter 2.

In chapter 1, we study $\mu$-scalar curvature not only from this perspective, which is well-studied aspect in the case of cscK metrics, but also from a different viewpoint based on Tian–Zhu's volume minimization argument on Kähler–Ricci solitons. In the latter viewpoint, we fix our parameter $\lambda \in \mathbb{R}$ and study constraints on $\xi$ for the existence of $\mu_\xi^\lambda$-cscK metrics. It turns out that for each $\lambda \leq 0$, there are only finitely many $\xi$ which has a chance to admit $\mu_\xi^\lambda$-cscK metrics and moreover such $\xi$ is unique when $\lambda \ll 0$, regardless of the actual existence of $\mu_\xi^\lambda$-cscK metrics. We also observe an intriguing new phenomenon as $\lambda$ tends to $-\infty$: the rescaled vectors $\lambda\xi_\lambda$ converge to extremal vector field as $\lambda \to -\infty$. This implies in chapter 2 that the existence of $\mu^\lambda$-cscK metrics for every $\lambda \ll 0$ implies the relative K-semistability of the polarized manifolds, which is related to the existence of extremal metrics. Conversely, we can also show the existence of extremal metrics implies the existence of $\mu^\lambda$-cscK metrics for every $\lambda \ll 0$. In this way, we conclude that $\mu^\lambda$-cscK metrics can be considered as a continuity path connecting to extremal metric.

In chapter 2, we formulate the $\mu$K-stability of polarized manifolds. A new parameter $\xi \in \mathfrak{t}$, which does not appear in the usual K-stability, prevents us to express '$\mu$-Futaki invariant' by an intersection formula similar to the usual Donaldson–Futaki invariant. Instead, we express it by an equivariant intersection formula. The formula enables us to observe the behaviors of $\mu$-Futaki invariants along the normalization and resolutions of test configurations. Using Lahdili's result on weighted K-semistability with respect to smooth test configurations, we conclude the $\mu$K-semistability of polarized manifolds with $\mu$-cscK metrics. On the other hand, we also construct an equivariant characteristic class $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda \in H^2_G(B, \mathbb{R})$ for equivariant families $(\mathcal{X}, \mathcal{L}) \to B$ of polarized schemes which generalizes Paul–Tian's CM line bundle. We develop basics on relative equivariant intersection on schemes to construct such characteristic class. This product will be used in section 4 of chapter II.

# Chapter 1

# Constant $\mu$-scalar curvature Kähler metric

In this chapter, we propose a new variant of scalar curvature of Kähler metric with a moment map, which we call $\mu$-scalar curvature, motivated by a version of Donaldson-Fujiki moment map picture on a weighted measure $e^{\theta_\xi}\omega^n$ associated to a holomorphic vector field $\xi^J$. We design our framework on constant $\mu$-scalar curvature Kähler metrics ($\mu$-cscK metrics for short) so that it fits into both of the frameworks on cscK metrics and Kähler-Ricci solitons. From the moment map picture, we are naturally motivated to study a family of $\mu$-cscK metrics parametrized by $\lambda \in \mathbb{R}$. It turns out that $\mu$-cscK metrics can also be regarded as a continuity path to/from extremal metrics.

We exhibit some fundamental constraints to the existence of $\mu$-cscK metrics by investigating a variant of Tian–Zhu's volume functional, which is closely related to Perelman's W-functional. A new K-energy is studied as an approach to the uniqueness problem of $\mu$-cscK metrics and as a prelude to new K-stability concept.

The content corresponds to the article [Ino2].

## 1.1 Main results

We simply begin with the definition of $\mu$-scalar curvature and the main results of this chapter.

### Setup

Let $X$ be a Kähler manifold and $\omega$ be a Kähler form on $X$. We call a smooth real vector field $\xi$ on $X$ *$\bar{\partial}$-Hamiltonian* with respect to $\omega$ if the complexified vector field $\xi^J := J\xi + \sqrt{-1}\xi$ is holomorphic ($\Leftrightarrow L_\xi J = 0$) and $i_{\xi^J}\omega$ is $\bar{\partial}$-exact. Note that $i_{\xi^J}\omega$ is $\bar{\partial}$-closed for any holomorphic $\xi^J$. As $\xi^J$ is holomorphic, we have $i_{\xi^J}(\omega + \sqrt{-1}\partial\bar{\partial}\phi) = i_{\xi^J}\omega + \sqrt{-1}\bar{\partial}\xi^J\phi$, so that the $\bar{\partial}$-Hamiltonian property does not depend on the choice of the Kähler form $\omega$ in the fixed Kähler class $[\omega]$. Moreover, it is known by [LS] that a vector field $\xi$ preserving $J$ on a compact Kähler manifold is $\bar{\partial}$-Hamiltonian with respect to $[\omega]$ if and only if it has a fixed point, thus in particular the $\bar{\partial}$-Hamiltonian property is even independent of the Kähler class $[\omega]$. We call a function $\theta$ satisfying $\sqrt{-1}\bar{\partial}\theta = i_{\xi^J}\omega$ a *$\bar{\partial}$-Hamiltonian potential* with respect to $\omega$, which is complex-valued in general. We call a $\bar{\partial}$-Hamiltonian vector field $\xi$ *properly $\bar{\partial}$-Hamiltonian* if $\xi$ generates a closed torus, i.e., the closure $\overline{\exp \mathbb{R}\xi} \subset \mathrm{Aut}(X)$ is compact.

We define the *$\mu_\xi$-scalar curvature* $s_\xi(\omega)$ of a Kähler metric $\omega$ and a $\bar{\partial}$-Hamiltonian vector $\xi$ by

$$s_\xi(\omega) = (s(\omega) + \bar{\Box}\theta) + (\bar{\Box}\theta - \xi^J\theta), \tag{1.1}$$

where $\theta$ is a $\bar{\partial}$-Hamiltonian potential of $\xi$ with respect to $\omega$ and $s(\omega)$ denotes the Kählerian scalar curvature: $s(\omega) := -g^{k\bar{l}}\partial_k\bar{\partial}_l \log\det g$. We can take a real-valued $\theta$ iff $\omega$ is $\xi$-invariant since we have $\sqrt{-1}(d\mathrm{Re}\theta - Jd\mathrm{Im}\theta) = \sqrt{-1}\bar{\partial}\theta - \overline{\sqrt{-1}\bar{\partial}\theta} = i_{\xi^J - \bar{\xi}^J}\omega = 2\sqrt{-1}i_\xi\omega$. In this case, $\theta$ is $\xi$-invariant, so $\xi^J\theta$ is also real valued.

A version of Donaldson-Fujiki moment map picture characterizes $\mu$-scalar curvature. As we will see this motivative interpretation in section 1.2.1, here we instead simply observe how the individual terms of the $\mu$-scalar curvature arise. The first term $s(\omega) + \bar{\Box}\theta$ is just the trace of the Bakry–Emery Ricci curvature $\mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}\theta$, which is well-studied in Riemannian and metric measure geometry. The second term $\bar{\Box}\theta - \xi^J\theta$ often arises as the Lie derivative of the weighted measure:

$$L_{\xi^J}(e^\theta\omega^n) = -(\bar{\Box}\theta - \xi^J\theta)e^\theta\omega^n.$$

Another important aspect is that this second term is a $\bar{\partial}$-Hamiltonian potential of the Bakry–Emery Ricci curvature $\mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}\theta$, i.e. $\sqrt{-1}\bar{\partial}(\bar{\Box}\theta - \xi^J\theta) = i_{\xi^J}(\mathrm{Ric}(\omega) - \sqrt{-1}\partial\bar{\partial}\theta)$.

We may also regard $s_\xi(\omega)$ as the trace of the following 'complex analogy of ($m$=1)-Bakry–Emery curvature' for an integrable complex structure $J$:

$$\mathrm{Ric}(\omega) + 2\sqrt{-1}\partial\bar{\partial}\theta - \sqrt{-1}\partial\theta \wedge \bar{\partial}\theta.$$

We can easily see that this $(1,1)$-form does not change by simultaneously replacing the equivariant form $\omega + \theta$ with $\tilde{\omega} + \tilde{\theta} = c(\omega + \theta)$ and $\xi$ with $\tilde{\xi} = c^{-1}\xi$ for a positive constant $c > 0$, so that we have

$$s_{c^{-1}\xi}(c\omega) = c^{-1} s_\xi(\omega) \tag{1.2}$$

for every positive constant $c > 0$. (Note $s_\xi(c\omega) \neq c^{-1} s_\xi(\omega)$ when $\xi \neq 0$. )

Next, introducing a parameter $\lambda \in \mathbb{R}$, we define the *$\mu_\xi^\lambda$-scalar curvature* of a Kähler metric $\omega$ by

$$s_\xi^\lambda(\omega) = (s(\omega) + \bar{\square}\theta) + (\bar{\square}\theta - \xi^J\theta) - \lambda\theta. \tag{1.3}$$

We call a Kähler metric $\omega$ a *constant $\mu_\xi^\lambda$-scalar curvature Kähler metric* (*$\mu_\xi^\lambda$-cscK metric* for short) if $s_\xi^\lambda(\omega)$ is constant. We may also use variant terminologies such as *$\mu^\lambda$-cscK metric* or *$\mu$-cscK metric* when the abbreviated parameters are determined/unimportant in the context. Since we have $\mathrm{Im}(s_\xi^\lambda(\omega)) = \Delta\mathrm{Im}\theta - 2J\xi(\mathrm{Im}\theta) - \lambda\mathrm{Im}\theta$, we get

$$\begin{aligned}\int_X \mathrm{Im}(s_\xi^\lambda(\omega))\mathrm{Im}\theta\, e^{\mathrm{Re}\theta}\omega^n &= \int_X (\Delta\mathrm{Im}\theta - 2J\xi(\mathrm{Im}\theta) - \lambda\mathrm{Im}\theta)\mathrm{Im}\theta\, e^{\mathrm{Re}\theta}\omega^n \\ &= \int_X |d\mathrm{Im}\theta|^2 e^{\mathrm{Re}\theta}\omega^n - \lambda\int_X (\mathrm{Im}\theta)^2 e^{\mathrm{Re}\theta}\omega^n\end{aligned}$$

by $(d\mathrm{Im}\theta, \mathrm{Re}\theta) = 2\xi(\mathrm{Re}\theta) = 2J\xi(\mathrm{Im}\theta)$. Thus we automatically obtain $L_\xi\omega = 0$ for any $\mu_\xi^\lambda$-cscK metric $\omega$ with $\lambda \leq 2\lambda_1$ for the positive first eigenvalue $\lambda_1$ of $\frac{1}{2}(\Delta - \nabla\mathrm{Re}\theta)$. In this thesis, we are mainly interested in this case, especially the case $\lambda \leq 0$. So from now on we always assume the $\xi$-invariance of the Kähler metric $\omega$. Hence $s_\xi^\lambda(\omega)$ is real-valued.

A cscK metric obviously gives an example of $\mu$-cscK metric for $\xi = 0$ and every $\lambda$. We will see in section 1.2.1 that a Kähler-Ricci soliton $\mathrm{Ric}(\omega) - L_{\xi^J}\omega = \lambda\omega$ gives an example of $\mu_\xi^\lambda$-cscK metric for $\lambda > 0$, which satisfies $\lambda \leq 2\lambda_1$. If $\omega$ is a $\mu_\xi^\lambda$-cscK metric, then for any positive constant $c > 0$, $\tilde{\omega} = c\omega$ is a $\mu_{\tilde{\xi}}^{\tilde{\lambda}}$-cscK metric for $\tilde{\xi} := c^{-1}\xi$ and $\tilde{\lambda} := c^{-1}\lambda$. The product $(X \times Y, \omega_X + \omega_Y)$ of a $\mu_{\xi_X}^\lambda$-cscK metric $\omega_X$ on $X$ and a $\mu_{\xi_Y}^\lambda$-cscK metric $\omega_Y$ on $Y$ gives a $\mu_{\xi_X+\xi_Y}^\lambda$-cscK metric.

## Main results

Now we collect the main results in this chapter. The first three results, except for Theorem B (3), are analogous to well-known foundational results on cscK metric and Kähler-Ricci soliton (cf. [Sze-book], [TZ2]. [AS] is a good survey. ).

In the following, $X$ denotes a compact Kähler manifold. Let us firstly recall the reduced automorphism group (cf. [Gau]). Put

$$\mathfrak{h}_0(X) := \{\xi \in \mathcal{X}_{\mathbb{R}}(X) \mid \xi \text{ is } \bar{\partial}\text{-Hamiltonian. }\}, \tag{1.4}$$

which is naturally a complex vector space by putting $\sqrt{-1}\xi := J\xi$. Denote by $\mathrm{Aut}^0(X/\mathrm{Alb})$ the connected subgroup of the group $\mathrm{Aut}(X)$ of biholomorphisms associated to $\mathfrak{h}_0(X)$. It is known by [LS] that $\mathfrak{h}_0(X)$ is the space of vector fields tangent to the Jacobi map $A_x : X \to \mathrm{Alb}(X)$, or equivalently, the space of vector fields with non-empty zero set. This group $\mathrm{Aut}^0(X/\mathrm{Alb})$ is called the *reduced automorphism group* of $X$. Similarly, we put $\mathfrak{h}_{0,\xi}(X) := \{\zeta \in \mathfrak{h}_0(X) \mid [\xi, \zeta] = 0\}$ for a vector $\xi \in \mathfrak{h}_0(X)$ and denote by $\mathrm{Aut}^0_\xi(X/\mathrm{Alb})$ the connected subgroup of $\mathrm{Aut}(X)$ associated to $\mathfrak{h}_{0,\xi}(X)$.

Note that for a line bundle $L$ on $X$, the identity component $\mathrm{Aut}^0(X, L)$ of the group of biholomorphisms lifting to $L$ is contained in the reduced automorphism group $\mathrm{Aut}^0(X/\mathrm{Alb})$. Moreover, it is known that $\mathrm{Aut}^0(X, L^{\otimes n})$ coincides with $\mathrm{Aut}^0(X/\mathrm{Alb})$ for some positive integer $n$ (because $\mathrm{Aut}^0(X/\mathrm{Alb})$ is linear algebraic). If $X$ has no holomorphic 1-form, or equivalently $b^1(X) = 0$, then the reduced automorphism group $\mathrm{Aut}^0(X/\mathrm{Alb})$ coincides with the identity component $\mathrm{Aut}^0(X)$ of the group of biholomorphisms of $X$.

**Theorem A** (Reductiveness)**.** Let $\omega$ be a constant $\mu^\lambda_\xi$-scalar curvature Kähler metric on a compact Kähler manifold $X$. Then

1. the group $\mathrm{Aut}^0_\xi(X/\mathrm{Alb})$ is the complexification of the compact connected subgroup ${}_{\mathrm{H}}\mathrm{Isom}^0_\xi(X, \omega)$ associated to the Lie algebra of Hamiltonian Killing vector fields with respect to the $\mu^\lambda_\xi$-cscK metric $\omega$ compatible with $\xi$. Especially, it is reductive. (Corollary 1.3.5)

2. When $\lambda \le 0$, $\mathrm{Aut}^0_\xi(X/\mathrm{Alb})$ is maximal among reductive subgroups of $\mathrm{Aut}^0(X/\mathrm{Alb})$. This fails in general when $\lambda \gg 0$. (Corollary 1.3.19)

When $b^1(X) = 0$, we can replace $\mathrm{Aut}^0_\xi(X/\mathrm{Alb})$ by the identity component $\mathrm{Aut}^0_\xi(X)$ of the group of biholomorphisms preserving $\xi$ and ${}_H\mathrm{Isom}^0_\xi(X, \omega)$ by the identity component $\mathrm{Isom}^0_\xi(X,\omega)$ of the group $\mathrm{Isom}_\xi(X,\omega)$ of isometries preserving $\xi$.

**Theorem B** ($\mu$-Futaki invariant and $\mu$-volume functional)**.** Let $X$ be a compact Kähler manifold, $[\omega]$ be a Kähler class and $\xi$ be a properly $\bar{\partial}$-Hamiltonian vector field on $X$.

1. There is a $\mathbb{C}$-linear functional $\mathrm{Fut}^\lambda_\xi : \mathfrak{h}_0(X) \to \mathbb{C}$ depending only on the quadruple $(X,[\omega],\xi,\lambda)$ such that $\mathrm{Fut}^\lambda_\xi$ vanishes if the Kähler class $[\omega]$ admits a $\mu^\lambda_\xi$-cscK metric. (Proposition 1.3.6)

2. For any compact Lie subgroup $K \subset \mathrm{Aut}^0(X/\mathrm{Alb})$ and $\lambda \in \mathbb{R}$, there always exists a vector $\xi \in \mathfrak{k}$ such that $\mathrm{Fut}^\lambda_\xi|_{\mathfrak{k}^c}$ vanishes, regardless of the existence of $\mu^\lambda$-cscK metrics. (Corollary 1.3.15)

3. We have the uniqueness of such $\xi$ for $\lambda \ll 0$. Moreover, the value

$$\lambda_{\mathrm{freeze}} := \sup\{\lambda \in \mathbb{R} \mid \forall \lambda' < \lambda \quad \mathrm{Fut}^\lambda_{\xi_{\lambda'}}|_{\mathfrak{k}^c} = 0 \text{ for a unique } \xi_{\lambda'}\}$$

   is never $\pm\infty$. (Proposition 1.5.3 and its remark)

The second claim in the above is a partial generalization of a volume minimization result in [TZ2], except for the uniqueness. Indeed, we will see in section 5.2 that vectors $\xi$ with $\mathrm{Fut}^\lambda_\xi \equiv 0$ are not unique for $\lambda \gg 0$ as claimed in the above (3). As for general theory, we are mainly interested in the case $\lambda \leq 0$ since in this case we have a nice compactness/finiteness results as in Corollary 1.3.18 and Corollary 1.3.19. The author suspects the above value $\lambda_{\mathrm{freeze}}$ is always slightly positive.

In section 4, we prove the following extension result, which generalizes the result of [Chen2]. This is the first step for studying the uniqueness of $\mu^\lambda_\xi$-cscK metrics and '$\mu$K-stability'.

**Theorem C** ($\mu$K-energy and geodesic)**.** Let $(X,[\omega])$ be a compact Kähler manifold with $T$-action. Fix a vector $\xi \in \mathfrak{t}$ and $\lambda \in \mathbb{R}$.

1. There is a functional $\mathcal{M}^\lambda_\xi$ on the space of $\xi$-invariant smooth Kähler metrics in the Kähler class $[\omega]$ such that the critical points of $\mathcal{M}^\lambda_\xi$ are precisely $\mu^\lambda_\xi$-cscK metrics and that $\mathcal{M}^\lambda_\xi$ is convex along smooth geodesics.

2. There is a canonical extension of this functional $\mathcal{M}_\xi^\lambda$ to the space $\overline{\mathcal{H}_{\omega,\xi}^{1,1}}$ of $\xi$-invariant *sub-Kähler metrics with $C^{1,1}$-potentials.* Here a sub-Kähler metric with $C^{1,1}$-potentials means a $(1,1)$-form $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ with $L^\infty$-coefficients given by a smooth Kähler metric $\omega$ and a $C^{1,1}$-smooth $\omega$-psh function $\phi$, i.e., a $C^{1,1}$-smooth function satisfying $\omega_\phi \geq 0$ as a current.

The following result illustrates an intriguing special aspect of $\mu$-scalar curvature. From this result, our parameter $\lambda$ can be thought as a continuity path connecting $\mu^0$-cscK metric/Kähler-Ricci soliton and extremal metric.

**Theorem D** (Behavior of K-optimal vectors)**.** Fix a compact subgroup $K \subset \mathrm{Aut}^0(X/\mathrm{Alb})$.

1. Let $\{(\xi_i, \lambda_i) \in \mathfrak{k} \times \mathbb{R}\}_{i \in \mathbb{N}}$ be a sequence satisfying $\mathrm{Fut}_{\xi_i}^{\lambda_i}|_{\mathfrak{k}^c} \equiv 0$ and $\lambda_i \to -\infty$. (Note such a sequence always exists by Theorem B. ) Then the rescaled sequence $\lambda_i \xi_i \in \mathfrak{k}$ converges to the extremal vector $\xi_{\mathrm{ext}}$ which is uniquely characterized by the property

$$\check{\mathrm{Fut}}_{\xi_{\mathrm{ext}}}^0(\zeta) := \int_X \Big( (s(\omega) - \bar{s}) - (\theta_{\xi_{\mathrm{ext}}} - \underline{\theta}_{\mathrm{ext}}) \Big) \theta_\zeta \omega^n = 0$$

for every $\zeta \in \mathfrak{k}$, where we put $\bar{s} = \int_X s(\omega)\omega^n / \int_X \omega^n$ and $\underline{\theta}_{\mathrm{ext}} := \int_X \theta_{\xi_{\mathrm{ext}}} \omega^n / \int_X \omega^n$. (Section 2.2 and Corollary 1.3.18)

2. If there are $\mu_{\xi_i}^{\lambda_i}$-cscK metrics $\omega_i$ with a uniform $C^{3,\alpha}$-bound of the Kähler potentials $\phi_i$ of $\omega_i = \omega + \sqrt{-1}\partial\bar{\partial}\phi_i$ and a uniform lower bound $C\omega \leq \omega_i$, then $\omega_i$ subconverges to an extremal metric $\omega_{\mathrm{ext}}$ on $X$. (Section 2.2)

3. Conversely, if there is an extremal metric on $\omega_{\mathrm{ext}}$ in a Kähler class $[\omega]$, then there are constants $\lambda_-$ and $\lambda_+$ such that there is a family of $\mu$-cscK metric $\{\omega_\lambda\}_{\lambda \in (-\infty, \lambda_-) \cup (\lambda_+, \infty)}$ where for each $\lambda \in (-\infty, \lambda_-) \cup (\lambda_+, \infty)$ the metric $\omega_\lambda$ is a $\mu_{\xi_\lambda}^\lambda$-cscK for some vector field $\xi_\lambda$ in the Kähler class $[\omega]$ such that $\omega_\lambda$ converges to $\omega_{\mathrm{ext}}$ smoothly as $\lambda \to \pm\infty$. (Theorem 1.5.5)

By Theorem B (3), the vector field $\xi_\lambda$ in the above Theorem D (3) is unique for each $\lambda \ll 0$, while we may have other solutions for $\lambda \gg 0$.

In section 5.1, we prove the following result analogous to one of the main results in [LS] on extremal metric.

**Theorem E** (Perturbation of Kähler class)**.** Let $\omega$ be a $\mu_\xi^\lambda$-cscK metric on a compact Kähler manifold $X$. Suppose we have $\lambda < 2\lambda_1$ for the first eigenvalue $\lambda_1 > 0$ of the operator $\bar{\Box} - J\xi = \frac{1}{2}(\Delta - \nabla \mathrm{Re}\theta_\xi)$ restricted to the space $C_\xi^\infty(X, \mathbb{R})$ of $\xi$-invariant real-valued functions. Then there exists a neighbourhood $U$ of $[\omega]$ in the Kähler cone and a positive constant $\epsilon > 0$ such that for every Kähler class $[\tilde{\omega}] \in U$ and $\tilde{\lambda} \in (\lambda - \epsilon, \lambda + \epsilon)$, there exists a vector $\tilde{\xi}$ and a constant $\mu_{\tilde{\xi}}^{\tilde{\lambda}}$-scalar curvature Kähler metric $\tilde{\omega}_{\tilde{\lambda}}$ in the Kähler class $[\tilde{\omega}]$.

It follows that if a Kähler class $[\omega]$ admits a cscK metric, then a small perturbation of $[\omega]$ admits both extremal metric and $\mu^\lambda$-cscK metrics for $\lambda < 2\lambda_1$. For example, the Kähler class $c_1(X)$ of $\mathbb{C}P^2$ blown up at three points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$ admits cscK metric, but a small perturbation of $c_1(X)$ does not admit cscK metrics (cf. [LS, Example 3.2]). This example shows that there exists a non-trivial path of $\mu^\lambda$-cscK metrics connecting extremal metric and $\mu^0$-cscK metric.

In section 6.2, we give more explicit examples of $\mu$-cscK metrics on ruled surfaces, using Calabi ansatz method. It turns out that there is a Kähler class that does not admit extremal metrics, but do admit $\mu^0$-cscK metrics. In particular, we observe that there exists a path of $\mu^\lambda$-cscK metrics connecting Kähler–Ricci soliton and extremal metric on $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in the Kähler class $c_1(X)$.

### Relation with Lahdili's work

Just after uploading the first version of [Ino2] on arXiv, the author was informed that the $\mu$-scalar curvature is a special part of weighted scalar curvature introduced in [Lah]. As there are some overlaps on the results, especially on $\mu$K-energy, we collect them here. Proposition 1.4.2 in this thesis should correspond to Theorem 5 in [Lah] and its corollary is mentioned in Remark 4. Proposition 1.4.1 is also covered in the proof of Proposition 1 in his paper. Proposition 1.3.6 corresponds to Proposition 2 in [Lah], but the statement is slightly different. Computing with $\mu$-Lichnerowicz operator, we can extend the domain of $\mathrm{Fut}_\xi^\lambda$ to $\mathfrak{h}_0(X)$ from the centralizer $\mathfrak{h}_{0,\xi}(X)$ of $\xi$, where the latter case is considered in [Lah]. While it is natural to consider only a torus equivariant test configurations to formulate weighted K-stability (or $\mu$K-stability), our slight deviation to non-equivariant direction $\mathfrak{h}_{0,\xi}(X)^\perp \subset \mathfrak{h}_0(X)$ enables us to conclude the maximality of $\mathrm{Aut}_\xi^0(X/\mathrm{Alb}) \subset$

$\mathrm{Aut}^0(X/\mathrm{Alb})$ among reductive subgroups in $\mathrm{Aut}^0(X/\mathrm{Alb})$ for $X$ admitting a $\mu^\lambda_\xi$-cscK metric for some $\lambda \leq 0$ (see Corollary 1.3.19). This is indeed not the case when $\lambda \gg 0$ as we see in the example of $\mathbb{C}P^1$. Lahdili also considers a weighted Futaki invariant for smooth test configurations, which should have an advantage towards an algebraic formulation of $\mu$K-stability (or weighted K-stability) for general test configurations (cf. section 1.4.2). The materials in the section 1.2.2, 1.3.3, 1.5.1 and 1.6 have different original flavors from these overlaps.

**Organization**

In section 1.2.1, we explain a motivative interpretation of $\mu$-scalar curvature as a moment map. We observe in section 1.2.2 how $\mu$-scalar curvature is related to extremal metric, assuming some results in section 1.3.3. We prove Theorem A in section 1.3.1 and check Theorem B (1) in section 1.3.2 using a formula obtained in section 1.3.1. Theorem B (2) is verified in section 1.3.3. We also prove Theorem D in this section, combining with the observation in section 1.2.2. Theorem C is demonstrated in section 1.4.1. We also present some naive stability notion which should fit into our '$\mu$'-framework. We note that Lahdili introduced weighted Futaki invariant for smooth test configurations in [Lah] in his weighted framework. This will be refined in the future study [Ino2] for general test configurations in our $\mu$-framework. In section 1.5, we prove Theorem E, Theorem B (3) and Theorem D (3). In section 1.6, we firstly observe there are non-trivial $\mu^\lambda$-cscK metrics on $\mathbb{C}P^1$ for $\lambda \gg 0$, and then we construct explicit examples of $\mu$-cscK metrics on ruled surfaces, using Calabi ansatz method.

## 1.2 Motivative observation on $\mu$-cscK metrics

### 1.2.1 $\mu$-scalar curvature and Donaldson-Fujiki picture

In this section, we fix a symplectic structure $\omega$ on a smooth manifold $M$ and a smooth vector field $\xi$ preserving $\omega$ and vary complex structures $J$ and the parameter $\lambda \in \mathbb{R}$.

In this setup, we can also consider all variation of Kähler metrics on a fixed complex manifold $X = (M, J)$ and in a fixed Kähler class as follows. Let $\omega'$ be another Kähler metric on $X$ in the Kähler class $[\omega]$. As $t\omega' + (1-t)\omega$ is nondegenerate for all $t \in [0,1]$, we can apply Moser's theorem to obtain a diffeomorphism $\phi$ of $M$ so that $\phi^*\omega' = \omega$. We obtain another $\omega$-compatible complex structure $\phi^* J$ on $M$. Conversely, if we have an $\omega$-compatible complex structure $J'$ which is biholomorphic to $J$ via some diffeomorphism $\phi$ of $M$, i.e. $J' = \phi^* J$, then we obtain another Kähler form $\omega' := (\phi^{-1})^*\omega$ on $X = (M, J)$. This gives a natural identification of the space of Kähler metrics in a fixed Kähler class $[\omega]$ on a complex manifold $X = (M, J)$ with the quotient space

$$\{J' \in \mathcal{J}(M, \omega) \mid (M, J') \text{ is biholomorphic to } (M, J)\}/\mathrm{Symp}(M, \omega).$$

The leaf $\{J' \in \mathcal{J}(M, \omega) \mid (M, J') \text{ is biholomorphic to } (M, J)\}$ can be psychologically regarded as 'the orbit of $J$ by the complexified action of $\mathcal{J}(M, \omega) \curvearrowleft \mathrm{Symp}(M, \omega)$', which enables us to interpret YTD-type conjecture as an infinite dimensional analogy of the finite dimensional Kempf-Ness theorem. (cf. [Don1], [Sze-book]. )

### Moment map

Let $(M, \omega)$ be a closed $C^\infty$-symplectic manifold. A smooth vector field $\xi$ on $M$ preserves $\omega$ if and only if $i_\xi \omega$ is closed, as $L_\xi \omega = d i_\xi \omega$. Put $\mathfrak{ham}(M, \omega)$ as

$$\mathfrak{ham}(M, \omega) := \{\xi \in \mathfrak{X}(M) \mid i_\xi \omega \text{ is exact } \}, \tag{1.5}$$

which is a Lie subalgebra of the Lie algebra $\mathfrak{X}(M)$ of smooth vector fields.

A smooth right action $M \curvearrowleft T$ by a Lie group $T$ is called *Hamiltonian* if the linearization $\mathfrak{t} \to \mathfrak{X}(M)$ factors through $\mathfrak{ham}(M, \omega)$. In this case, we have an equivariant smooth map $\mu : M \to \mathfrak{t}^*$ satisfying $-d\mu_\xi = i_\xi \omega$ called a *moment map*, where we consider the coadjoint action on $\mathfrak{t}^*$ and $\mu_\xi$ is a real valued function on $M$ defined by $\mu_\xi(x) := \langle \mu(x), \xi \rangle$. For two moment maps $\mu, \mu'$ with respect to the same action and the same symplectic form $\omega$, we know that $\mu - \mu'$ is constant as $d(\mu - \mu') = 0$ and moreover $\mu - \mu' \in (\mathfrak{t}^*)^T = \{\nu \in \mathfrak{t}^* \mid \nu.t = \nu \text{ for every } t \in T\}$ by the equivariance of the maps. In other words, moment maps are unique modulo $(\mathfrak{t}^*)^T$. We will mainly consider an action by a closed real torus $T \cong (U(1))^k$ and have $(\mathfrak{t}^*)^T = \mathfrak{t}^*$ in this case.

There is an associated element $[\omega + \mu] \in H^2_T(M; \mathbb{R})$ of the equivariant de Rham cohomology. For another $T$-invariant symplectic form $\omega'$ and a moment map $\mu'$, we have $[\omega + \mu] = [\omega' + \mu']$ if and only if there exists a $T$-invariant 1-form $\phi$ such that $\omega = \omega' + d\phi$ and $\mu_\xi = \mu'_\xi + i_\xi \phi$. In particular, we have $[\omega+\mu]=[\omega+\mu']$ if and only if $\mu=\mu'$ as $\xi$ has a zero. The push-forward measure $\mu_*(\omega^n/n!)$ on $\mathfrak{t}^*$ is called the *Duistermaat–Heckman measure*, which defines the same measure independent of the choice of $\omega+\mu$ in the same equivariant cohomology class [GGK].

### Vector fileds

For an $\omega$-compatible almost complex structure $J$, we put

$$\xi^J := J\xi + \sqrt{-1}\xi, \quad \bar{\xi}^J := J\xi - \sqrt{-1}\xi \tag{1.6}$$

and $\theta_\xi := -2\mu_\xi$. Then we have

$$\sqrt{-1}\bar{\partial}\theta_\xi = i_{\xi^J}\omega, \quad \sqrt{-1}\partial\theta_\xi = -i_{\bar{\xi}^J}\omega, \tag{1.7}$$

where $\bar{\partial} := (d+\sqrt{-1}Jd)/2$ and $\partial := (d-\sqrt{-1}Jd)/2$. In other words, we have $\xi^J = g^{p\bar{q}}\theta_{\bar{q}}\partial_p$ and $\bar{\xi}^J = g^{p\bar{q}}\theta_p\bar{\partial}_q$ in the usual Kählerian notation. We also have $J\xi = -\nabla_{g_J}\mu_\xi$ and thus

$$\xi^J\theta_\xi = -2(J\xi)\mu_\xi = 2|\xi|^2_{g_J} = |\xi^J|^2_{g_J} \tag{1.8}$$

$$= |\bar{\partial}\theta_\xi|^2_{g_J} = \mathrm{tr}_{g_J}(\sqrt{-1}\partial\theta_\xi \wedge \bar{\partial}\theta_\xi). \tag{1.9}$$

### $\mu$-scalar curvature

Let $(M,\omega)$ be a closed $C^\infty$-symplectic manifold with a Hamiltonian action by a closed real torus $T$ and $\mu : M \to \mathfrak{t}^*$ be a moment map. For an $\omega$-compatible almost complex structure $J$, we denote by $s(J)$ the hermitian scalar curvature defined by Donaldson [Don1], which coincides with the usual Kähler (the half of the Riemannian) scalar curvature $s_{\mathrm{Kä}}(g_J) = \frac{1}{2}s_{\mathrm{Rm}}(g_J)$ for integrable $J$. Note that this $s(J)$ differs from the half of the Riemannian scalar curvature $\frac{1}{2}s_{\mathrm{Rm}}(g_J)$ for non-integrable $J$ in general.

The *$\mu$-scalar curvature* $s_\xi(g_J)$ of a metric $g_J(\cdot,\cdot)=\omega(\cdot,J\cdot)$ (associated to a $T$-invariant $\omega$-compatible almost complex structure $J$ on $M$) with respect to a vector $\xi \in \mathfrak{t}$ is defined as follows:

$$s_\xi(g_J) := (s(J) - \Delta_{g_J}\mu_\xi) + (-\Delta_{g_J}\mu_\xi + 2\xi^J\mu_\xi), \tag{1.10}$$

where $\Delta_{g_J}$ denotes the usual Riemannian Laplacian $\Delta_{g_J} = d^*d$ with respect to $g_J$ Since two moment maps with respect to the same symplectic form only differ by a constant, (1.10) is independent of the choice of the moment map $\mu$. When $\xi = 0$ and $J$ is integrable, the $\mu$-scalar curvature is of course nothing but the usual Kählerian scalar curvature.

As $\Delta_{g_J}$ is the twice of $\bar{\partial}/\partial$-Laplacians $\bar{\square} = \square = -g^{i\bar{j}}\partial_i\bar{\partial}_j$ when $J$ is integrable, we can express (1.10) as

$$\begin{aligned} s_\xi(g_J) &= (s_{\mathrm{K\ddot{a}}}(g_J) + \bar{\square}\theta_\xi) + (\bar{\square}\theta_\xi - \xi^J\theta_\xi) \\ &= (\bar{\square}\log\det g - \xi^J \log\det g + \sum_{i=1}^n \partial_i\xi^i) + (\bar{\square}\theta_\xi - \xi^J\theta_\xi) \\ &= (\bar{\square} - \xi^J)\log(e^{\theta_\xi}\det g) + \sum_{i=1}^n \partial_i\xi^i, \end{aligned} \tag{1.11}$$

using $\theta_\xi = -2\mu_\xi$. Note that $\xi^J \log\det g - \sum \partial_i\xi^i = -\bar{\square}\theta_\xi$ is a globally-defined function while $\xi^J\log\det g$ is just locally-defined on a holomorphic chart.

Put

$$\begin{aligned} \bar{s}_\xi(J) &:= \int_X s_\xi(g_J)e^{-2\mu_\xi}\omega^n \Big/ \int_X e^{-2\mu_\xi}\omega^n \qquad (1.12) \\ &= \int_X (s(g_J) - \Delta_{g_J}\mu_\xi)e^{-2\mu_\xi}\omega^n \Big/ \int_X e^{-2\mu_\xi}\omega^n. \end{aligned}$$

A similar calculation as in the proof of Proposition 3.1 in [Ino1] (cf. section 3.3.1) shows

$$\begin{aligned} \frac{d}{dt}\bar{s}_\xi(J_t) &= \Big(\frac{1}{4}\frac{d}{dt}(4s(J_t), e^{-2\mu_\xi}) + \int_X 2|\xi|^2_{g_J}\Big) \Big/ \int_X e^{-2\mu_\xi}\omega^n \\ &= \Big(\frac{1}{4}(L_{-2e^{-2\mu_\xi}\xi}J_t, J_t\dot{J}_t) + \int_X 2\omega(\xi, \dot{J}_t\xi)e^{-2\mu_\xi}\omega^n\Big) \Big/ \int_X e^{-2\mu_\xi}\omega^n \\ &= \int_X \Big((-Jd\mu_\xi\otimes\xi + d\mu_\xi\otimes J\xi, J_t\dot{J}_t) + 2\omega(\xi, \dot{J}_t\xi)\Big)e^{-2\mu_\xi}\omega^n \Big/ \int_X e^{-2\mu_\xi}\omega^n \\ &= 0. \end{aligned}$$

So $\bar{s}_\xi(J)$ is a constant independent of $J$ compatible with $\omega$. For an integrable complex structure $J$, we can compute it as

$$\begin{aligned} \bar{s}_\xi &= \Big(\int_X ne^{-2\mu_\xi}\mathrm{Ric}(\omega)\wedge\omega^{n-1} + \int_X \bar{\square}(-2\mu_\xi)e^{-2\mu_\xi}\omega^n\Big) \Big/ \int_X e^{-2\mu_\xi}\omega^n \\ &= \int_X (\mathrm{Ric}(\omega) + \bar{\square}\theta_\xi)\ e^{\omega+\theta_\xi} \Big/ \int_X e^{\omega+\theta_\xi} \\ &= 2\pi(c_1(X,\xi)\cdot e^{c_1(L,\xi)})/e^{c_1(L,\xi)}, \end{aligned}$$

where the last expression depends only on the equivariant Chern classes.

For each $\lambda \in \mathbb{R}$, we define the $\mu^\lambda$*-scalar curvature* $s^\lambda_\xi(g_J)$ of a metric $g_J$ by

$$s^\lambda_\xi(g_J) = s_\xi(g_J) + 2\lambda\mu_\xi. \tag{1.13}$$

We put

$$\bar{\mu}_\xi := \int_M \mu_\xi e^{-2\mu_\xi}\omega^n \Big/ \int_M e^{-2\mu_\xi}\omega^n \tag{1.14}$$

and

$$\hat{\mu}_\xi := \mu_\xi - \bar{\mu}_\xi, \tag{1.15}$$

$$\bar{s}^\lambda_\xi := \bar{s}_\xi + 2\lambda\bar{\mu}_\xi, \tag{1.16}$$

$$\hat{s}^\lambda_\xi(g_J) := s^\lambda_\xi(g_J) - \bar{s}^\lambda_\xi. \tag{1.17}$$

Then the constant $\bar{s}^\lambda_\xi$ depends only on the equivariant Chern classes $c^T_1(X), c^T_1(L)$ and $s^\lambda_\xi(g_J)$ is constant iff $\hat{s}^\lambda_\xi(g_J) = 0$.

### Relation with Kähler-Ricci soliton

There are two fundamental examples of constant $\mu$-scalar curvature Kähler metric:

- A constant scalar curvature Kähler metric is also a constant $\mu$-scalar curvature Kähler metric with respect to $\xi = 0$ and any $\lambda \in \mathbb{R}$.
- A Kähler-Ricci soliton $g_J$ with respect to $\xi$, i.e. $\mathrm{Ric}(g_J) - L_{\xi^J} g_J = \lambda g_J$, is a constant $\mu$-scalar curvature Kähler metric with respect to $\xi$ and $\lambda$.

The second claim follows from a standard calculation in [TZ2] (cf. [Ino1]). For the readers' convenience, we exhibit the proof here. Remember that Kähler-Ricci soliton with nontrivial $\xi \neq 0$ could exist only when $\lambda > 0$ and $[\lambda\omega] \in 2\pi c_1(X)$. In particular, $X$ is a Fano manifold in this case. Take a Ricci potential $h$ of $\omega$, i.e. $\mathrm{Ric}(\omega) - \lambda\omega = \sqrt{-1}\partial\bar{\partial}h$, and consider a moment map $\mu$ with respect to $\omega$ normalized as

$$\int_X \mu e^h \omega^n = 0. \tag{1.18}$$

Taking the Lie derivative $L_{\xi^J}$ of $\mathrm{Ric}(\omega) - \lambda\omega = \sqrt{-1}\partial\bar{\partial}h$, we have

$$\sqrt{-1}\partial\bar{\partial}(\bar{\Box}\theta_\xi - \lambda\theta_\xi) = \sqrt{-1}\partial\bar{\partial}((\bar{\partial}^\sharp h)\theta_\xi),$$

where we used that $\xi^J$ is holomorphic and $(\bar{\partial}^\sharp h)\theta_\xi = \xi^J h$. Note that the operator $\bar{\Box} - \bar{\partial}^\sharp h$ is formally self-adjoint with respect to the weighted measure $e^h\omega^n$, therefore $(\bar{\Box} - \bar{\partial}^\sharp h)f = \varphi$ has a solution $f$ (unique up to constant) iff $\int_X \varphi e^h \omega^n = 0$. So under the normalization (1.18), we obtain

$$\bar{\Box}\theta_\xi - \xi^J h - \lambda\theta_\xi = 0. \tag{1.19}$$

Then we can express $\bar{s}_\xi$ as

$$\begin{aligned} \bar{s}_\xi &= \int_X (\bar{\Box}(-h + \theta_\xi) + \lambda n) e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n \\ &= \lambda n + \int_X (-\bar{\Box} h + \xi^J h + \lambda\theta_\xi) e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n \\ &= \lambda n + \lambda \int_X \theta_\xi e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n, \end{aligned} \tag{1.20}$$

where we again used that $\bar{\Box} - \xi^J$ is formally self-adjoint with respect to the weighted measure $e^{\theta_\xi}\omega^n$. Now suppose $\omega$ is a Kähler-Ricci soliton, then taking the trace of $\mathrm{Ric}(g_J) - L_{\xi'} g_J = \lambda g_J$, we obtain

$$s(g_J) + \bar{\Box}\theta_\xi = \lambda n.$$

As $h$ is equal to $\theta_\xi$ up to constant, we have $\bar{\theta}_\xi = \int_X \theta_\xi e^{\theta_\xi}\omega^n = 0$ under the normalization (1.18) and $\bar{\Box}\theta_\xi - \xi^J\theta_\xi - \lambda\theta_\xi = 0$. Therefore, we conclude

$$s_\xi(g_J) - \lambda\theta_\xi = (s(g_J) + \bar{\Box}\theta_\xi) + (\bar{\Box}\theta_\xi - \xi^J\theta_\xi) - \lambda\theta_\xi = \bar{s}_\xi.$$

The normalization (1.18) of the moment map $\mu$ is equivalent to $[\omega + \mu] = c_1^T(X)$ where $c_1^T(X)$ denotes the equivariant Chern class of the anticanonical bundle $-K_X$, which can be represented by the equivariant closed form $\mathrm{Ric}(\omega) + \bar{\Box}\theta$ in the equivariant deRham cohomology.

### Donaldson-Fujiki picture for $\mu$-scalar curvature

Now we explain the moment map picture for $\mu$-scalar curvature. Let $(M, \omega)$ be a real $2n$-dimensional $C^\infty$-symplectic manifold. Denote by $\mathcal{J}_\xi(M, \omega)$ the space of all $\xi$-invariant almost complex structures compatible with $\omega$, which admits the structure of an infinite dimensional Fréchet manifold and is path-connected. We have the following symplectic structure $\Omega_\xi$ on $\mathcal{J}_\xi(M, \omega)$:

$$\Omega_\xi(A, B) := \int_M \mathrm{Tr}(JAB) e^{-2\mu_\xi}\omega^n \tag{1.21}$$

for each $A, B \in T_J\mathcal{J}_\xi(M,\omega) \subset \mathrm{End}TM$.

For simplicity, we assume the first Betti number of $M$ is zero. In this case, we can identify the Lie algebra $\mathfrak{symp}_\xi(M,\omega)$ of the Fréchet Lie group $\mathrm{Symp}_\xi(M,\omega)$ of symplectic diffeomorphisms preserving $\xi$ with the space $C^\infty_\xi(M)/\mathbb{R}$ of real-valued $\xi$-invariant $C^\infty$-functions on $M$ modulo constant. We identify a $2n$-form $\varphi$ on $M$ satisfying $\int_M \varphi = 0$ and $L_\xi\varphi = 0$ with the following element of the dual of $\mathfrak{symp}_\xi(M,\omega)$: $f \mapsto \int_M f\varphi$.

Now define a smooth map $\mathcal{S}^\lambda_\xi : \mathcal{J}_\xi(M,\omega) \to \mathfrak{symp}_\xi(M,\omega)^*$ of Fréchet manifolds by

$$\mathcal{S}^\lambda_\xi(J) := 4\hat{s}^\lambda_\xi(g_J)e^{-2\mu_\xi}\omega^n. \tag{1.22}$$

Then we have the following. The proof will be given in Proposition 3.3.1.

**Proposition 1.2.1** ([Ino1])**.** The map $\mathcal{S}^\lambda_\xi : \mathcal{J}_\xi(M,\omega) \to \mathfrak{symp}_\xi(M,\omega)^*$ is a moment map with respect to the symplectic structure $\Omega_\xi$ and the action of $\mathrm{Symp}_\xi(M,\omega)$ on $\mathcal{J}_\xi(M,\omega)$. Namely, $\mathcal{S}^\lambda_\xi$ is a $\mathrm{Symp}_\xi(M,\omega)$-equivariant smooth map satisfying

$$-\frac{d}{dt}\Big|_{t=0}\langle \mathcal{S}^\lambda_\xi(J_t), f\rangle = \Omega_\xi(L_{X_f}J_0, \dot{J}_0) \tag{1.23}$$

for every smooth curve $J_t \in \mathcal{J}_\xi(M,\omega)$ and $f \in C^\infty_\xi(M)$, where $X_f$ is the Hamiltonian vector field of $f$: $df = -i_{X_f}\omega$.

Note that moment maps with respect to the symplectic structure $\Omega_\xi$ is unique up to $\mathrm{Symp}_\xi(M,\omega)$-invariant elements of $\mathfrak{symp}_\xi(M,\omega)$. In particular, the map $J \mapsto (\hat{s}_\xi(g_J) + \mu_\zeta - \int_M \mu_\zeta e^{-2\mu_\xi}\omega^n / \int_M e^{-2\mu_\xi}\omega^n)e^{-2\mu_\xi}\omega^n$ also gives a moment map for any $\zeta$ tangent to the action of the closed torus generated by $\xi$. In this thesis, we restrict our interest to the proportional one, i.e. $\zeta = -2\lambda\xi$ for some $\lambda \in \mathbb{R}$.

The following invariant gives a constraint on $\lambda$ for each fixed $\xi$ and conversely a constraint on $\xi$ for each fixed $\lambda$ for the non-emptiness of the moduli space $(\mathcal{S}^\lambda_\xi)^{-1}(0)/\mathrm{Symp}_\xi(M,\omega)$. We will study these constraints in the next section and section 1.3.3, respectively.

**Corollary 1.2.2** ($\mu$-Futaki invariant)**.** Let $\mathfrak{t}$ be the Lie algebra of the closed torus generated by $\xi$. The following linear map $\mathrm{Fut}^\lambda_\xi : \mathfrak{t} \to \mathbb{R}$,

$$\mathrm{Fut}^\lambda_\xi(\zeta) := \int_M \hat{s}^\lambda_\xi(g_J)(-2\mu_\zeta)e^{-2\mu_\xi}\omega^n \Big/ \int_M e^{-2\mu_\xi}\omega^n$$

is independent of the choice of $J \in \mathcal{J}_\xi(M,\omega)$ and the moment map $\mu$ (as we divide it by $\int_M e^{-2\mu_\xi}\omega^n$).

If we fix a complex structure $J$, it is independent of the choice of the Kähler metric $\omega'$ in the Kähler class $[\omega]$ (by Moser's theorem). So in particular, $\mathrm{Fut}^\lambda_\xi$ can be regarded as an invariant of the quadruple $(X, [\omega], \xi, \lambda)$ where $X = (M, J)$ is a complex manifold. As observed in [Wang1], the moment map picture further expects that $\mathrm{Fut}^\lambda_\xi$ extends to $\mathfrak{h}_{0,\xi}(X)$. In section 1.3.2, we further show that $\mathrm{Fut}^\lambda_\xi$ extends to $\mathfrak{h}_0(X)$, which is larger than $\mathfrak{h}_{0,\xi}(X)$. Such an extension is out of expectations coming from the moment map picture.

Note that the above corollary also shows that this complex invariant $\mathrm{Fut}^\lambda_\xi$ (restricted to $\mathfrak{t}$) is also a $T$-equivariant deformation invariant.

#### Weighted cscK metrics and $\mu$-cscK metrics

For a smooth positive function $v$ on $P$, Lahdili [Lah] defines the *weighted scalar curvature* $s_v(\omega)$ by

$$s_v(\omega) := s(\omega) \cdot (v \circ \mu^\omega) + \Delta_\omega (v \circ \mu^\omega) - \frac{1}{2} \sum_{1 \le i,j \le k} (J\xi_i) \mu^\omega_{\xi_j} \cdot \Big(\frac{\partial^2 v}{\partial x^i \partial x^j} \circ \mu^\omega\Big). \tag{1.24}$$

As observed in [Lah], weighted scalar curvature has a moment map picture similar to that for $\mu$-scalar curvature in the previous section.

When $v$ is of the form $v(x) = \tilde{v}(\langle x, \xi \rangle)$ with some smooth positive function $\tilde{v}$ on $\mathbb{R}$ and $\xi \in \mathfrak{t}$, we can simplify it as

$$\begin{aligned} s_v(\omega) &= s(\omega) \cdot (\tilde{v} \circ \mu^\omega_\xi) + \big(\Delta_\omega \mu^\omega_\xi \cdot (\tilde{v}' \circ \mu^\omega_\xi) - (\nabla \mu^\omega_\xi, \nabla \mu^\omega_\xi) \cdot (\tilde{v}'' \circ \mu^\omega_\xi)\big) - \frac{1}{2} (J\xi) \mu^\omega_\xi \cdot (\tilde{v}'' \circ \mu^\omega_\xi) \\ &= s(\omega) \cdot (\tilde{v} \circ \mu^\omega_\xi) + \Delta_\omega \mu^\omega_\xi \cdot (\tilde{v}' \circ \mu^\omega_\xi) + \frac{1}{2} (J\xi) \mu^\omega_\xi \cdot (\tilde{v}'' \circ \mu^\omega_\xi) \end{aligned}$$

Substituting $v(x) = e^{\langle x, -2\xi \rangle}$ yields our $\mu$-scalar curvature $s_\xi(\omega)$:

$$s_v(\omega) = \big((s(\omega) + \bar{\Box}\theta_\xi) + (\bar{\Box}\theta_\xi - (J\xi)\theta_\xi)\big) e^{\theta_\xi} =: s_\xi(\omega) e^{\theta_\xi}.$$

So $\mu^0_\xi$-cscK metrics are equivalent to weighted cscK metrics with the weight $v(x) = e^{\langle x, -2\xi \rangle}$. For general $\lambda \in \mathbb{R}$, $\mu^\lambda_\xi$-cscK metrics are regarded as a special case of weighted extremal metrics.

### 1.2.2 From $\mu$-cscK metrics to extremal metric: $\lambda \searrow -\infty$

In this section, we fix a complex structure $J$ on $M$ and a Kähler class $[\omega]$. We observe some intriguing features of $\mu^\lambda$-cscK, assuming some results in the rest of this chapter.

#### Constraint on $\lambda$

There is an a priori constraint on $\lambda$ for each fixed $\xi \neq 0$ to admit a $\mu$-cscK metric in a fixed Kähler class $[\omega]$. If there is a $\mu_\xi^\lambda$-cscK metric $\omega'$, i.e. $\hat{s}_\xi^\lambda(\omega') = 0$, in the Kähler class $[\omega]$, then we must have

$$0 = \mathrm{Fut}_\xi^\lambda(\xi) = \mathrm{Fut}_\xi^0(\xi) - \lambda\left(\int_X \theta_\xi^2 e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n - \Big(\int_X \theta_\xi e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n\Big)^2\right). \tag{1.25}$$

For $\zeta \in \mathfrak{t}$, we put

$$\nu_\xi(\zeta) := \int_X \theta_\zeta^2 e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n - \Big(\int_X \theta_\zeta e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n\Big)^2. \tag{1.26}$$

This is invariant when we add a constant $c$ on $\theta_\zeta$, so it must be positive when $\zeta \neq 0$ since it is obviously positive when normalizing $\theta_\zeta$ so that $\int_X \theta_\zeta e^{\theta_\xi}\omega^n = 0$. The function $\nu_\xi$ is an invariant of the equivariant deRham class $[\omega + \mu]$ and $\xi$, since it can be expressed as

$$\nu_\xi(\zeta) = \frac{\int_P \langle m, -2\zeta\rangle^2 e^{\langle m, -2\xi\rangle} DH(m)}{\int_P e^{\langle m, -2\xi\rangle} DH(m)} - \Big(\frac{\int_P \langle m, -2\zeta\rangle e^{\langle m, -2\xi\rangle} DH(m)}{\int_P e^{\langle m, -2\xi\rangle} DH(m)}\Big)^2,$$

using the Duistermaat-Heckman measure $DH = \mu_* \omega^n$, which is an invariant of the equivariant deRham class $[\omega + \mu]$ associated to the moment map. Here $P$ denotes the support of the measure $DH$.

Thus from (1.25) we can determine $\lambda$ as

$$\lambda = \lambda_\xi := \mathrm{Fut}_\xi^0(\xi)/\nu_\xi(\xi), \tag{1.27}$$

where the right hand side is an invariant of the triple $(X, [\omega], \xi)$ (also an invariant of the symplectic triple $(M, \omega, \xi)$). The sign of $\lambda_\xi$ coincides with that of $\mathrm{Fut}_\xi^0(\xi)$.

#### $\lambda$ as a function on the real blowing-up $\hat{\mathfrak{t}}$

While the function $\lambda_\xi$ is well-defined and continuous just on the punctured space $\mathfrak{t} \setminus \{0\}$, the following functional

$$\xi \mapsto |\xi| \cdot \lambda_\xi \tag{1.28}$$

continuously extends to the real blowing-up

$$\hat{\mathfrak{t}} := \{(\xi, \Xi) \mid \xi \in \Xi = [0,\infty) \cdot v \subset \mathfrak{t},\ v \in \mathfrak{t} \setminus \{0\}\} \xrightarrow{\pi} \mathfrak{t} : (\xi, \Xi) \mapsto \xi$$

of $\mathfrak{t}$ at the origin, where we take the norm on $\mathfrak{t}$ as $|\xi|^2 := \int_X \theta_\xi^2 \omega^n$. Indeed, as $|\xi|$ tends to 0, the function $|\xi|^{-2}\nu_\xi(\xi)$ on $\mathfrak{t} \setminus \{0\}$ approaches to a positive continuous function $\hat{\nu}(0,\Xi) = 1/\int_X \omega^n - (\int_X \theta_\Xi \omega^n / \int_X \omega^n)^2$ on the boundary sphere $\pi^{-1}(0)$, where we put $\theta_\Xi := \theta_v$ for a unique vector $v \in \Xi$ with $|v| = 1$ and similarly $|\xi|^{-1} \cdot \mathrm{Fut}^0_\xi(\xi)$ approaches to a continuous function $\widehat{\mathrm{Fut}}(0,\Xi) = \mathrm{Fut}(v) = \int_X (s - \bar{s})\theta_\Xi \omega^n / \int_X \omega^n$ on $\pi^{-1}(0)$. Here the positivity of $\hat{\nu}$ again follows by the Cauchy-Schwartz inequality.

We will see in section 1.3.3 that $\lambda_\xi$, i.e. $\mathrm{Fut}^0_\xi(\xi)$, is always positive sufficiently away from the origin. Assuming this, it follows that any sequence $\xi_i \in \mathfrak{t}$ with $\lambda_{\xi_i} \to -\infty$ must converge to the origin $0 \in \mathfrak{t}$. Moreover, as the function $|\xi| \cdot \lambda_\xi$ is bounded near the origin, we have a uniform bound $|\lambda_i \xi_i| \le C$, so that there is a subsequence such that $\lambda_i \xi_i$ converges to some vector $\check{\xi} \in \mathfrak{t}$. Now suppose $\mathrm{Fut}^{\lambda_i}_{\xi_i} = 0$ for every $i$. Since we can compute all $\mathrm{Fut}^{\lambda_i}_{\xi_i}$ by a fixed $T$-invariant Kähler metric $\omega$, the limit of this functional is given as

$$\check{\mathrm{Fut}}^0_{\check{\xi}}(\zeta) := \int_X \Big( (s(\omega) - \bar{s}) - (\theta_{\check{\xi}} - \underline{\theta}_{\check{\xi}}) \Big) \theta_\zeta \omega^n \Big/ \int_X \omega^n,$$

where we put $\underline{\theta}_{\check{\xi}} := \int_X \theta_{\check{\xi}} \omega^n / \int_X \omega^n$. We must have $\check{\mathrm{Fut}}^0_{\check{\xi}} \equiv 0$ for the limit vector $\check{\xi}$.

Such a vector $\check{\xi}$ is uniquely characterized as the critical point of the following strictly convex functional on $\mathfrak{t}$:

$$C(\xi) := \int_X \Big( (s(\omega) - \bar{s}) - (\theta_\xi - \underline{\theta}_\xi) \Big)^2 \omega^n \Big/ \int_X \omega^n - \int_X (s(\omega) - \bar{s})^2 \omega^n \Big/ \int_X \omega^n, \tag{1.29}$$

whose derivative at $\xi$ is $2\check{\mathrm{Fut}}^0_\xi$. (We add the second term so that the functional is independent of the choice $\omega \in [\omega]$. ) The minimizer of this functional is called *the extremal vector*. We denote it by $\xi_{\mathrm{ext}}$. From the above observation, we obtain $\check{\xi} = \xi_{\mathrm{ext}}$ for the limit vector $\check{\xi}$, independent of the choice of the subsequence of $\{i\}$. It follows that the original sequence $\lambda_i \xi_i$ also converges to $\xi_{\mathrm{ext}}$.

**Extremal metric in the limit of $\lambda \to -\infty$**

Suppose there is a sequence of $\mu_{\xi_i}^{\lambda_i}$-cscK metrics $\omega_i$ in the fixed Kähler class $[\omega]$ with $\lambda_i \to -\infty$:

$$(s(\omega_i) + \bar{\square}_{\omega_i}\theta_{\xi_i}(\omega_i)) + (\bar{\square}_{\omega_i}\theta_{\xi_i}(\omega_i) - \xi_i^J\theta_{\xi_i}(\omega_i)) - \lambda_i\theta_{\xi_i}(\omega_i) = \bar{s}_{\xi_i}^{\lambda_i},$$

where $\theta_{\xi_i}(\omega_i)$ denotes the $\bar{\partial}$-Hamiltonian potential with respect to $\omega_i$ in the same equivariant class. Fix a reference metric $\omega$ and take a Kähler potential $\phi_i$ of $\omega_i$ so that $\max \phi_i = 0$.

Suppose we have a uniform $C^{3,\alpha}$-bound of $\phi_i$ and a uniform bound $C\omega \le \omega_i$, then the limit of the metrics gives a metric $\omega_{-\infty} \in [\omega]$ after taking a subsequence. Remember that the vectors $\xi_i$ must converge to 0 and the sequence $\lambda_i\xi_i$ converges to the extremal vector $\xi_{\mathrm{ext}}$ (by the observation in the last subsection). It follows that $\theta_{\xi_i}(\omega_i) = \theta_{\xi_i}(\omega) - \xi_i^J\phi_i$ converges to 0 in $C^{2,\alpha}$ and the limit metric $\omega_{-\infty}$ must satisfy the following equation

$$s(\omega_{-\infty}) - \theta_{\xi_{\mathrm{ext}}}(\omega_{-\infty}) = \mathrm{const},$$

which is nothing but the equation of extremal metric.

Conversely, we will see in section 1.5 the following:

- If there exists an extremal metric, there also exists $\mu^\lambda$-cscK metrics in the same Kähler class for $\lambda$ sufficiently small or large.
- If there is a $\mu^\lambda$-cscK metric for $\lambda \le 0$, then we can find a $\mu^{\lambda'}$-cscK metric in the same Kähler class for small perturbations $\lambda' \in (\lambda - \epsilon, \lambda + \epsilon)$.

Thus the problem of connecting $\mu^0$-cscK metric/Kähler–Ricci soliton and extremal metric (when both of them exist) reduces to the problem on the a priori estimate.

Though we firstly introduced the parameter $\lambda$ so that we can include Kähler-Ricci soliton in our study on $\mu$-cscK metric, the above observation now tells us that the parameter $\lambda$ can be regarded as a continuity path connecting $\mu^0$-cscK metric/Kähler–Ricci soliton and extremal metric.

## 1.3 $\mu$-Futaki invariant, $\mu$-volume functional and automorphism group

### 1.3.1 $\mu$-Lichnerowicz operator and reductiveness

In this section, we fix a complex structure $J$ on $M$, a Kähler metric $\omega$ and a function $\theta$ on $M$. We prove the reductiveness of the automorphism group of a Kähler manifold admitting $\mu$-cscK. This result is a first step to construct a good moduli space of the complex structures of Kähler manifolds admitting $\mu$-cscK metrics, in order to apply GIT locally. We firstly begin with basic calculations for the readers' and the author's convenience.

#### Warming up for calculations

Let $(X, \omega)$ be a Kähler manifold, $\theta$ be a smooth real-valued function on $X$ and $(E, h)$ be a hermitian (not necessarily holomorphic, so far) vector bundle on $X$. Define an $L^2$-norm $\langle \cdot, \cdot \rangle_\theta$ by

$$\langle \alpha, \beta \rangle_\theta := \int_X h(\alpha, \beta)\ e^\theta \omega^n$$

for smooth sections $\alpha, \beta \in \Omega^0(E)$. For a differential operator $D : \Omega^0(E) \to \Omega^0(F)$ from $E$ to $F$, denote by $D^{\theta *} : \Omega^0(F) \to \Omega^0(E)$ the formal left adjoint of $D$ with respect to such pairing, i.e.

$$\langle D^{\theta *} \alpha, \beta \rangle_{E,\theta} = \langle \alpha, D\beta \rangle_{F,\theta}$$

for all sections $\alpha \in \Omega^0(F), \beta \in \Omega^0(E)$. As usual, we denote by $\Lambda : \Omega^{p,q}(E) \to \Omega^{p-1,q-1}(E)$ the adjoint operator of $\omega \wedge$:

$$h^{p-1,q-1}(\Lambda(\alpha), \beta) = h^{p,q}(\alpha, \omega \wedge \beta),$$

where $h^{p,q}$ is the induced hermitian metric on $\Lambda^{p,q} \otimes E$ defined as

$$(u_{1\cdots p} \wedge u_{\bar{1}\cdots\bar{q}} \otimes \sigma, v_{1\cdots p} \wedge v_{\bar{1}\cdots\bar{q}} \otimes \tau) := h(\sigma, \tau) \cdot \det \begin{pmatrix} g(u_i, \overline{v_{\bar{j}}}) & 0 \\ 0 & g(u_{\bar{k}}, \overline{v_{\bar{l}}}) \end{pmatrix}_{k,l=1,\dots,q}^{i,j=1,\dots,p}$$

for $u_{1\cdots p} = u_1 \wedge \cdots \wedge u_p, v_{1\cdots p} = v_1 \wedge \cdots \wedge v_p \in \Lambda^{p,0}_x X$, $u_{\bar{1}\cdots\bar{q}} = u_{\bar{1}} \wedge \cdots \wedge u_{\bar{q}}, v_{\bar{1}\cdots\bar{q}} = v_{\bar{1}} \wedge \cdots \wedge v_{\bar{q}} \in \Lambda^{0,q}_x X$ and $\sigma, \tau \in E_x$.

*Example* 1.3.1. For $\alpha = \alpha_{i\bar{j}} \otimes dz^i \wedge d\bar{z}^j \in \Omega^{1,1}(E)$, we have

$$\Lambda(\alpha) = -\sqrt{-1} g^{i\bar{j}} \alpha_{i\bar{j}}.$$

For $\gamma = \gamma_{i\bar{j}\bar{k}} \otimes dz^i \wedge d\bar{z}^j \wedge d\bar{z}^k \in \Omega^{1,2}(E)$, we have

$$\Lambda(\gamma) = -\sqrt{-1} g^{i\bar{j}} (\gamma_{i\bar{j}\bar{k}} - \gamma_{i\bar{k}\bar{j}}) d\bar{z}^k.$$

For a hermitian connection $\nabla$ on $(E, h)$, the following local expressions yield global operators.

$$\nabla' := \sum_{i=1}^{n} dz^i \wedge \nabla^{\wedge}_{\partial_i} : \Omega^{p,q}(E) \to \Omega^{p+1,q}(E), \tag{1.30}$$

$$\nabla'' := \sum_{i=1}^{n} d\bar{z}^i \wedge \nabla^{\wedge}_{\bar{\partial}_i} : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E), \tag{1.31}$$

where

$$\nabla^{\wedge} : \Omega^0(\Lambda^{p,q} \otimes E) \to \Omega^1(\Lambda^{p,q} \otimes E)$$

is the induced connection on $\Lambda^{p,q} \otimes E$. These operators $\nabla', \nabla''$ are the first order differential operators from $\Lambda^{p,q} \otimes E$ to $\Lambda^{p+1,q} \otimes E$ and $\Lambda^{p,q} \otimes E$ to $\Lambda^{p,q+1} \otimes E$, respectively, and $\nabla' + \nabla''$ is the exterior covariant derivative of $\nabla$.

Put $\theta_p := \partial\theta/\partial z^p, \theta_{\bar{q}} := \partial\theta/\partial\bar{z}^q$ on a holomorphic chart of $X$ and denote by $\xi', \xi''$ the following global vector fields associated to $\theta$

$$\xi' := \partial^{\sharp}\theta = g^{p\bar{q}} \theta_{\bar{q}} \partial_p, \quad \xi'' := \bar{\partial}^{\sharp}\theta = g^{p\bar{q}} \theta_p \bar{\partial}_q.$$

Then the formal adjoints $\nabla'^{\theta *}, \nabla''^{\theta *}$ of $\nabla', \nabla''$ with respect to the pairing $\langle \cdot, \cdot \rangle_\theta$ can be written as

$$\nabla'^{\theta *} = \nabla'^{*} - i_{\xi'} = \sqrt{-1}\Big(\Lambda\nabla'' - \nabla''\Lambda\Big) - i_{\xi'}, \tag{1.32}$$

$$\nabla''^{\theta *} = \nabla''^{*} - i_{\xi''} = -\sqrt{-1}\Big(\Lambda\nabla' - \nabla'\Lambda\Big) - i_{\xi''}. \tag{1.33}$$

Indeed, using $\langle \alpha, \beta \rangle_\theta = \langle \alpha, \beta e^\theta \rangle$, we compute

$$\begin{aligned} \langle \nabla'^{\theta *}\alpha, \beta \rangle_\theta &= \langle \alpha, \nabla'\beta \rangle_\theta \\ &= \langle \alpha, \nabla'(e^\theta \beta) \rangle - \langle \alpha, \partial\theta \wedge \beta \rangle_\theta \\ &= \langle \nabla'^{*}\alpha, \beta \rangle_\theta - \langle i_{\xi'}\alpha, \beta \rangle_\theta. \end{aligned}$$

### Weitzenböck formula

Let $(L, e^{-\phi})$ be a holomorphic hermitian line bundle on $X$, where we denote the hermitian connection by a local expression $e^{-\phi}$. We denote by $\nabla$ the Chern connection on $L$. The Chern curvature is given by $\partial\bar{\partial}\phi$. Put

$$\nabla^{\sharp} := \nabla'_{T^{1,0}X\otimes L} \circ \sharp : \Omega^{0,1}(L) \to \Omega^{1,0}(T^{1,0}X \otimes L),$$
$$\nabla^{\sharp\sharp} := \nabla''_{T^{1,0}X\otimes L} \circ \sharp : \Omega^{0,1}(L) \to \Omega^{0,1}(T^{1,0}X \otimes L),$$

where $\sharp : \Omega^{0,1}(L) \to \Omega^{0}(T^{1,0}X \otimes L)$ is given by $\sharp(\alpha_{\bar{j}} d\bar{z}^j) = g^{i\bar{j}}\alpha_{\bar{j}}\partial_i$. Consider the following four variants of weighted Laplacian acting on $\Omega^{0,1}(L)$:

$$\begin{aligned}
\Box^{\theta} &:= \nabla'^{\theta *}\nabla' + \nabla'\ \nabla'^{\theta *} = \nabla'^{\theta *}\nabla',\\
\bar{\Box}^{\theta} &:= \nabla''^{\theta *}\nabla'' + \nabla''\ \nabla''^{\theta *},\\
\Box^{\theta}_{\#} &:= \nabla^{\sharp^{\theta *}}\nabla^{\sharp} = \flat(\nabla'^{\theta *}_{TX\otimes L}\nabla'_{TX\otimes L})\sharp,\\
\bar{\Box}^{\theta}_{\#} &:= \nabla^{\sharp\sharp^{\theta *}}\nabla^{\sharp\sharp} = \flat(\nabla''^{\theta *}_{TX\otimes L}\nabla''_{TX\otimes L})\sharp,
\end{aligned}$$

where $\flat : \Omega^{0}(T^{1,0}X \otimes L) \to \Omega^{0,1}(L)$ is given by $\flat(\eta^i\partial_i) = g_{i\bar{j}}\eta^i d\bar{z}^j$.

**Lemma 1.3.2** (Weighted Laplacians)**.** The above weighted Laplacians can be expressed by the usual Laplacians as follows.

$$\Box^{\theta} = \Box - \nabla^{\wedge}_{\xi'}, \tag{1.34}$$
$$\bar{\Box}^{\theta} = \bar{\Box} - \nabla^{\wedge}_{\xi''} - g^{i\bar{j}}\theta_{i\bar{k}} d\bar{z}^k \otimes \bar{\partial}_j, \tag{1.35}$$
$$\Box^{\theta}_{\#} = \Box_{\#} - \nabla^{\wedge}_{\xi'}, \tag{1.36}$$
$$\bar{\Box}^{\theta}_{\#} = \bar{\Box}_{\#} - \nabla^{\wedge}_{\xi''}, \tag{1.37}$$

where $\theta_{i\bar{k}} = \partial^2\theta/\partial z^i\partial\bar{z}^k$ and $g^{i\bar{j}}\theta_{i\bar{k}} d\bar{z}^k \otimes \bar{\partial}_j \in \mathrm{End}(T^{0,1}X)$ is identified with the operator acting on $\Omega^{0,1}(L)$.

*Proof.* Let $\alpha_{\bar{k}} \otimes d\bar{z}^k$ be an element of $\Omega^{0,1}(L)$ expressed by local sections $\alpha_{\bar{k}}$ of $L$. Then from (1.32), we have

$$\begin{aligned}
\Box^{\theta}(\alpha_{\bar{k}} \otimes d\bar{z}^k) &= \nabla'^{\theta *}\nabla'(\alpha_{\bar{k}} \otimes d\bar{z}^k)\\
&= \Big(\nabla'^{*}\nabla' - i_{\xi'}\nabla'\Big)(\alpha_{\bar{k}} \otimes d\bar{z}^k)\\
&= (\Box - \nabla^{\wedge}_{\xi'})(\alpha_{\bar{k}} \otimes d\bar{z}^k).
\end{aligned}$$

We can do similarly as for $\Box^\theta_\#$ and $\bar\Box^\theta_\#$.

As for $\bar\Box^\theta$, we calculate as follows.

$$\begin{aligned}\bar\Box^\theta(\alpha_{\bar k}\otimes d\bar z^k) &= (\nabla''^{\theta *}\nabla'' + \nabla''\nabla''^{\theta *})(\alpha_{\bar k}\otimes d\bar z^k)\\ &= \Big((\nabla''^{*}\nabla'' + \nabla''\nabla''^{*}) - (i_{\xi''}\nabla'' + \nabla'' i_{\xi''})\Big)(\alpha_{\bar k}\otimes d\bar z^k)\\ &= \bar\Box(\alpha_{\bar k}\otimes d\bar z^k) - \nabla^\wedge_{\xi''}(\alpha_{\bar k}\otimes d\bar z^k) + d\bar z^q\wedge i_{\xi''}\nabla^\wedge_{\bar\partial_q}(\alpha_{\bar k}\otimes d\bar z^k) - \bar\partial_L(g^{i\bar j}\theta_i\alpha_{\bar j})\\ &= (\bar\Box - \nabla^\wedge_{\xi''})(\alpha_{\bar k}\otimes d\bar z^k) - g^{i\bar j}\theta_{i\bar k}\alpha_{\bar j}d\bar z^k,\end{aligned}$$

where we transform $d\bar z^q\wedge i_{\xi''}\nabla^\wedge_{\bar\partial_q}(\alpha_{\bar k}\otimes d\bar z^k)$ as

$$\begin{aligned}d\bar z^q\wedge i_{\xi''}\nabla^\wedge_{\bar\partial_q}(\alpha_{\bar k}\otimes d\bar z^k) &= d\bar z^q\wedge(\xi^{\bar k}\bar\partial_{L,\bar q}\alpha_{\bar k} + \alpha_{\bar k}(-\xi^{\bar j}\Gamma^{\bar k}_{\bar q\bar j}))\\ &= g^{l\bar k}\theta_l\bar\partial_L\alpha_{\bar k} - \alpha_{\bar k}\theta_l g^{l\bar j}g^{p\bar k}g_{p\bar j,\bar q}d\bar z^q\\ &= \theta_l\bar\partial_L(g^{l\bar k}\alpha_{\bar k}).\end{aligned}$$

□

**Corollary 1.3.3** (Weitzenböck formula)**.** Write $\mathrm{Ric}(\omega) = \sqrt{-1}\bar\partial\partial\log\det(g_{p\bar q})$ as $\sqrt{-1}R_{i\bar j}dz^i\wedge d\bar z^j$ and put $\xi := (\xi' - \xi'')/2\sqrt{-1}$. Then we have the following.

$$\Box^\theta - \bar\Box^\theta = \Lambda(\sqrt{-1}\partial\bar\partial\phi) - 2\sqrt{-1}\nabla^\wedge_\xi + g^{i\bar j}\theta_{i\bar k}d\bar z^k\otimes\bar\partial_j, \tag{1.38}$$

$$\Box^\theta_\# - \bar\Box^\theta_\# = \Lambda(\sqrt{-1}\partial\bar\partial\phi) + g^{i\bar j}R_{i\bar k}d\bar z^k\otimes\bar\partial_j - 2\sqrt{-1}\nabla^\wedge_\xi, \tag{1.39}$$

$$\Box^\theta_\# - \Box^\theta = 0, \tag{1.40}$$

$$\bar\Box^\theta_\# - \bar\Box^\theta = -g^{i\bar j}(R_{i\bar k} - \theta_{i\bar k})d\bar z^k\otimes\bar\partial_j. \tag{1.41}$$

*Proof.* The first two equalities follow from the above lemma combined with the usual Kodaira-Nakano formula

$$\begin{aligned}\Box - \bar\Box &= \Lambda(\sqrt{-1}\partial\bar\partial\phi),\\ \Box_\# - \bar\Box_\# &= \flat(g^{i\bar j}R_p{}^k{}_{i\bar j}dz^p\otimes\partial_k + g^{i\bar j}\phi_{i\bar j}dz^p\otimes\partial_p)\sharp\\ &= g^{p\bar q}R_{p\bar l}d\bar z^l\otimes\bar\partial_q + g^{i\bar j}\phi_{i\bar j}d\bar z^q\otimes\bar\partial_q.\end{aligned}$$

Put $\alpha_{\bar k,p} := \nabla_p\alpha_{\bar k}$ and $\alpha_{\bar k,\bar q} := \nabla_{\bar q}\alpha_{\bar k}$. Then using (1.32) and (1.33), we obtain

$$\nabla^\sharp(\alpha_{\bar k}d\bar z^k) = g^{l\bar k}\alpha_{\bar k,p}dz^p\otimes\partial_l, \tag{1.42}$$

$$\nabla^{\sharp\theta *}(\beta^l_p dz^p\otimes\partial_l) = -g_{l\bar j}g^{p\bar q}(\beta^l_{p,\bar q} + \beta^l_p\theta_{\bar q})d\bar z^j, \tag{1.43}$$

$$\nabla^{\sharp\sharp}(\alpha_{\bar k}d\bar z^k) = (g^{l\bar k}\alpha_{\bar k})_{\bar q}d\bar z^q\otimes\partial_l, \tag{1.44}$$

$$\nabla^{\sharp\sharp\theta *}(\beta^l_{\bar q}d\bar z^q\otimes\partial_l) = \nabla^{\sharp\sharp *}(\beta^l_{\bar q}d\bar z^q\otimes\partial_l) - g_{l\bar j}g^{p\bar q}\theta_p\beta^l_{\bar q}d\bar z^j. \tag{1.45}$$

$\square$

**$\mu$-Lichnerowicz operator and Reductiveness**

**Proposition 1.3.4** ($\mu$-Lichnerowicz operator)**.** Put $\mathcal{D} := \nabla^{\sharp\sharp}\bar{\partial} : C^\infty_{\mathbb{C}}(X) \to \Omega^{0,1}(T^{1,0}X)$. Suppose $\xi' = \partial^\sharp\theta$ is a holomorphic vector field, then

$$\begin{aligned}(\mathcal{D}^{\theta*}\mathcal{D})f &= (\bar{\partial}^{\theta*}\bar{\square}^\theta_\#\bar{\partial})f \\ &= (\bar{\partial}^{\theta*}\bar{\square}^\theta\bar{\partial})f - (\bar{\partial}^{\theta*}(g^{i\bar{j}}(R_{i\bar{k}} - \theta_{i\bar{k}})d\bar{z}^k \otimes \bar{\partial}_j)\bar{\partial})f \\ &= (\bar{\square} - \xi'')^2 f + (\mathrm{Ric}(\omega) - L_{\xi'}\omega, \sqrt{-1}\partial\bar{\partial}f) \\ &\qquad + (\bar{\partial}^\sharp s_\xi(\omega))(f),\end{aligned} \tag{1.46}$$

where $s_\xi(\omega) = (s(\omega) + \bar{\square}\theta) + (\bar{\square}\theta - \xi'\theta)$.

*Proof.* It suffices to show the third equality. As $\bar{\partial}\bar{\partial} = 0$, we have $\bar{\partial}^{\theta*}\bar{\square}^\theta\bar{\partial} = (\bar{\partial}^{\theta*}\bar{\partial})(\bar{\partial}^{\theta*}\bar{\partial}) = (\bar{\square} - \xi'')(\bar{\square} - \xi'')$. The second term in the second formula can be simplified as

$$-(\bar{\partial}^{\theta*}(g^{i\bar{j}}(R_{i\bar{k}} - \theta_{i\bar{k}})d\bar{z}^k \otimes \bar{\partial}_j)\bar{\partial})f = (\sqrt{-1}(\Lambda\partial) + i_{\xi''})(g^{i\bar{j}}(R_{i\bar{k}} - \theta_{i\bar{k}})f_{\bar{j}}d\bar{z}^k). \tag{1.47}$$

As for $\sqrt{-1}(\Lambda\partial)(g^{i\bar{j}}(R_{i\bar{k}} - \theta_{i\bar{k}})f_{\bar{j}}d\bar{z}^k)$,

$$\begin{aligned}\sqrt{-1}(\Lambda\partial)(g^{i\bar{j}}(R_{i\bar{k}} - \theta_{i\bar{k}})f_{\bar{j}}d\bar{z}^k) &= \sqrt{-1}(-\sqrt{-1}g^{l\bar{k}})(g^{i\bar{j}}(R_{i\bar{k}} - \theta_{i\bar{k}})f_{\bar{j}})_l \\ &= g^{l\bar{k}}g^{i\bar{j}}(R_{i\bar{k}} - \theta_{i\bar{k}})f_{l\bar{j}} + g^{l\bar{k}}(g^{i\bar{j}}(R_{i\bar{k}} - \theta_{i\bar{k}}))_l f_{\bar{j}} \\ &= (\mathrm{Ric}(\omega) - L_{\xi'}\omega, \sqrt{-1}\partial\bar{\partial}f) \\ &\qquad + g^{l\bar{k}}g^{i\bar{j}}{}_{,l}(R_{i\bar{k}} - \theta_{i\bar{k}})f_{\bar{j}} + g^{l\bar{k}}g^{i\bar{j}}(R_{i\bar{k},l} - \theta_{i\bar{k},l})f_{\bar{j}},\end{aligned} \tag{1.48}$$

where $R_{i\bar{k},l} = \partial R_{i\bar{k}}/\partial z^l$ and $\theta_{i\bar{k},l} = \partial^3\theta/\partial z^i\partial\bar{z}^k\partial z^l$. As $R_{i\bar{k},l} - \theta_{i\bar{k},l} = R_{l\bar{k},i} - \theta_{l\bar{k},i}$, the last term of (1.48) is equal to

$$g^{i\bar{j}}(g^{l\bar{k}}(R_{l\bar{k}} - \theta_{l\bar{k}}))_i f_{\bar{j}} - g^{i\bar{j}}g^{l\bar{k}}{}_{,i}(R_{l\bar{k}} - \theta_{l\bar{k}})f_{\bar{j}}. \tag{1.49}$$

As $g^{l\bar{k}}g^{i\bar{j}}{}_{,l} = -g^{l\bar{k}}g^{i\bar{q}}g_{p\bar{q},l}g^{p\bar{j}} = -g^{l\bar{k}}g^{i\bar{q}}g_{l\bar{q},p}g^{p\bar{j}} = g^{i\bar{k}}{}_{,p}g^{p\bar{j}}$, the second term of (1.48) is distinguished by the second term of (1.49). So we obtain

$$\sqrt{-1}(\Lambda\partial)(g^{i\bar{j}}(R_{i\bar{k}} - \theta_{i\bar{k}})f_{\bar{j}}d\bar{z}^k) = (\mathrm{Ric}(\omega) - L_{\xi'}\omega, \sqrt{-1}\partial\bar{\partial}f) + (\bar{\partial}^\sharp(s + \bar{\square}\theta))f.$$

The rest term in (1.47) is

$$\begin{aligned} i_{\xi''}(g^{i\bar{j}}(R_{i\bar{k}}-\theta_{i\bar{k}})f_{\bar{j}}d\bar{z}^k) &= \xi^{\bar{k}}g^{i\bar{j}}(R_{i\bar{k}}-\theta_{i\bar{k}})f_{\bar{j}} \\ &= g^{l\bar{k}}g^{i\bar{j}}\theta_l(R_{i\bar{k}}-\theta_{i\bar{k}})f_{\bar{j}} \end{aligned}$$

and the following calculations show (1.46). As $\xi' = g^{q\bar{p}}\theta_{\bar{p}}$ is holomorphic, we have $(g^{p\bar{q}}\theta_p)_i = \overline{(g^{\bar{p}q}\theta_{\bar{p}})_{\bar{i}}} = \overline{\bar{\partial}_i\xi^q} = 0$. It follows that

$$\begin{aligned} (\bar{\square}\theta)_i = -(g^{p\bar{q}}\theta_{p\bar{q}})_i &= -(g^{p\bar{q}}\theta_p)_{i\bar{q}} + (g^{p\bar{q}}{}_{,\bar{q}}\theta_p)_i \\ &= (-g^{p\bar{k}}g^{l\bar{q}}g_{l\bar{k},\bar{q}}\theta_p)_i \\ &= -(g^{l\bar{q}}g_{l\bar{q},\bar{k}})_i g^{p\bar{k}}\theta_p - g^{l\bar{q}}g_{l\bar{q},\bar{k}}(g^{p\bar{k}}\theta_p)_i \\ &= g^{p\bar{k}}\theta_p R_{i\bar{k}} \end{aligned}$$

and

$$(\xi'\theta)_i = (g^{l\bar{k}}\theta_{\bar{k}}\theta_l)_i = (g^{l\bar{k}}\theta_l)_i\theta_{\bar{k}} + g^{l\bar{k}}\theta_l\theta_{i\bar{k}} = g^{l\bar{k}}\theta_l\theta_{i\bar{k}}.$$

□

**Corollary 1.3.5** (Reductiveness)**.** Suppose there exists a $\mu^\lambda_\xi$-cscK metric $\omega$ on $X$, then the identity component $\mathrm{Aut}^0_\xi(X/\mathrm{Alb})$ of the subgroup of the reduced automorphism group $\mathrm{Aut}(X/\mathrm{Alb})$ preserving $\xi$ is the complexification of the group ${}_H\mathrm{Isom}^0_\xi(X,\omega)$ of the Hamiltonian isometries of the $\mu$-cscK metric $\omega$ preserving $\xi$, especially, it is reductive.

*Proof.* If $\omega$ is a $\mu$-cscK, then the operator $\mathcal{D}^{\theta_*}\mathcal{D}$ restricted to $C^\infty_\xi(X,\mathbb{C}) = \{f \in C^\infty(X,\mathbb{C}) \mid \xi f = 0\}$ is a real operator. It follows that

$$\{f \in C^\infty_\xi(X,\mathbb{C}) \mid \mathcal{D}f = 0\} = \{g + \sqrt{-1}h \mid \mathcal{D}g = \mathcal{D}h = 0,\ g,h \in C^\infty_\xi(X,\mathbb{R})\},$$

which are respectively isomorphic to $\mathfrak{aut}_\xi(X,[\omega])$ and $\mathfrak{isom}_\xi(X,g)\oplus\sqrt{-1}\mathfrak{isom}_\xi(X,g)$ as we have $\mathcal{D} = \nabla^{\sharp\sharp}\bar{\partial} = \bar{\partial}_{TX}\partial^\sharp$. □

### 1.3.2 $\mu$-Futaki invariant

In this section, we fix a complex structure $J$ on $M$, a Kähler class $[\omega]$, the properly $\bar{\partial}$-Hamiltonian vector field $\xi$ and the parameter $\lambda \in \mathbb{R}$.

Let $\xi$ be a properly $\bar{\partial}$-Hamiltonian vector field on a Kähler manifold $X$. Taking a $\xi$-invariant Kähler metric $\omega \in [\omega]$, we define a $\mathbb{C}$-linear functional $\mathrm{Fut}^\lambda_\xi : \mathfrak{h}_0(X) \to \mathbb{C}$ by

$$\mathrm{Fut}^\lambda_\xi(\zeta) := \int_X \hat{s}^\lambda_\xi(\omega)\theta_\zeta\, e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n. \tag{1.50}$$

Remember that

$$\hat{s}^\lambda_\xi(\omega) = (s(\omega) + \bar{\Box}\theta_\xi) + (\bar{\Box}\theta_\xi - \xi^J\theta_\xi) - \lambda\theta_\xi - \bar{s}^\lambda_\xi,$$
$$\bar{s}^\lambda_\xi = \int_X (s + \bar{\Box}\theta_\xi - \lambda\theta_\xi)e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n.$$

The following proposition proves Theorem B (1).

**Proposition 1.3.6.** The linear functional $\mathrm{Fut}^\lambda_\xi$ is independent of the choice of the $\xi$-invariant Kähler metric $\omega$ in the fixed Kähler class $[\omega]$ and of the normalization of the moment map $\theta$ (independent of the equivariant cohomology class $[\omega + \theta]$).

*Proof.* Take two $\xi$-invariant Kähler forms $\omega, \omega' \in [\omega]$ and take a smooth function $\phi$ so that $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\phi$. Put $\omega_t := \omega + t\sqrt{-1}\partial\bar{\partial}\phi$. Then the moment map $\mu^t$ with respect to $\omega_t$ with $\omega_t + \mu^t \in [\omega + \mu]$ is given by $\mu^t_\xi = \mu_\xi - t\xi^J\phi/2$. We put $\theta^t_\zeta := \theta_\zeta + t\zeta^J\phi$ for $\zeta \in \mathfrak{h}_0(X)$, which satisfies $\bar{\partial}\theta^t_\zeta = i_{\zeta^J}\omega_t$ and is complex-valued in general and becomes real-valued when $\zeta \in \mathfrak{h}_0(X,\omega)$. As we already know that $\int_X e^{\theta^t_\xi}\omega^n$ is invariant, it is sufficient to see

$$\frac{d}{dt}\int_X \hat{s}^\lambda_\xi(g_t)\theta^t_\zeta \; e^{\theta^t_\xi}\omega^n_t = 0$$

for every $t \in [0,1]$. Firstly, we compute

$$\begin{aligned}
\frac{d}{dt}\hat{s}^\lambda_\xi(\omega_t) &= \frac{d}{dt}\Big((g^{i\bar{j}}_t R^t_{i\bar{j}} - g^{i\bar{j}}_t\theta^t_{\xi,i\bar{j}}) + (-g^{i\bar{j}}_t\theta^t_{\xi,i\bar{j}} - \xi^J\theta_{t,\xi}) - \lambda\theta_{t,\xi}\Big) \\
&= -g^{i\bar{q}}_t\dot{g}^t_{p\bar{q}}g^{p\bar{j}}_t R^t_{i\bar{j}} - g^{i\bar{j}}_t(g^{k\bar{l}}\dot{g}^t_{k\bar{l}})_{i\bar{j}} + g^{i\bar{q}}_t\dot{g}^t_{p\bar{q}}g^{p\bar{j}}_t\theta^t_{\xi,i\bar{j}} - g^{i\bar{j}}_t(\xi^J\phi)_{i\bar{j}} \\
&\quad + g^{i\bar{q}}_t\dot{g}^t_{p\bar{q}}g^{p\bar{j}}\theta^t_{\xi,i\bar{j}} - g^{i\bar{j}}(\xi^J\phi)_{i\bar{j}} - \xi^J\xi^J\phi - \lambda\xi^J\phi \\
&= -\bar{\Box}_t\bar{\Box}_t\phi - (\mathrm{Ric}(\omega_t) - L_{\xi^J}\omega_t, \sqrt{-1}\partial\bar{\partial}\phi) + \bar{\Box}_t\xi^J\phi \\
&\quad + (L_{\xi^J}\omega_t, \sqrt{-1}\partial\bar{\partial}\phi) + (\bar{\Box}_t - \xi^J)(\xi^J\phi) - \lambda\xi^J\phi \\
&= -((\bar{\Box}_t - \bar{\xi}^J)^2\phi + (\mathrm{Ric}(\omega_t) - L_{\xi^J}\omega_t, \sqrt{-1}\partial\bar{\partial}\phi)) - \bar{\xi}^J\bar{\Box}_t\phi - \bar{\Box}_t\bar{\xi}^J\phi + \bar{\xi}^J\bar{\xi}^J\phi \\
&\quad + \bar{\Box}_t\xi^J\phi + (L_{\xi^J}\omega_t, \sqrt{-1}\partial\bar{\partial}\phi) + (\bar{\Box}_t - \xi^J)(\xi^J\phi) - \lambda\xi^J\phi \\
&= -\mathcal{D}^{\theta *}_t\mathcal{D}_t\phi + (\bar{\partial}^\sharp s_\xi(g_t))(\phi) - \lambda\xi^J\phi \\
&\qquad\qquad - \bar{\xi}^J\bar{\Box}_t\phi + \bar{\Box}_t\xi^J\phi + (L_{\xi^J}\omega_t, \sqrt{-1}\partial\bar{\partial}\phi) \\
&= -\mathcal{D}^{\theta *}_t\mathcal{D}_t\phi + (\bar{\partial}^\sharp\hat{s}^\lambda_\xi(g_t))(\phi) = -\overline{\mathcal{D}^{\theta *}_t\mathcal{D}_t}\phi + (\partial^\sharp\hat{s}^\lambda_\xi(g_t))(\phi),
\end{aligned}$$

where we used the $\xi$-invariance of metrics for $\bar{\xi}^J\phi = \xi^J\phi$ etc. and compute the last line by

$$\begin{aligned}(L_{\xi^J}\omega_t, \sqrt{-1}\partial\bar{\partial}\phi) &= g_t^{i\bar{l}} g_t^{k\bar{j}} \theta^t_{\xi,i\bar{j}} \phi_{k\bar{l}} \\ &= g_t^{i\bar{l}}(\xi^k\phi_{k\bar{l}})_i - g_t^{i\bar{l}} g_{t,i}^{k\bar{j}} \theta^t_{\xi,\bar{j}}\phi_{k\bar{l}} - g_t^{i\bar{l}}\xi^k\phi_{k\bar{l}i} \\ &= g_t^{i\bar{l}}((\xi^k\phi_k)_{\bar{l}})_i - g_{t,p}^{k\bar{l}} g_t^{p\bar{j}}\theta^t_{\xi,\bar{j}}\phi_{k\bar{l}} - \xi^k(g_t^{i\bar{l}}\phi_{i\bar{l}})_k + g_t^{k\bar{j}}\theta^t_{\xi,\bar{j}} g_{t,k}^{i\bar{l}}\phi_{i\bar{l}} \\ &= -\bar{\Box}\xi^J\phi + \xi^J\bar{\Box}\phi.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{d}{dt}\int_X \hat{s}^\lambda_\xi(g_t)\theta^t_\zeta\ e^{\theta^t_\xi}\omega^n_t &= \int_X -\overline{\mathcal{D}_t^{\theta*}\mathcal{D}_t}\phi\ \theta^t_\zeta e^{\theta^t_\xi}\omega^n_t + \int_X (\partial^\sharp \hat{s}^\lambda_\xi(g_t))(\phi)\ \theta^t_\zeta e^{\theta^t_\xi}\omega^n_t \\ &\quad + \int_X \hat{s}^\lambda_\xi(g_t)\zeta^J\phi\ e^{\theta^t_\xi}\omega^n_t - \int_X \hat{s}^\lambda_\xi(g_t)\theta^t_\zeta(\Box_t - \xi^J)(\phi)e^{\theta^t_\xi}\omega^n_t \\ &= -\int_X \phi\ \mathcal{D}_t^{\theta*}\mathcal{D}_t\theta^t_\zeta e^{\theta^t_\xi}\omega^n_t + \int_X (\partial^\sharp \hat{s}^\lambda_\xi(g_t))(\phi)\ \theta^t_\zeta e^{\theta^t_\xi}\omega^n_t \\ &\quad + \int_X \hat{s}^\lambda_\xi(g_t)\zeta^J\phi\ e^{\theta^t_\xi}\omega^n_t - \int_X \partial^\sharp(\hat{s}^\lambda_\xi(g_t)\theta^t_\zeta)(\phi)e^{\theta^t_\xi}\omega^n_t \\ &= -\int_X \phi\ \mathcal{D}_t^{\theta*}\mathcal{D}_t\theta^t_\zeta e^{\theta^t_\xi}\omega^n_t\end{aligned}$$

and the last term vanishes as $\zeta \in \mathfrak{h}_0(X)$. $\Box$

By the definition of $\mathrm{Fut}^\lambda_\xi$, if there is a $\mu^\lambda_\xi$-cscK metric in the Kähler class $[\omega]$, $\mathrm{Fut}^\lambda_\xi$ must vanish.

We put

$$\widetilde{\mathscr{F}}^\lambda_\xi(\zeta) := \int_X \hat{s}^\lambda_\xi(\omega)\theta_\zeta\ e^{\theta_\xi}\omega^n = \mathrm{Fut}^\lambda_\xi(\zeta)\int_X e^{\theta_\xi}\omega^n. \tag{1.51}$$

In contrast to $\mathrm{Fut}^\lambda_\xi$, $\widetilde{\mathscr{F}}^\lambda_\xi$ depends on the choice of the moment map $\theta$ while it is independent of the choice of the Kähler metric in the fixed Kähler class $[\omega]$.

When $X$ is a Fano manifold and $[\omega] = 2\pi c_1(X)$, $\mathrm{Fut}^1_\xi$ reduces to the following well-known form:

$$\mathrm{Fut}^1_\xi(\zeta) = -\int_X \zeta^J(h-\theta^v_\xi)e^{\theta^v_\xi}\omega^n = -\int_X \theta_\zeta e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n,$$

where $h$ is a Ricci potential and $\theta^v_\xi$ denotes the normalization of $\theta_\xi$ satisfying $\int_X e^{\theta^v_\xi}\omega^n = 1$. This invariant was investigated in [TZ2].

### 1.3.3 $\mu$-volume functional

Here we introduce a generalization of a functional considered in [TZ2].

Let $X$ be a compact Kähler manifold with a Hamiltonian holomorphic action of a compact Lie group $K$ and $\omega$ be a $K$-invariant Kähler form on $X$. Define the *$\mu$-volume functional* $\mathrm{Vol}^\lambda$ with respect to $\omega$ on $\mathfrak{k}$ by

$$\mathrm{Vol}^\lambda(\xi) := e^{\bar{s}^\lambda_\xi} \Big( \int_X e^{\theta_\xi} \omega^n \Big)^\lambda \tag{1.52}$$

using a real-valued Hamiltonian potential $\theta_\xi : \sqrt{-1}\bar{\partial}\theta_\xi = i_{\xi^J}\omega$. We can easily check that $\mathrm{Vol}^\lambda(\xi)$ is independent of the choice of the Hamiltonian potential. Remember again that the constant $\bar{s}^\lambda_\xi$ is given by

$$\bar{s}^\lambda_\xi = \int_X (s + \bar{\square}\theta_\xi - \lambda\theta_\xi) e^{\theta_\xi} \omega^n \Big/ \int_X e^{\theta_\xi} \omega^n.$$

As $\bar{s}^\lambda_\xi$ and $\int_X e^{\theta_\xi}\omega^n$ is independent of the choice of the $\xi$-invariant Kähler metric, the $\mu$-volume functional $\mathrm{Vol}^\lambda$ is also independent of the choice of the $K$-invariant Kähler metric in the fixed Kähler class $[\omega]$.

When $X$ is a Fano manifold and the Kähler class $[\omega]$ is equal to $2\pi c_1(X)$, we have $\bar{s}^1_\xi = n$ by (1.20) under the normalization (1.18) and thus obtain $\mathrm{Vol}^1 = e^n \int_X e^{\theta_\xi}\omega^n$, which is equivalent to the volume functional considered in [TZ2]. We can easily see the properness and the convexity of $\mathrm{Vol}^1$ in this case and thus obtain a unique critical point $\xi$ of $\mathrm{Vol}^1$, which is equivalent to $\mathrm{Fut}_\xi \equiv 0$.

In [Ino1], the author used this result in order to formulate an appropriate moduli problem for Fano manifolds admitting Kähler-Ricci solitons, which is equivalent to detect a sensible moduli stack, and to construct the moduli space of them. It is important that we have such a result for all Fano manifolds, not only for Fano manifolds admitting Kähler-Ricci solitons, as we must include 'K-semistable' manifolds in the member of the moduli stack in order to ensure the openness of the interested families in general families, which corresponds to the Artinness of the moduli stack.

#### Variational formulas

**Proposition 1.3.7.** The derivative $d_\xi \mathrm{Vol}^\lambda$ of $\mathrm{Vol}^\lambda$ at $\xi \in \mathfrak{k}$ is given by

$$(d_\xi \mathrm{Vol}^\lambda)(\zeta) = \mathrm{Vol}^\lambda(\xi) \cdot \mathrm{Fut}^\lambda_\xi(\zeta).$$

*Proof.* We calculate the derivative of $\log \mathrm{Vol}^\lambda(\xi) = \bar{s}^\lambda_\xi + \lambda \log \int_X e^{\theta_\xi} \omega^n$. We have the following basic calculations:

$$\frac{d}{dt}\Big|_{t=0} \int_X (s + \bar{\square}\theta_{\xi+t\zeta} - \lambda\theta_{\xi+t\zeta}) e^{\theta_{\xi+t\zeta}} \omega^n \tag{1.53}$$

$$= \int_X \Big( (s + \bar{\square}\theta_\xi - \lambda) + (\bar{\square}\theta_\xi - \xi^J \theta_\xi - \lambda\theta_\xi) \Big) \theta_\zeta e^{\theta_\xi} \omega^n$$

$$= \widetilde{\mathscr{F}}^\lambda_\xi(\zeta) + (\bar{s}^\lambda_\xi - \lambda) \int_X \theta_\zeta e^{\theta_\xi} \omega^n,$$

$$\frac{d}{dt}\Big|_{t=0} \int_X e^{\theta_{\xi+t\zeta}} \omega^n = \int_X \theta_\zeta e^{\theta_\xi} \omega^n. \tag{1.54}$$

It follows that

$$\frac{d}{dt}\Big|_{t=0} \bar{s}^\lambda_{\xi+t\zeta} = \mathrm{Fut}^\lambda_\xi(\zeta) - \lambda \int_X \theta_\zeta e^{\theta_\xi} \omega^n \Big/ \int_X e^{\theta_\xi} \omega^n.$$

So we obtain

$$\frac{d}{dt}\Big|_{t=0} \log \mathrm{Vol}^\lambda(\xi + t\zeta) = \frac{d}{dt}\Big|_{t=0} \Big( \bar{s}^\lambda_{\xi+t\zeta} + \lambda \log \int_X e^{\theta_{\xi+t\zeta}} \omega^n \Big)$$

$$= \mathrm{Fut}^\lambda_\xi(\zeta).$$

□

*Remark* 1.3.8. The log of the $\mu$-volume functional is given by

$$\log \mathrm{Vol}^\lambda = \int_X (s + \bar{\square}\theta^v_\xi - \lambda\theta^v_\xi) e^{\theta^v_\xi} \omega^n,$$

where we put $\theta^v_\xi := \theta_\xi - \log \int_X e^{\theta_\xi} \omega^n$ so that $\int_X e^{\theta^v_\xi} \omega^n = 1$. As we have $\int_X \bar{\square}\theta^v_\xi e^{\theta^v_\xi} \omega^n = \int_X |\bar{\partial}\theta^v_\xi|^2 e^{\theta^v_\xi} \omega^n$, this functional has the same expression with the Perelman's $W$-functional:

$$W(\omega, f, \lambda^{-1}) := \int_X (2\lambda^{-1}(s + |\bar{\partial} f|^2) + f) e^{-f} \Big( \frac{\lambda\omega}{4\pi} \Big)^n.$$

While we usually consider the $W$-functional for positive $\lambda > 0$ (and for general smooth function $f$ with the normalization $\int_X e^{-f} (\frac{\lambda\omega}{4\pi})^n = 1$), we are mainly interested in $\lambda \leq 0$ for our $\mathrm{Vol}^\lambda$ in the context of $\mu$-cscK.

Next, we exhibit the second variational formula of $\mathrm{Vol}^\lambda$. Define a smooth map $\mathrm{DVol}^\lambda : \mathfrak{k} \to \mathfrak{k}^*$ by

$$\mathrm{DVol}^\lambda(\xi) = d_\xi \mathrm{Vol}^\lambda = \mathrm{Vol}^\lambda(\xi) \cdot \mathrm{Fut}^\lambda_\xi. \tag{1.55}$$

**Proposition 1.3.9.** The derivative $d_\xi \mathrm{DVol}^\lambda : \mathfrak{k} \to \mathfrak{k}^*$ of $\mathrm{DVol}^\lambda$ at $\xi \in \mathfrak{k}$ is given by

$$\begin{aligned}
\langle d_\xi \mathrm{DVol}^\lambda(\zeta), \bullet\rangle &= \mathrm{Vol}^\lambda(\xi)^2 \cdot \mathrm{Fut}^\lambda_\xi(\zeta) \cdot \mathrm{Fut}^\lambda_\xi(\bullet) \\
&\quad - \mathrm{Vol}^\lambda(\xi) \cdot \frac{\int_X \theta_\zeta e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} \mathrm{Fut}^\lambda_\xi(\bullet) - \mathrm{Vol}^\lambda(\xi) \cdot \frac{\int_X \theta_\bullet e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} \mathrm{Fut}^\lambda_\xi(\zeta) \\
&\quad + \mathrm{Vol}^\lambda(\xi) \cdot \left( \int_X e^{\theta_\xi}\omega^n \right)^{-1} \int_X (\hat{s}^\lambda_\xi \theta_\zeta \theta_\bullet + 2\zeta^J \theta_\bullet) e^{\theta_\xi}\omega^n \\
&\quad - \lambda \mathrm{Vol}^\lambda(\xi) \cdot \left( \int_X e^{\theta_\xi}\omega^n \right)^{-1} \left( \int_X \theta_\zeta \theta_\bullet e^{\theta_\xi}\omega^n - \frac{\int_X \theta_\zeta e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} \int_X \theta_\bullet e^{\theta_\xi}\omega^n \right).
\end{aligned}$$

*Proof.* Using the first variational formula, we have

$$\begin{aligned}
\langle d_\xi \mathrm{DVol}^\lambda(\zeta), \bullet\rangle &= \mathrm{Vol}^\lambda(\xi)^2 \cdot \mathrm{Fut}^\lambda_\xi(\zeta) \cdot \mathrm{Fut}^\lambda_\xi(\bullet) - \mathrm{Vol}^\lambda(\xi) \cdot \frac{\int_X \theta_\zeta e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} \mathrm{Fut}^\lambda_\xi(\bullet) \\
&\quad + \mathrm{Vol}^\lambda(\xi) \left( \int_X e^{\theta_\xi}\omega^n \right)^{-1} \frac{d}{dt}\Big|_{t=0} \widetilde{\mathscr{F}}^\lambda_{\xi+t\zeta}(\bullet).
\end{aligned}$$

The claim follows by the following computation.

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} \widetilde{\mathscr{F}}^\lambda_{\xi+t\zeta}(\bullet) &= \frac{d}{dt}\Big|_{t=0} \int_X (s + 2\bar{\Box}\theta_{\xi+t\zeta} - (\xi+t\zeta)^J \theta_{\xi+t\zeta} - \lambda\theta_{\xi+t\zeta} - \bar{s}^\lambda_{\xi+t\zeta}) \theta_\bullet e^{\theta_{\xi+t\zeta}}\omega^n \\
&= \int_X (2\bar{\Box}\theta_\zeta - 2\xi^J\theta_\zeta - \lambda\theta_\zeta)\theta_\bullet e^{\theta_\xi}\omega^n - \frac{\widetilde{\mathscr{F}}^\lambda_\xi(\zeta) - \lambda \int_X \theta_\zeta e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} \int_X \theta_\bullet e^{\theta_\xi}\omega^n \\
&\quad + \int_X \hat{s}^\lambda_\xi \theta_\bullet \theta_\zeta e^{\theta_\xi}\omega^n \\
&= 2\int_X \zeta^J \theta_\bullet e^{\theta_\xi}\omega^n - \frac{\int_X \theta_\bullet e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} \widetilde{\mathscr{F}}^\lambda_\xi(\zeta) + \int_X \hat{s}^\lambda_\xi \theta_\bullet \theta_\zeta e^{\theta_\xi}\omega^n \\
&\quad - \lambda \left( \int_X \theta_\zeta \theta_\bullet e^{\theta_\xi}\omega^n - \frac{\int_X \theta_\zeta e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} \int_X \theta_\bullet e^{\theta_\xi}\omega^n \right).
\end{aligned}$$

∎

### Corollaries of the second variational formula

Using the second variational formula, we obtain a criterion for $\xi$ to be a local minimizer.

**Corollary 1.3.10.** Let $\omega$ be a $\mu_\xi^\lambda$-cscK metric on $X$. If $\xi$ is a local minimizer of $\mathrm{Vol}^\lambda$, then

$$\lambda \leq \frac{2\int_X |\zeta^J|_g^2 e^{\theta_\xi}\omega^n / \int_X e^{\theta_\xi}\omega^n}{\nu_\xi(\zeta)} \tag{1.56}$$

for every $\zeta \in \mathfrak{k} \setminus \{0\}$ (with $|\zeta| = 1$).

Conversely, if we have

$$\lambda < \frac{2\int_X |\zeta^J|_g^2 e^{\theta_\xi}\omega^n / \int_X e^{\theta_\xi}\omega^n}{\nu_\xi(\zeta)} \tag{1.57}$$

for every $\zeta \in \mathfrak{k}\setminus\{0\}$ (with $|\zeta| = 1$), then $\xi \in \mathfrak{k}$ is an isolated local minimizer of the functional $\mathrm{Vol}^\lambda$. (Note that $2\int_X |\zeta^J|_g^2 e^{\theta_\xi}\omega^n / \int_X e^{\theta_\xi}\omega^n$ depends on the $\mu$-cscK metric $\omega$ and so on $\lambda$. ) **Especially, $\xi$ is an isolated local minimizer when $\lambda \leq 0$.**

Moreover, let $\lambda_1$ be the first eigenvalue of the weighted $\bar{\partial}$-Laplacian $\bar{\Box}_g - \xi^J$ (restricted to the space of $\xi$-invariant real functions) with respect to the $\mu$-cscK metric $\omega$ and suppose $\lambda < 2\lambda_1$, then $\xi \in \mathfrak{k}$ is an isolated local minimizer.

*Proof.* If $\xi$ is a local minimizer, then we should have $(d/dt)^2|_{t=0}\mathrm{Vol}^\lambda(\xi + t\zeta) = d_\xi D\mathrm{Vol}^\lambda(\zeta)(\zeta) \geq 0$ for every $\zeta \neq 0$. On the other hand, if we have $(d/dt)^2|_{t=0}\mathrm{Vol}^\lambda(\xi + t\zeta) = d_\xi D\mathrm{Vol}^\lambda(\zeta)(\zeta) > 0$ for every $\zeta \neq 0$, then $\xi$ is an isolated minimizer. Then the first two claims follow by the second variational formula of $\mathrm{Vol}^\lambda$. The last statement follows by the Poincare's inequality. $\square$

Note that the origin $0 \in \mathfrak{k}$ is a critical point of $\mathrm{Vol}^\lambda$ if and only if the usual Futaki invariant Fut vanishes, which is independent of $\lambda$. So we also obtain the following corollary, which will give a non-uniqueness of critical points in the next subsection.

**Corollary 1.3.11.** Suppose $\mathrm{Fut} \equiv 0$. Then the origin $0 \in \mathfrak{k}$ is an isolated local minimizer of $\mathrm{Vol}^\lambda$ if

$$\lambda < \frac{\int_X ((s - \bar{s})\theta_\zeta^2 + 2|\zeta^J|^2)\omega^n / \int_X \omega^n}{\nu_0(\zeta)}$$

and $0 \in \mathfrak{k}$ is a local minimizer only when

$$\lambda \leq \frac{\int_X ((s-\bar{s})\theta_\zeta^2 + 2|\zeta^J|^2)\omega^n / \int_X \omega^n}{\nu_0(\zeta)}$$

for every $\zeta \in \mathfrak{k} \setminus \{0\}$, where the right hand side is independent of the choices of the Kähler metric in the fixed Kähler class and the moment map.

*Proof.* Note $\hat{s}_0^\lambda = s - \bar{s}$. The claim follows by the second variational formula. We can express $\int_X ((s-\bar{s})\theta_\zeta^2 + 2|\zeta^J|^2)\omega^n$ by the integral of equivariant closed forms as

$$\int_X ((s-\bar{s})\theta_\zeta^2 + 2|\zeta^J|^2)\omega^n = \frac{2}{n+1}\Big(\int (\mathrm{Ric}(\omega) + \bar{\square}\theta_\zeta)(\omega+\theta_\zeta)^{n+1} - \frac{\bar{s}}{n+2}\int_X (\omega+\theta_\zeta)^{n+2}\Big),$$

which proves the independence from $\omega$. As for the independence of the normalization of the moment map, it follows from $\mathrm{Fut} = \int_X (s-\bar{s})\theta\omega^n \equiv 0$. □

For a Kähler manifold $(X, [\omega])$ with $\mathrm{Fut} \equiv 0$, we put

$$\begin{aligned}\lambda_\infty(X, [\omega]) &:= \sup\{\lambda \in \mathbb{R} \mid \mathrm{Vol}^\lambda \text{ is locally minimized at the origin }\} \\ &= \min_{|\zeta|=1} \frac{\int_X ((s-\bar{s})\theta_\zeta^2 + 2|\zeta^J|^2)\omega^n}{\nu_0(\zeta)}.\end{aligned} \tag{1.58}$$

Note that for every $\lambda \leq \lambda_\infty(X, [\omega])$, $\mathrm{Vol}_{[\omega]}^\lambda$ is locally minimized at the origin. By the Poincare's inequality, we have the following lower bound:

$$\sup_{\omega \in [\omega]} (\min_X s(\omega) + 2\lambda_1(\omega)) - \bar{s} \leq \lambda_\infty(X, [\omega]), \tag{1.59}$$

where $\lambda_1(\omega)$ denotes the first eigenvalue of $\bar{\square}_\omega$.

Now suppose $X$ is a Fano manifold and there is a Kähler metric $\omega$ in a fixed Kähler class on $X$ with a lower bound on the Ricci curvature $\mathrm{Ric}(\omega) \geq \delta\omega$ for $\delta > 0$. Then by Lichnerowicz–Obata's theorem, we obtain a lower bound on the first eigenvalue $\lambda_1(\omega) \geq \frac{n}{2n-1}\delta$ (note that $\bar{\square} = \frac{1}{2}\Delta$). On the other hand, we have $s(\omega) \geq \delta n$. It follows that if the Futaki invariant of $X$ vanishes, then by (1.59) we obtain a lower bound $\lambda_\infty(X, [\omega]) \geq \delta(n + \frac{2n}{2n-1}) - \bar{s}$. So in particular, in this case, the origin $0 \in \mathfrak{k}$ is an isolated local minimizer of $\mathrm{Vol}^\lambda$ for all $\lambda \leq 0$ if $\delta > \frac{2n-1}{(2n+1)n}\bar{s}$.

As for $[\omega] = 2\pi c_1(X)$, we can explicitly compute as

$$\lambda_\infty(X, 2\pi c_1(X)) = 2. \tag{1.60}$$

This follows by the equality of equivariant classes $[\mathrm{Ric} + \bar{\Box}\theta_\zeta] = [\omega + \theta_\zeta]$ and the formula in the proof of the above corollary. We can also deduce this by using $s - \bar{s} = -\bar{\Box}h$ and $\bar{\Box}\theta_\zeta - \zeta^J h - \theta_\zeta = 0$ as in (1.19).

*Question* 1.3.12. Is $\lambda_\infty(X, [\omega])$ positive for every Kähler manifold $X$ and Kähler class $[\omega]$ with vanishing Futaki invariant?

### Properness of $\mathrm{Vol}^\lambda$

Now we show that $\mathrm{Vol}^\lambda$ is proper for general $X$, not necessarily a Fano manifold, and thus always have a critical point.

**Lemma 1.3.13.** Let $M$ be a closed manifold and $f$ be a Morse-Bott function. Normalize $f$ so that $\max f = 0$ by adding a constant and suppose $f^{-1}(0)$ is connected of codimension $k$. Then for any smooth measure $dm$, the parametrized measure $t^{k/2}e^{tf}dm$ converges to a non-zero finite measure supported on $f^{-1}(0)$ as $t$ tends to $+\infty$.

Moreover, the parametrized measure $(-1)^p t^{k/2+p} f^p e^{tf} dm$ converges to a non-zero finite measure supported on $f^{-1}(0)$ for every non-negative integer $p$.

*Proof.* On any compact set $K \subset M \setminus f^{-1}(0)$, the parametrized measure $t^{k/2}e^{tf}dm$ converges to zero in the order $o(t^{k/t}e^{-\epsilon t})$ as $f$ is smaller than some $-\epsilon < 0$ on $K$.

For a point $p$ of $f^{-1}(0)$, we can take a local coordinate of $p$ so that $f(x)$ can be written as $= -(x_1^2 + \cdots + x_k^2)$. Then we can write $t^{k/2}e^{tf}dm$ as

$$t^{k/2}e^{tf}dm = t^{k/2}e^{-t(x_1^2+\cdots+x_k^2)}m(x)dx_1\cdots dx_n$$

for a positive function $m(x)$ on this coordinate. It suffices to prove that the parametrized measure $t^{k/2}e^{-t(x_1^2+\cdots x_k^2)}dx_1\cdots dx_n$ converges to a non-zero finite measure supported on $\{x_1 = \cdots = x_k = 0\}$. As we only need to check the convergence of the integration of all the test functions of boxes, the claim follows by the Gaussian integral.

As for $(-1)^p t^{k/2+p} f^p e^{tf} dm = t^{k/2+p}(x_1^2+\cdots+x_k^2)^p e^{-t(x_1^2+\cdots+x_k^2)}m(x)dx_1\cdots dx_n$, we have

$$\int x_1^{2p_1}\cdots x_k^{2p_k}e^{-t(x_1^2+\cdots+x_k^2)}dx_1\cdots dx_k = \prod_{i=1}^k \int x_i^{2p_i}e^{-tx_i^2}dx_i$$

for $p_i$ with $p_1 + \cdots + p_k = p$. Integrating by parts, we obtain

$$\int_0^a x_i^{2p_i} e^{-tx_i^2} dx_i = -\frac{1}{2t} a^{2p_i - 1} e^{-ta^2} + \frac{2p_i - 1}{2t} \int_0^a x_i^{2p_i - 2} e^{-tx_i^2} dx_i$$
$$= \cdots = o(e^{-ta^2}) + Ct^{-p_i} \int_0^a e^{-tx_i^2} dx_i = o(e^{-ta^2}) + Ct^{-p_i} t^{-1/2}.$$

This proves the claim. □

**Proposition 1.3.14.** Let $X$ be a compact Kähler manifold and $K$ be a compact Lie group acting on $X$. The limit $\lim_{t \to \infty} t^{-1} \log \mathrm{Vol}^\lambda(t\xi)$ exists. It is moreover independent of $\lambda \in \mathbb{R}$ and is strictly positive for each $\xi \in \mathfrak{k} \setminus \{0\}$. In particular, $\mathrm{Vol}^\lambda$ is proper on $\mathfrak{k}$ for each $\lambda \in \mathbb{R}$.

*Proof.* Recall that the Hamiltonian potential $\theta_\xi$ is a Morse-Bott function with only even indices and co-indices. In particular, $\theta_\xi^{-1}(c)$ is a connected submanifold for every $c \in \mathbb{R}$ (cf. [MS]). As $\mathrm{Vol}^\lambda$ is independent of the normalization of $\theta_\xi$, we can suppose $\max \theta_\xi = 0$. Note that we have $\theta_{t\xi} = t\theta_\xi$ for $t \geq 0$ with respect to this normalization, while it is not linear on $\xi$. Let $2k$ be the real codimension of $\Sigma := \theta_\xi^{-1}(0)$.

We can write the log of the $\mu$-volume functional as

$$\log \mathrm{Vol}^\lambda(\xi) = \bar{s}^0_\xi + \lambda \log \int_X e^{\theta_\xi} \omega^n - \lambda \int_X \theta_\xi e^{\theta_\xi} \omega^n \Big/ \int_X e^{\theta_\xi} \omega^n.$$

As for the first term, we can write as

$$\bar{s}^0_{t\xi}/t = \int_X (s(x)/t) e^{t\theta_\xi} \omega^n \Big/ \int_X e^{t\theta_\xi} \omega^n + \int_X \bar{\Box} \theta_\xi e^{t\theta_\xi} \omega^n \Big/ \int_X e^{t\theta_\xi} \omega^n.$$

Since $\max_{x \in X} |s(x)/t|$ goes to 0 and $e^{t\theta_\xi}\omega^n / \int_X e^{t\theta_\xi}\omega^n$ is a probability measure for any $t$, the first term converges to zero as $t$ tends to infinity. Thanks to the above lemma, the second term converges to the integration of $\bar{\Box}\theta_\xi$ with respect to a non-zero finite measure supported on $\Sigma$. Since $\theta_\xi$ is a Morse–Bott function, the Hessian at critical points are non-degenerate to the normal direction, so that we obtain a strict positivity of $\bar{\Box}\theta_\xi$ on $\Sigma$. It follows that $\bar{s}^0_{t\xi}/t$ converges to a positive constant $\lim_{t\to\infty} \int_X \bar{\Box}\theta_\xi t^k e^{t\theta_\xi} \omega^n / \int_X t^k e^{t\theta_\xi} \omega^n = \int_\Sigma \bar{\Box}\theta_\xi dm_\infty / \int_X dm_\infty$.

It suffices to show the rest terms converge to zero as $t$ tends to infinity. Again by the above lemma, $t^k \int_X e^{t\theta_\xi} \omega^n$ converges to a positive constant, so that we have

$$t^{-1} \log \int_X e^{t\theta_\xi} \omega^n = O(t^{-1} \log t) \to 0$$

as $t \to \infty$. Similarly, we have

$$t^{-1} \int_X \theta_{t\xi} e^{\theta_{t\xi}} \omega^n \Big/ \int_X e^{\theta_{t\xi}} \omega^n = t^{-1} \int_X t^{k+1} \theta_\xi e^{t\theta_\xi} \omega^n \Big/ \int_X t^k e^{t\theta_\xi} \omega^n = O(t^{-1}) \to 0$$

as $t \to \infty$. $\square$

We obtain Theorem B (2).

**Corollary 1.3.15.** There exists a vector $\xi \in \mathfrak{k}$ for which the $\mu$-Futaki invariant $\mathrm{Fut}_\xi$ restricted to $\mathfrak{k}^c$ vanishes.

*Remark* 1.3.16. From Corollary 1.3.11 in the last subsection, we conclude that critical points of $\mathrm{Vol}^\lambda$ are **not** unique for a Kähler class $[\omega]$ with vanishing Futaki invariant $\mathrm{Fut} = 0$ and sufficiently large $\lambda$.

**Proposition 1.3.17.** For each $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{k} \backslash \{0\}$, the limit of $t^{-1}\mathrm{Fut}^\lambda_{t\xi}(t\xi) = \mathrm{Fut}^\lambda_{t\xi}(\xi)$ as $t \to \infty$ exists and is strictly positive. In particular, the functional $\xi \mapsto \mathrm{Fut}^\lambda_\xi(\xi)$ is proper on $\mathfrak{k}$ for each $\lambda \in \mathbb{R}$.

*Proof.* Remember that $\mathrm{Fut}^\lambda_{t\xi}(t\xi) = \int_X \hat{s}^\lambda_{t\xi} \theta_{t\xi} e^{\theta_{t\xi}} \omega^n / \int_X e^{\theta_{t\xi}} \omega^n$. As $t^k e^{\theta_{t\xi}} \omega^n$ converges to a positive measure, it suffices to prove that $t^{k-1} \int_X \hat{s}^\lambda_{t\xi} \theta_{t\xi} e^{\theta_{t\xi}} \omega^n$ converges to a positive constant for any $\xi \in \mathfrak{k}$ ($k$ depends on $\xi$). Similarly as before, we can suppose $\max \theta_\xi = 0$. Put $\Sigma := \theta_\xi^{-1}(0)$. We can compute as

$$\begin{aligned} t^{k-1} \int_X \hat{s}^\lambda_{t\xi} \theta_{t\xi} e^{\theta_{t\xi}} \omega^n &= t^{k+1} \int_X (s/t + \bar{\square}\theta_\xi) \theta_\xi e^{t\theta_\xi} \omega^n + t^k \int_X \bar{\square}\theta_\xi e^{t\theta_\xi} \omega^n \\ &\quad - \lambda t^{k+1} \int_X \theta_\xi^2 e^{t\theta_\xi} \omega^n - t^{k+1} (\bar{s}_{t\xi}/t - \lambda \bar{\theta}_{t\xi}/t) \int_X \theta_\xi e^{t\theta_\xi} \omega^n. \end{aligned}$$

By the above lemma, the third term and $\bar{\theta}_{t\xi}/t = \int_X \theta_\xi e^{t\theta_\xi} \omega^n / \int_X e^{t\theta_\xi} \omega^n$ converges to zero, so that the limit can be computed as the limit of

$$\int_X (s/t + \bar{\square}\theta_\xi - \bar{s}_{t\xi}/t) t^{k+1} \theta_\xi e^{t\theta_\xi} \omega^n + \int_X \bar{\square}\theta_\xi t^k e^{t\theta_\xi} \omega^n.$$

Let $d\varpi$ denote the probability measure on $\Sigma$ given as the limit of the measures $e^{t\theta_\xi} \omega^n / \int_X e^{t\theta_\xi} \omega^n = t^k e^{t\theta_\xi} \omega^n / \int_X t^k e^{t\theta_\xi} \omega^n$. Then the integrand $s/t + \bar{\square}\theta_\xi - \bar{s}_{t\xi}/t$ of the first term uniformly converges to $\bar{\square}\theta_\xi - \int_\Sigma \bar{\square}\theta_\xi d\varpi$. Again thanks to the above lemma, we have non-zero finite measures $dm'_\infty = \lim_{t\to\infty} (-1) t^{k+1} \theta_\xi e^{t\theta_\xi} \omega^n$

and $dm_\infty = \lim_{t\to\infty} t^k e^{t\theta_\xi}\omega^n$ supported on $\Sigma$. (We have $d\varpi = dm_\infty / \int_\Sigma dm_\infty$. ) It follows that the limit is given by

$$-\int_\Sigma \Big(\bar{\Box}\theta_\xi - \int_\Sigma \bar{\Box}\theta_\xi d\varpi\Big) dm'_\infty + \int_\Sigma \bar{\Box}\theta_\xi dm_\infty.$$

Since we have $\sqrt{-1}\bar{\partial}(\bar{\Box}\theta_\xi) = i_{\xi^J}\mathrm{Ric}(\omega)$, $\bar{\Box}\theta_\xi$ is constant along each connected critical manifold. It follows that $\bar{\Box}\theta_\xi$ is constant along $\Sigma$ (thanks to the connectedness of $\Sigma$, as we noted in the proof of the last proposition) and so the integrand of the first term is identically zero. So the limit is $\int_\Sigma \bar{\Box}\theta_\xi dm_\infty$, which is strictly positive as $\theta_\xi$ has a non-degenerate Hessian to the normal direction. $\Box$

Now we obtain the following expected result, which shows that critical points of $\mathrm{Vol}^\lambda$ must converge to the origin as $\lambda$ tends to $-\infty$ as we observed in subsection 1.2.2.

**Corollary 1.3.18.** The set $\{\xi \in \mathfrak{k} \mid \lambda_\xi \leq 0\}$ is compact for the functional $\lambda_\xi$ considered in section 1.2.2. As a consequence, $\{\xi \in \mathfrak{k} \mid \mathrm{Fut}^\lambda_\xi \equiv 0$ for some $\lambda \leq 0\}$ is compact.

*Proof.* It follows from

$$\{\xi \in \mathfrak{k} \mid \mathrm{Fut}^\lambda_\xi \equiv 0 \text{ for some } \lambda \leq 0\} \subset \{\xi \in \mathfrak{k} \mid \lambda_\xi \leq 0\} = \{\xi \in \mathfrak{k} \mid \mathrm{Fut}^0_\xi(\xi) \leq 0\}.$$

$\Box$

The following is a partial evidence for the uniqueness of the candidates of $\xi$ for $\mu^\lambda$-cscK metrics.

**Corollary 1.3.19.** For each $\lambda \leq 0$, the set $\{\xi \mid \exists \omega \in [\omega]$ is a $\mu^\lambda_\xi$-cscK metric $\}$ is finite and is in the centralizer of $\mathfrak{k}$. In particular, $\mathrm{Aut}^0_\xi(X/\mathrm{Alb}) \subset \mathrm{Aut}^0(X/\mathrm{Alb})$ is a maximal reductive subgroup if there exists a $\mu^\lambda_\xi$-cscK for some $\lambda \leq 0$.

*Proof.* The set $\kappa$ of isolated local minimizers of $\mathrm{Vol}^\lambda$ with the non-degenerate Hessians is a zero dimensional compact submanifold of $\mathfrak{k}$ and thus consists of finitely many points. As we saw in the last subsection, a vector $\xi$ of a $\mu^\lambda_\xi$-cscK metric must be an element of $\kappa$ when $\lambda \leq 0$. This proves the first claim.

For each $g \in K$, we have $\mathrm{Fut}^\lambda_{g_*\xi}(\zeta) = \mathrm{Fut}^\lambda_\xi(g_*^{-1}\zeta)$. It follows that $K$ fixes the set $\kappa$ and thus $\kappa$ must be in the centralizer of $\mathfrak{k}$. We can see the maximal reductiveness of $\mathrm{Aut}^0_\xi(X/\mathrm{Alb})$ from Corollary 1.3.5 and by taking a maximal compact subgroup $K$. We already know that the properly $\bar{\partial}$-Hamiltonian vector $\xi$ must be tangent to the centralizer of a maximal compact subgroup $K$. (It is essential that $\mu$-Futaki invariant is defined on $\mathfrak{h}_0(X)$ rather than on $\mathfrak{h}_{0,\xi}(X)$ and vanishes on $\mathfrak{h}_0(X)$ rather than on the complexification $\mathfrak{k}'^c \subset \mathfrak{h}_0(X)$ of the Lie algebra $\mathfrak{k}'$ of the isometry group of the $\mu^\lambda_\xi$-cscK metric. It is a priori not evident that we can find a $K$-invariant $\mu^\lambda_\xi$-cscK for a maximal compact subgroup $K \subset \mathrm{Aut}^0(X/\mathrm{Alb})$, however, the claim indeed holds from this corollary and Corollary 1.3.5 as for $\lambda \leq 0$. ) Therefore, the subgroup $\mathrm{Aut}^0_\xi(X/\mathrm{Alb})$ contains the complexification of $K$, which is a maximal reductive subgroup of $\mathrm{Aut}^0(X/\mathrm{Alb})$. □

## 1.4 $\mu$K-energy and $\mu$K-stability

### 1.4.1 $\mu$K-energy functional

We introduce $\mu$K-energy functional and observe some fundamental properties of it.

#### Space of Kähler metrics and geodesics

Let $\omega$ be a Kähler metric on a Kähler manifold $X$ and $\xi$ be a properly $\bar{\partial}$-Hamiltonian vector field preserving $\omega$. We denote by $\mathcal{H}_{\omega,\xi}$ the space of $\xi$-invariant smooth Kähler potentials with respect to $\omega$ and $\ddot{\mathcal{H}}_{\omega,\xi}$ the space of $\xi$-invariant Kähler metrics in the fixed cohomology class $[\omega]$. Namely, we put

$$\mathcal{H}_{\omega,\xi} := \{\phi \in C^\infty_\xi(X;\mathbb{R}) \mid \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0\}, \tag{1.61}$$

$$\ddot{\mathcal{H}}_{\omega,\xi} := \{\omega_\phi \in [\omega] \mid \omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi > 0,\ \xi\phi = 0\}. \tag{1.62}$$

We consider the following Riemannian metric on $\ddot{\mathcal{H}}_{\omega,\xi}$:

$$(\psi_1, \psi_2)_\xi = \int_X \psi_1\psi_2 \, e^{\theta_\xi(\phi)}\omega_\phi^n, \tag{1.63}$$

where we identify the tangent space $T_{\omega_\phi}\ddot{\mathcal{H}}_{\omega,\xi}$ with $\{\psi \in C^\infty_\xi(X) \mid \int_X \psi\, e^{\theta_\xi(\phi)}\omega^n_\phi = 0\}$. This pairing is real-valued as $\omega_\phi$ is $\xi$-invariant.

A path in $\ddot{\mathcal{H}}_{\omega,\xi}$ corresponds to a path of $\xi$-invariant functions $\phi_t$ normalized as $\int_X \dot{\phi}_t e^{\theta_\xi(\phi_t)} \omega_{\phi_t}^n = 0$. The energy of a finite path $\{\phi_t\}_{t \in [a,b]}$ with respect to the Riemannian metric $(\cdot,\cdot)_\xi$ is given by

$$E(\phi_t) = \int_a^b \int_X |\dot{\phi}_t|^2 e^{\theta_\xi(\phi_t)} \omega_{\phi_t}^n.$$

A *geodesic* is by definition a critical point of the energy functional on the space of paths with fixed initial and terminal points. Computing the first derivative of the energy functional shows that geodesic paths precisely correspond to paths satisfying the following equation

$$\nabla_X(\ddot{\phi}_t - |\bar{\partial}\dot{\phi}_t|^2_{g_{\phi_t}}) = 0 \tag{1.64}$$

under the normalization $\int_X \dot{\phi}_t e^{\theta_\xi(\phi_t)} \omega_{\phi_t}^n = 0$. As the equation does not change by adding a function depending only on $t$, we can find a geodesic $\phi_t$ by solving the equation

$$\ddot{\varphi}_t - |\bar{\partial}\dot{\varphi}_t|^2_{g_{\varphi_t}} = 0$$

and putting $\phi_t := \varphi_t - \int_0^t dt \int_X \dot{\varphi}_t e^{\theta_\xi(\varphi_t)} \omega_{\varphi_t}^n$. Note that the geodesic equation itself does not depend on $\xi$, however, the normalization of paths does depend on $\xi$.

**$\mu$K-energy**

Define the *$\mu K$-energy* $\mathcal{M}_\xi^\lambda$ on the space $\mathcal{H}_{\omega,\xi}$ of smooth Kähler potentials by

$$\mathcal{M}_\xi^\lambda(\phi) := -\int_0^1 dt \int_X \hat{s}_\xi^\lambda(g_{\phi_t})\dot{\phi}_t \, e^{\theta_\xi(\phi_t)} \omega_{\phi_t}^n, \tag{1.65}$$

where $\phi_t$ is a path connecting $0$ and $\phi$, i.e. $\phi_0 = 0$ and $\phi_1 = \phi$. It is independent of the choice of the smooth path $\phi_t$ connecting $0$ and $\phi$. Indeed, let $\phi_{t,0}$ and $\phi_{t,1}$ be two paths connecting $0$ and $\phi$ and take an interpolating path $\phi_{t,s}$ of paths, then we can calculate as

$$
\begin{aligned}
&\frac{d}{ds}\int_0^1 dt \int_X \hat{s}^\lambda_\xi(g_{t,s})\frac{d\phi_{t,s}}{dt}e^{\theta_\xi^{t,s}}\omega_{t,s}^n \\
&\qquad = \int_0^1 dt \int_X \Big((-\mathcal{D}^{\theta*}_{t,s}\mathcal{D}_{t,s}\frac{d\phi_{t,s}}{ds} + (\bar{\partial}^\sharp \bar{s}^\lambda_\xi(g_{t,s}))\frac{d\phi_{t,s}}{ds})\frac{d\phi_{t,s}}{dt} \\
&\qquad\qquad + \hat{s}^\lambda_\xi(g_{t,s})\frac{d^2\phi_{t,s}}{dsdt} - \hat{s}^\lambda_\xi(g_{t,s})\frac{d\phi_{t,s}}{dt}(\bar{\Box}_{g_{t,s}} - \xi^J)\frac{d\phi_{t,s}}{ds}\Big)e^{\theta_\xi^{t,s}}\omega_{t,s}^n \\
&\qquad = \int_0^1 dt \int_X \Big( - (\mathcal{D}_{t,s}\frac{d\phi_{t,s}}{ds}, \mathcal{D}_{t,s}\frac{d\phi_{t,s}}{dt}) + \hat{s}^\lambda_\xi(g_{t,s})(\frac{d^2\phi_{t,s}}{dtds} - (\bar{\partial}\frac{d\phi_{t,s}}{dt}, \bar{\partial}\frac{d\phi_{t,s}}{ds}))\Big)e^{\theta_\xi^{t,s}}\omega_{t,s}^n \\
&\qquad = \int_0^1 dt \frac{d}{dt}\int_X \hat{s}^\lambda_\xi(g_{t,s})\frac{d\phi_{t,s}}{ds}e^{\theta_\xi^{t,s}}\omega_{t,s}^n \\
&\qquad = \int_X \hat{s}^\lambda_\xi(g_{1,s})\frac{d\phi_{1,s}}{ds}e^{\theta_\xi^{1,s}}\omega_{1,s}^n - \int_X \hat{s}^\lambda_\xi(g_{0,s})\frac{d\phi_{0,s}}{ds}e^{\theta_\xi^{0,s}}\omega_{0,s}^n \\
&\qquad = 0.
\end{aligned}
$$

Here the third equality follows by the symmetry of the second expression with respect to $s$ and $t$ and the last equality follows just by $(d/ds)\phi_{1,s} = (d/ds)\phi_{0,s} = 0$.

The $\mu$K-energy $\mathcal{M}^\lambda_\xi$ descends to the space of Kähler metrics $\ddot{\mathcal{H}}_{\omega,\xi}$ and the critical points of $\mathcal{M}^\lambda_\xi$ precisely correspond to $\mu^\lambda_\xi$-cscK metrics.

In the proof of the finite dimensional Kempf-Ness theorem for a moment map $\mu : X \to \mathfrak{k}^*$, we make use of the convexity of the Kempf-Ness functional $\mathfrak{k}/\mathfrak{k}_x \to \mathbb{R}$ to prove that $\mu^{-1}(0) \cap x.K^c = x.K$, which is analytically analogous to the uniqueness of ($\mu$-)cscK in a given Kähler class and geometrically corresponds to the injectivity of the map to the GIT quotient $\mu^{-1}(0)/K \to X^{ss} /\!\!/ K^c$. In order to study the uniqueness of $\mu$-cscK in the same spirit of the Kempf-Ness theorem, we should have the following result.

**Proposition 1.4.1** (Convexity along smooth geodesics)**.** The $\mu$K-energy $\mathcal{M}^\lambda_\xi$ is convex along smooth geodesics.

*Proof.* For a smooth path $\phi_t$ in $\ddot{\mathcal{H}}_{\omega,\xi}$, we compute

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{M}^\lambda_\xi(\phi_t) &= -\frac{d}{dt}\int_X \hat{s}^\lambda_\xi(g_{\phi_t})\dot{\phi}_t e^{\theta_\xi(\phi_t)}\omega^n_{\phi_t} \\ &= -\int_X \Big( -\mathcal{D}^{\theta *}_t \mathcal{D}_t\dot{\phi}_t + (\partial^\sharp \hat{s}^\lambda_\xi(g_{\phi_t}))(\dot{\phi}_t)\Big)\dot{\phi}_t e^{\theta_\xi(\phi_t)}\omega^n_{\phi_t} \\ &\quad -\int_X \hat{s}^\lambda_\xi(g_{\phi_t})\ddot{\phi}_t e^{\theta_\xi(\phi_t)}\omega^n_{\phi_t} + \int_X \hat{s}^\lambda_\xi(g_{\phi_t})\dot{\phi}_t(\bar{\Box}_t - \xi')\dot{\phi}_t e^{\theta_\xi(\phi_t)}\omega^n_{\phi_t} \\ &= \int_X |\mathcal{D}_t\dot{\phi}_t|^2_{g_t} e^{\theta_\xi(\phi_t)}\omega^n_{\phi_t} - \int_X \hat{s}^\lambda_\xi(g_{\phi_t})(\ddot{\phi}_t - |\bar{\partial}\dot{\phi}_t|^2_{g_{\phi_t}})e^{\theta_\xi(\phi_t)}\omega^n_{\phi_t}.\end{aligned}$$

It follows that for a smooth geodesic $\phi_t$, we have

$$\frac{d^2}{dt^2}\mathcal{M}^\lambda_\xi(\phi_t) = \int_X |\mathcal{D}_t\dot{\phi}_t|^2_{g_t} e^{\theta_\xi(\phi_t)}\omega^n_{\phi_t} \geq 0.$$

$\square$

**Extension to $C^{1,1}$-potentials**

We show that $\mathcal{M}^\lambda_\xi$ can be extended to the space

$$\overline{\mathcal{H}^{1,1}_{\omega,\xi}} := \{\phi \in C^{1,1}_\xi(X) \mid \omega + \sqrt{-1}\partial\bar{\partial}\phi \geq 0\}$$

of $C^{1,1}$-smooth sub-Kähler potentials, which generalizes the result of [Chen2] known as Chen-Tian's formula.

It is known by [Chen] that for any two smooth Kähler metrics there always exists a unique connecting $C^{1,1}$-smooth geodesic in $\overline{\mathcal{H}^{1,1}_{\omega,\xi}}$, where one interprets the geodesic equation as a solution of a Monge-Ampère equation on the complex manifold $X \times \{a \leq |z| \leq b\}$ with boundary. Using the $C^{1,1}$-extension of the usual K-energy, Berman and Berndtsson [BB] proves the uniqueness of cscK and extremal metrics.

**Proposition 1.4.2.** The $\mu$K-energy $\mathcal{M}^\lambda_\xi$ can be expressed as follows.

$$\begin{aligned}\mathcal{M}^\lambda_\xi(\phi) = &\int_X \log\frac{\omega_\phi^n}{\omega^n} e^{\theta_\xi(\phi)}\omega_\phi^n - n!\int_0^1 dt\int_X \dot\phi_t(\mathrm{Ric}(\omega)+\bar\Box_g\theta_\xi)e^{\omega_{\phi_t}+\theta_\xi(\phi_t)}\\
&+\bar s_\xi\int_0^1 dt\int_X \dot\phi_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n + \lambda\int_0^1 dt\Big(\int_X \theta_\xi(\phi_t)\dot\phi_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n - \bar\theta_\xi\int_X\dot\phi_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n\Big).\\
= &\int_X \log\frac{e^{\theta_\xi(\phi)}\omega_\phi^n}{e^{\theta_\xi}\omega^n} e^{\theta_\xi(\phi)}\omega_\phi^n - n!\int_0^1 dt\int_X \dot\phi_t\big((\mathrm{Ric}(\omega)-\sqrt{-1}\partial\bar\partial\theta_\xi)+(\bar\Box_g\theta_\xi-\xi^J\theta_\xi)\big)e^{\omega_{\phi_t}+\theta_\xi(\phi_t)}\\
&+\bar s_\xi\int_0^1 dt\int_X \dot\phi_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n + \lambda\int_0^1 dt\Big(\int_X \theta_\xi(\phi_t)\dot\phi_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n - \bar\theta_\xi\int_X\dot\phi_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n\Big),\end{aligned} \tag{1.66}$$

where we put $\bar\theta_\xi := \int_X \theta_\xi e^{\theta_\xi}\omega^n/\int_X e^{\theta_\xi}\omega^n$ (independent of $\phi_t$).

*Proof.* Recall the definition

$$\hat s^\lambda_\xi(g_{\phi_t}) = (s(g_{\phi_t})+\bar\Box_{\phi_t}\theta_\xi(\phi_t)) + (\bar\Box_{\phi_t}\theta_\xi(\phi_t)-\xi^J\theta_\xi(\phi_t)) - \lambda\theta_\xi(\phi_t) - (\bar s_\xi - \lambda\bar\theta_\xi).$$

Firstly, we transform $s(g_{\phi_t})$ as follows:

$$\begin{aligned}s(g_{\phi_t}) &= \mathrm{tr}_{g_{\phi_t}}(\sqrt{-1}\bar\partial\partial\log\det\omega_{\phi_t})\\
&= \bar\Box_{\phi_t}\log\frac{\omega_{\phi_t}^n}{\omega^n} + \mathrm{tr}_{g_{\phi_t}}(\sqrt{-1}\bar\partial\partial\log\det\omega)\\
&= (\bar\Box_{\phi_t}-\xi^J)\log\frac{\omega_{\phi_t}^n}{\omega^n} + \xi^J\log\frac{\omega_{\phi_t}^n}{\omega^n} + \mathrm{tr}_{g_{\phi_t}}(\mathrm{Ric}(\omega)).\end{aligned}$$

For the second term, we have

$$\xi^J\log\frac{\omega_{\phi_t}^n}{\omega^n} = \frac{\omega^n}{\omega_{\phi_t}^n}\xi^J\Big(\frac{\omega_{\phi_t}^n}{\omega^n}\Big) = \frac{L_{\xi^J}\Big(\frac{\omega_{\phi_t}^n}{\omega^n}\cdot\omega^n\Big)}{\omega_{\phi_t}^n} - \frac{\omega_{\phi_t}^n}{\omega^n}\frac{L_{\xi^J}\omega^n}{\omega_{\phi_t}^n} = -\bar\Box_{g_{\phi_t}}\theta_\xi(\phi_t)+\bar\Box_g\theta_\xi.$$

The integration of the first term yields the following entropy term

$$\begin{aligned}\int_X\Big((\bar\Box_{\phi_t}-\xi^J)\log\frac{\omega_{\phi_t}^n}{\omega^n}\Big)\dot\phi_t\ e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n &= \int_X\log\frac{\omega_{\phi_t}^n}{\omega^n}\Big((\bar\Box_{\phi_t}-\xi^J)\dot\phi_t\Big)\ e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n\\
&= -\frac{d}{dt}\Big(\int_X\log\frac{\omega_{\phi_t}^n}{\omega^n}e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n\Big) - \int_X\bar\Box_{\phi_t}\dot\phi_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n\\
&= -\frac{d}{dt}\Big(\int_X\log\frac{\omega_{\phi_t}^n}{\omega^n}e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n\Big) - \int_X\xi^J\dot\phi_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n.\end{aligned}$$

The second term of the last expression removes the following second term of the minus of the $\mu$K-energy

$$
\begin{aligned}
\int_0^1 dt \int_X (\bar{\Box}_{\phi_t}\theta_\xi(\phi_t) - \xi^J\theta_\xi(\phi_t))\dot{\phi}_t \; e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n \\
&= \int_0^1 dt \int_X (\bar{\partial}\theta_\xi(\phi_t), \bar{\partial}\dot{\phi}_t)_{g_{\phi_t}} \; e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n \\
&= \int_0^1 dt \int_X \xi^J\dot{\phi}_t \; e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n.
\end{aligned}
$$

Thus we have the following expression of the minus of the $\mu$K-energy:

$$
\begin{aligned}
-\mathcal{M}_\xi^\lambda(\phi) = -\int_X \log\frac{\omega_\phi^n}{\omega^n}e^{\theta_\xi(\phi)}\omega_\phi^n + \int_0^1 dt \int_X \Big( -\bar{\Box}_{g_{\phi_t}}\theta_\xi(\phi_t) + \bar{\Box}_g\theta_\xi + \mathrm{tr}_{g_{\phi_t}}(\mathrm{Ric}(\omega))\Big)\dot{\phi}_t \; e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n \\
+ \int_0^1 dt \int_X \bar{\Box}_{g_{\phi_t}}\theta_\xi(\phi_t)\dot{\phi}_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n - \bar{s}_\xi \int_0^1 dt \int_X \dot{\phi}_t \; e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n \\
- \lambda \int_0^1 dt \Big( \int_X \theta_\xi(\phi_t)\dot{\phi}_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n - \bar{\theta}_\xi \int_X \dot{\phi}_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n \Big)
\end{aligned}
$$

and obtain the first expression of the $\mu$K-energy by $\mathrm{tr}_{g_{\phi_t}}(\mathrm{Ric}(\omega))\omega_{\phi_t}^n = n\mathrm{Ric}(\omega)\wedge \omega_{\phi_t}^{n-1}$. The second expression follows by

$$
\begin{aligned}
\int_X \xi^J\phi e^{\theta_\xi(\phi)}\omega_\phi^n &= \int_0^1 dt \frac{d}{dt}\int_X \xi^J\phi_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n \\
&= \int_0^1 dt \int_X \xi^J\dot{\phi}_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n - \int_0^1 dt \int_X \xi^J\phi_t(\bar{\Box}_{\phi_t} - \xi^J)\dot{\phi}_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n \\
&= \int_0^1 dt \int_X (\bar{\Box}_{\phi_t} - \xi^J)(\theta_\xi(\phi_t) - \xi^J\phi_t)\dot{\phi}_t e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n \\
&= n!\int_0^1 dt \int_X (-\sqrt{-1}\partial\bar{\partial}\theta_\xi - \xi^J\theta_\xi)e^{\omega_{\phi_t}+\theta_\xi(\phi_t)},
\end{aligned}
$$

where we applied

$$
\int_X \xi^J\varphi e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n = \int_X (\bar{\partial}\theta_\xi, \bar{\partial}\varphi)e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n = \int_X (\bar{\Box}_{\phi_t}\theta_\xi(\phi_t) - \xi^J\theta_\xi(\phi_t))\varphi e^{\theta_\xi(\phi_t)}\omega_{\phi_t}^n
$$

and $\bar{\Box}_{\phi_t}\theta_\xi\omega_{\phi_t}^n = \mathrm{tr}_{g_{\phi_t}}(-\sqrt{-1}\partial\bar{\partial}\theta_\xi)\omega_{\phi_t}^n = -n\sqrt{-1}\partial\bar{\partial}\theta_\xi \wedge \omega_{\phi_t}^{n-1}$.

□

The first term in the second expression (1.66) of $\mathcal{M}_\xi^\lambda$ is known as the entropy

$$H_\mu(\nu) = \int_X \frac{d\nu}{d\mu} \log\Big(\frac{d\nu}{d\mu}\Big) d\mu$$

for the probability measures $\nu = \frac{1}{V_\xi} e^{\theta_\xi(\phi)} \omega_\phi^n, \mu = \frac{1}{V_\xi} e^{\theta_\xi} \omega^n$. Here, for general probability measures, $d\nu/d\mu$ denotes the Radon–Nykodim derivative, which is a measurable function, and the value of the function $(d\nu/d\mu)\log(d\nu/d\mu)$ is defined to be zero on which $d\nu/d\mu$ is zero. The total mass $V_\xi = \int_X e^{\theta_\xi}\omega^n = \int_X e^{\theta_\xi(\phi)}\omega_\phi^n$ is independent of the choice of $\phi$ as the Duistermaat–Heckman measure is an invariant of $[\omega + \mu]$. Applying the Jensen's inequality with respect to the convex function $\phi(t) = t\log t$ on $[0,\infty)$, we get

$$\int_X \frac{d\nu}{d\mu} \log\Big(\frac{d\nu}{d\mu}\Big) d\mu \geq \phi\Big(\int_X \frac{d\nu}{d\mu} d\mu\Big) = \phi(1) = 0.$$

For any $C^{1,1}$-smooth path of $C^{1,1}$-smooth sub-Kähler potentials $\phi_t$, the current $\omega_{\phi_t}$ is just a differential form with $L^\infty$-coefficient and $\theta_\xi(\phi_t)$ and $\dot{\phi}_t$ are Lipschitz functions on $X$. As for $\mathrm{Ric}(\omega)$, $\sqrt{-1}\partial\bar{\partial}\theta_\xi, \bar{\Box}_g\theta_\xi$ and $\xi^J\theta_\xi$, they are constructed from the initial smooth metric $\omega$, so are smooth. Thus we obtain the following corollary.

**Corollary 1.4.3** (Extension to the space of $C^{1,1}$-smooth sub-Kähler potentials)**.** The $\mu$K-energy $\mathcal{M}_\xi^\lambda$ can be uniquely extended to the space $\overline{\mathcal{H}_{\omega,\xi}^{1,1}}$ of $C^{1,1}$-smooth sub-Kähler potentials so that $\mathcal{M}_\xi^\lambda - \mathcal{H}_\xi$ is continuous, where $\mathcal{H}_\xi$ is the lower-semi continuous function

$$\mathcal{H}_\xi(\phi) = \int_X \log \frac{e^{\theta_\xi(\phi)}\omega_\phi^n}{e^{\theta_\xi}\omega^n} e^{\theta_\xi(\phi)}\omega_\phi^n$$

on $\overline{\mathcal{H}_{\omega,\xi}^{1,1}}$.

### 1.4.2 A prelude to $\mu$K-stability

In this section, we discuss on '$\mu$K-stability' which should fit into the existence problem on $\mu$-cscK metrics.

For a geodesic ray $\phi : [0,\infty) \to \overline{\mathcal{H}_{\omega,\xi}^{1,1}}$, we put

$$M_\xi^{\lambda,\mathrm{NA}}(\phi) := \liminf_{t\to\infty} \frac{\mathcal{M}_\xi^\lambda(\phi_t)}{t},$$

which might take the value $\infty$ for a general geodesic.

For a vector $\zeta \in \mathfrak{h}_0(X)$, the following path $\phi_t$ gives a smooth geodesic:

$$\sqrt{-1}\partial\bar{\partial}\phi_t = f_t^*\omega - \omega, \quad \int_X \dot{\phi}_t e^{\theta_\xi(\phi_t)} (f_t^*\omega)^n = 0,$$

where $f_t$ is the one parameter subgroup generated by the vector field $J\zeta$. As for this geodesic ray $\phi$, we can easily see that $M_\xi^{\lambda,NA}(\phi)$ exists along this ray and is nothing but the $\mu$-Fuatki invariant $-\mathrm{Fut}_\xi^\lambda(\zeta)$.

If the $\mu$K-energy is bounded from below, then we must have $M_\xi^{\lambda,NA}(\phi) \geq 0$. The most naive and pretty analytic formulation of $\mu$K-stability is that we call a quadruple $(X, [\omega], \xi, \lambda)$ *$\mu$K-semistable* (with respect to geodesics) if we have $M_\xi^{\lambda,NA}(\phi) \geq 0$ for all geodesics $\phi$ and call it *$\mu$K-polystable* (with respect to geodesics) if it is $\mu$K-semistable and we have $M_\xi^{\lambda,NA}(\phi) = 0$ iff $\phi$ is a geodesic given by a vector $\zeta \in \mathfrak{h}_0(X)$. Then we conjecture there exists a $\mu_\xi^\lambda$-cscK metric in the Kähler class $[\omega]$ if and only if the quadruple $(X, [\omega], \xi, \lambda)$ is $\mu$K-polystable. (cf. [Lah, Theorem 7])

Of course, it is desirable that we can reformulate this quite naive $\mu$K-stability notion to fit into a more algebraic formalism. Namely, we should

- exhibit $M_\xi^{\lambda,NA}(\phi)$ for a geodesic $\phi$ associated to a test configuration by an equivariant intersection formula using the equivariant polarization $\mathcal{L}$ and the equivariant relative canonical sheaf $\omega_{\bar{\mathcal{X}}/\mathbb{P}^1}$ of the compactified test configuration. (cf. [Lah] and [Ino3])

- detect the candidate vector $\xi$ for the solution of the $\mu$-cscK equation uniquely in a torus action, in order to formulate a sensible notion of families of $\mu$K-semistable $T$-varieties enjoying the separation property.

The detection of the candidate (called *K-optimal* in [Ino1]) vector $\xi$ follows from, for instance, the uniqueness of local minimizers of $\mathrm{Vol}^\lambda$ when $\lambda \leq 0$. If this is the case, we can formulate the $\mu^\lambda$K-stability for a $T$-equivariant polarized manifold $(X, [\omega])$ by using the local minimizer $\xi$ of $\mathrm{Vol}^\lambda$. It is interesting to ask if there is a wall-crossing phenomena, namely, if the $\mu^\lambda$K-stability of $(X, [\omega])$ with a torus action jumps at some $\lambda \leq 0$. We will see in the next section the behavior of the existence of the $\mu^\lambda$-cscK metric when perturbing $\lambda$.

## 1.5 Perturbation and propagation

### 1.5.1 Perturbation of Kähler class and $\lambda$

#### Regularity

We firstly check an elliptic regularity for constant $\mu$-scalar curvature Kähler metric. Remember that the $\mu$-scalar curvature of a Kähler metric $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ can be written as

$$s_\xi(\omega_\phi) = (\bar{\square}_\phi - \xi^J)\Big(\log(e^{\theta_\xi(\phi)}\det(g_{k\bar{l}} + \phi_{k\bar{l}}))\Big) + \sum_{i=1}^n \partial_i \xi^i.$$

Using this, the equation of constant $\mu$-scalar curvature

$$s_\xi(\omega + \sqrt{-1}\partial\bar{\partial}\phi) - \lambda\theta_\xi(\phi) = \bar{s}_\xi - \lambda\bar{\theta}_\xi$$

reduces to the following coupled equation

$$\begin{cases} F = \log \frac{e^{\theta_\xi(\phi)}\det(g_{k\bar{l}}+\phi_{k\bar{l}})}{e^{\theta_\xi}\det g_{k\bar{l}}} \\ (\bar{\square}_\phi - \xi^J)F = \bar{s}_\xi - \lambda\bar{\theta}_\xi + \lambda\theta_\xi(\phi) - (\bar{\square}_\phi - \xi^J)\log(e^{\theta_\xi}\det g) + \sum_{i=1}^n \partial_i\xi^i. \end{cases} \tag{1.67}$$

Take a $C^\infty$-smooth initial Kähler metric $\omega$ and a $C^{2,\alpha}$-smooth function $\phi$ so that $\omega_\phi = \omega + \sqrt{-1}\partial\bar{\partial}\phi$ is a $C^{0,\alpha}$-smooth Kähler metric. Then $\theta_\xi(\phi) = \theta_\xi - \xi^J\phi$ is a $C^{1,\alpha}$-smooth function and the equation (1.67) makes sense for a $C^2$-smooth function $F$.

Suppose $F \in C^2$ satisfies the equation (1.67). By differentiating the first equation in (1.67), we obtain a local equation

$$\bar{\square}_\phi(\partial_i\phi) = \partial_i F - \partial_i(\xi^J\phi) - g_\phi^{k\bar{l}}(\partial_i g_{k\bar{l}}) + g^{k\bar{l}}(\partial_i g_{k\bar{l}}). \tag{1.68}$$

Since the right hand side of this equation is $C^{0,\alpha}$-smooth and the elliptic operator $\bar{\square}_\phi$ has $C^{0,\alpha}$-coefficients, the elliptic regularity shows that $\partial_i\phi$ should be $C^{2,\alpha}$-smooth. By taking all the derivative $\partial_i$, we obtain the $C^{3,\alpha}$-smoothness of $\phi$. Then the right hand side of the second equation in (1.67) becomes $C^{1,\alpha}$-smooth and the elliptic operator $\bar{\square}_\phi - \xi^J$ has $C^{1,\alpha}$-coefficients, so again the elliptic regularity shows that $F$ is actually $C^{3,\alpha}$-smooth. Now the bootstrapping argument shows that the function $\phi$ and $F$ must be $C^\infty$-smooth functions.

### Perturbation

Let $\omega$ be a $\mu^\lambda_\xi$-cscK metric on a compact Kähler manifold $X$. By Corollary 1.3.5, the metric $\omega$ is preserved by some maximal closed torus $T \subset \mathrm{Aut}^0_\xi(X/\mathrm{Alb})$ containing the torus generated by $\xi$. The centralizer of $T$ in $\mathrm{Aut}^0_\xi(X/\mathrm{Alb})$ is the complexified algebraic torus $T^c$. We denote by $\mathcal{H}^{1,1}(X,\mathbb{R})$ the space of harmonic real $(1,1)$-form with respect to $\Delta_g = d^*d + dd^*$ associated to the metric $g = \omega J$, i.e. $\mathcal{H}^{1,1}(X,\mathbb{R}) = \{\alpha \in \Omega^{1,1}(X,\mathbb{R}) \mid \Delta_g \alpha = 0\}$, which is isomorphic to $H^{1,1}(X,\mathbb{R})$ by the projection. The action of the maximal torus $T$ on $\mathcal{H}^{1,1}(X,\mathbb{R})$ is trivial as the action extends to the action on $H^2(X,\mathbb{R})$, which is trivial as it preserves the integral lattice and $T$ is connected, so that each $\alpha \in \mathcal{H}^{1,1}$ is $T$-invariant.

Let $\mathcal{U} \subset \mathcal{H}^{1,1}(X,\mathbb{R}) \times C^{k+4,\alpha}(X,\mathbb{R})^T$ be an open neighbourhood of the origin on which we have $\omega + \alpha + \sqrt{-1}\partial\bar{\partial}\phi > 0$. For $(\alpha,\phi) \in \mathcal{U}$, we denote by $g_{\alpha,\phi}$ the Kähler metric associated to the Kähler form $\omega_{\alpha,\phi} := \omega + \alpha + \sqrt{-1}\partial\bar{\partial}\phi$ and by $\theta^{\alpha,\phi}_\eta$ the real-valued function satisfying $\sqrt{-1}\bar{\partial}\theta^{\alpha,\phi}_\eta = i_{\eta^J}\omega_{\alpha,\phi}$ and $\int_X \theta^{\alpha,\phi}_\eta e^{\theta^{\alpha,\phi}_\xi}\omega^n_{\alpha,\phi} = 0$. (This normalization is well-defined since for any constant $c \in \mathbb{R}$ we have $\int_X \theta^{\alpha,\phi}_\eta e^{\theta^{\alpha,\phi}_\xi + c}\omega^n_{\alpha,\phi} = 0$ iff $\int_X \theta^{\alpha,\phi}_\eta e^{\theta^{\alpha,\phi}_\xi}\omega^n_{\alpha,\phi} = 0$. ) The function $\theta^{\alpha,\phi}_\eta$ linearly depends on $\eta$, so that $\theta^{\alpha,\phi}$ is a moment map with respect to $\omega_{\alpha,\phi}$. Now consider a smooth map $\mathscr{S}^\lambda_\xi : \mathbb{R} \times \mathfrak{t} \times \mathcal{U} \to C^{k,\alpha}(X,\mathbb{R})^T$ defined by

$$\begin{aligned} \mathscr{S}^\lambda_\xi(\delta,\zeta,\alpha,\phi) &= s^{\lambda+\delta}_{\xi+\zeta}(\omega + \alpha + \sqrt{-1}\partial\bar{\partial}\phi) \\ &= (s(g_{\alpha,\phi}) + \bar{\Box}_{g_{\alpha,\phi}}\theta^{\alpha,\phi}_{\xi+\zeta}) + (\bar{\Box}_{g_{\alpha,\phi}}\theta^{\alpha,\phi}_{\xi+\zeta} - (\xi+\zeta)^J\theta^{\alpha,\phi}_{\xi+\zeta}) - (\lambda+\delta)\theta^{\alpha,\phi}_{\xi+\zeta}. \end{aligned} \tag{1.69}$$

The linearization of this smooth map $\mathscr{S}^\lambda_\xi$ at $(0,0,0,0) \in \mathfrak{t} \times \mathcal{U}$ is given by

$$(0,\zeta,0,0) \mapsto 2(\bar{\Box} - \xi^J)\theta_\zeta - \lambda\theta_\zeta, \tag{1.70}$$

$$(0,0,0,\phi) \mapsto -\mathcal{D}^{\theta *}_\omega \mathcal{D}_\omega \phi + (\bar{\partial}^\sharp_\omega s^\lambda_\xi(\omega))(\phi) \tag{1.71}$$

with respect to a general $T$-invariant initial metric $\omega$, which is not necessarily a $\mu^\lambda_\xi$-cscK metric. We do not need the derivative to the directions $(\delta,0,0,0)$ and $(0,0,\alpha,0)$.

Now we show the following Theorem E.

**Theorem 1.5.1.** Let $\omega$ be a $\mu$-cscK metric on a compact Kähler manifold $X$ with respect to $\xi$ and $\lambda \in \mathbb{R}$. Suppose we have $\lambda < 2\lambda_1$ for the first eigenvalue $\lambda_1$ of the weighted $\bar{\partial}$-Laplacian $\bar{\square}_\omega - \xi^J$ restricted to the space $C^\infty(X)^T$, where $T$ is a maximal torus contained in ${}_{\mathrm{H}}\mathrm{Isom}^0_\xi(X,\omega)$. Then there exists a neighbourhood $U$ of $[\omega]$ in the Kähler cone and a positive constant $\epsilon > 0$ such that for each $\tilde{\lambda} \in (\lambda-\epsilon, \lambda+\epsilon)$, every Kähler class $[\tilde{\omega}]$ in $U$ admits a $\mu$-cscK metric $\tilde{\omega}_{\tilde{\lambda}}$ with respect to some vector $\tilde{\xi}_{\tilde{\lambda}} \in \mathfrak{t}$ and the given $\tilde{\lambda}$. The vector $\tilde{\xi}_{\tilde{\lambda}}$ is in the center of a maximal compact of $\mathrm{Aut}^0(X/\mathrm{Alb})$ when $\tilde{\lambda} \leq 0$.

*Proof.* Let $\bar{\mathscr{S}}^\lambda_\xi : \mathbb{R} \times \mathfrak{t} \times \mathcal{U} \to C^{k,\alpha}(X,\mathbb{R})^T/\mathbb{R}$ be the projection of $\mathscr{S}^\lambda_\xi$. By the implicit function theorem, it suffices to show that the derivative operator $d_0\bar{\mathscr{S}}^\lambda_\xi : \mathbb{R} \times \mathfrak{t} \times \mathcal{H}^{1,1}(X,\mathbb{R}) \times C^{k+4,\alpha}(X,\mathbb{R}) \to C^{k,\alpha}(X,\mathbb{R})/\mathbb{R}$ is Fredholm and surjective when restricted to $\{0\} \times \mathfrak{t} \times \{0\} \times C^{k+4,\alpha}(X,\mathbb{R})$.

As $\omega$ is a $\mu^\lambda_\xi$-cscK metric, we have $d_0\mathscr{S}^\lambda_\xi(0,0,0,\phi) = -\mathcal{D}^{\theta*}\mathcal{D}\phi$. Since $\mathcal{D}^{\theta*}\mathcal{D}$ is an elliptic operator and $\mathbb{R} \times \mathfrak{t} \times \mathcal{H}^{1,1}(X,\mathbb{R})$ is finite dimensional, both $d_0\mathscr{S}^\lambda_\xi$ and $d_0\bar{\mathscr{S}}^\lambda_\xi$ are Fredholm operators.

The cokernel (the $L^2(e^{\theta_\xi}\omega^n)$-orthogonal complement of the image) of the operator $-\mathcal{D}^{\theta*}\mathcal{D}$ is given by

$$
\begin{aligned}
\{\psi \in C^{k,\alpha}(X,\mathbb{R})^T \mid \int_X (\mathcal{D}^{\theta*}\mathcal{D}\phi)\psi \; e^{\theta_\xi}\omega^n = 0 \text{ for all } \phi \in C^{k+4,\alpha}(X,\mathbb{R})^T\} \\
= \{\psi \in C^{k,\alpha}(X,\mathbb{R})^T \mid \mathcal{D}\psi = 0\} = \mathbb{R} \oplus \mathfrak{t},
\end{aligned}
$$

where the last equality holds as $T$ is maximal. For each non-zero element $\theta_\zeta \in \mathfrak{t}$, which is normalized as $\int_X \theta_\zeta e^{\theta_\xi}\omega^n = 0$, we have

$$
\int_X (d_0\mathscr{S}^\lambda_\xi(0,\zeta,0,0))\theta_\zeta e^{\theta_\xi}\omega^n = \int_X (2|\bar{\partial}\theta_\zeta|^2 - \lambda\theta_\zeta^2)e^{\theta_\xi}\omega^n > 0
$$

by our assumption $\lambda < 2\lambda_1$ and the Poincare inequality. Therefore the image $d_0\mathscr{S}^\lambda_\xi(0,\zeta,0,0)$ is non-constant and the composition $D = p \circ d_0\mathscr{S}^\lambda_\xi|_{\{0\}\times\mathfrak{t}\times\{0\}\times\{0\}} : \mathfrak{t} \to \mathbb{R} \oplus \mathfrak{t}$ with the $L^2(e^{\theta_\xi}\omega^n)$-orthogonal projection $p : C^{k,\alpha}(X,\mathbb{R})^T \to \mathbb{R} \oplus \mathfrak{t}$ is injective. It follows that $\mathbb{R} \oplus \mathrm{Im}D = \mathbb{R} \oplus \mathfrak{t}$ and so $d_0\bar{\mathscr{S}}^\lambda_\xi$ is surjective when restricted to $\{0\} \times \mathfrak{t} \times \{0\} \times C^{k+4,\alpha}(X,\mathbb{R})$. $\square$

The perturbed vector $\tilde{\xi}$ is a local minimizer of $\mathrm{Vol}^{\tilde{\lambda}}_{[\tilde{\omega}]}$ by Corollary 1.3.10 in the above theorem.

*Remark* 1.5.2. As a cscK metric $\omega$ is a $\mu^\lambda_0$-cscK metric for every $\lambda \in \mathbb{R}$, we in particular obtain a $\mu^\lambda_\xi$-cscK metric for every $\lambda \leq 0$ and in every Kähler class $[\tilde{\omega}]$ in a neighbourhood $U_\lambda$ of $[\omega]$.

It is proved in [LS] that there is also a neighbourhood $U_{-\infty}$ of $[\omega]$ such that $[\tilde{\omega}]$ admits an extremal metric. Note that a $\mu^\lambda_\xi$-cscK metric (or an extremal metric) is not a cscK metric iff $[\tilde{\omega}]$ has non-trivial Futaki invariant $\mathrm{Fut}_{[\tilde{\omega}]} \neq 0$.

In the next section, we show that we can take such a neighbourhood $U_{-\infty}$ so that $U_{-\infty} \subset U_\lambda$ for every $\lambda \leq 0$.

### 1.5.2 Propagation from infinity

#### $\mu$-volume functional and Möbius bundles

Consider a funcitonal $W(\xi, \lambda) = W^\lambda(\xi) = \log(\mathrm{Vol}^\lambda(\xi)/(\int \omega^n)^\lambda) - \bar{s}$ on $\mathfrak{k} \times \mathbb{R}$. When $\kappa \to 0$, we have the limit of $\kappa^{-1} W(\kappa\eta, \kappa^{-1})$ as follows:

$$
\begin{aligned}
\kappa^{-1} W(\kappa\eta, \kappa^{-1}) &= \kappa^{-1} \left( \int_X (s + \bar{\square}\theta_{\kappa\eta}) e^{\theta_{\kappa\eta}} \omega^n \Big/ \int_X e^{\theta_{\kappa\eta}} \omega^n - \bar{s} \right) \\
&\quad - \kappa^{-2} \left( \int_X \theta_{\kappa\eta} e^{\theta_{\kappa\eta}} \omega^n \Big/ \int_X e^{\theta_{\kappa\eta}} \omega^n - \log \int_X e^{\theta_{\kappa\eta}} \omega^n \Big/ \int_X \omega^n \right) \\
&= \kappa^{-1} \left( \int_X (s + \bar{\square}\theta_{\kappa\eta}) e^{\theta_{\kappa\eta}} \omega^n \Big/ \int_X e^{\theta_{\kappa\eta}} \omega^n - \bar{s} \right) \\
&\quad - \kappa^{-1} \left( \int_X \theta_\eta e^{\theta_{\kappa\eta}} \omega^n \Big/ \int_X e^{\theta_{\kappa\eta}} \omega^n - \int_X \theta_\eta \omega^n \Big/ \int_X \omega^n \right) \\
&\quad + \kappa^{-2} \left( \log \int_X e^{\theta_{\kappa\eta}} \omega^n \Big/ \int_X \omega^n - \int_X \theta_{\kappa\eta} \omega^n \Big/ \int_X \omega^n \right) \\
&\longrightarrow \frac{d}{d\kappa}\Big|_{\kappa=0} \left( \int_X (s + \bar{\square}\theta_{\kappa\eta}) e^{\theta_{\kappa\eta}} \omega^n \Big/ \int_X e^{\theta_{\kappa\eta}} \omega^n - \int_X \theta_\eta e^{\kappa\eta} \omega^n \Big/ \int_X e^{\theta_{\kappa\eta}} \omega^n \right) \\
&\qquad + \lim_{\kappa\to 0} (2\kappa)^{-1} \left( \int_X \theta_\eta e^{\theta_{\kappa\eta}} \omega^n \Big/ \int_X e^{\theta_{\kappa\eta}} \omega^n - \int_X \theta_\eta \omega^n \Big/ \int_X \omega^n \right) \\
&= \left( \int_X \omega^n \right)^{-2} \left( \int_X s\theta_\eta \omega^n \cdot \int_X \omega^n - \int_X s\omega^n \cdot \int_X \theta_\eta \omega^n \right) \\
&\qquad - \frac{1}{2} \left( \int_X \omega^n \right)^{-2} \left( \int_X \theta_\eta^2 \omega^n \cdot \int_X \omega^n - \left( \int_X \theta_\eta \omega^n \right)^2 \right) \\
&= -\frac{1}{2} \int_X \Big( (s - \bar{s}) - (\theta_\eta - \underline{\theta}_\eta) \Big)^2 \omega^n \Big/ \int_X \omega^n + \frac{1}{2} \int_X (s - \bar{s})^2 \omega^n \Big/ \int_X \omega^n.
\end{aligned}
$$

The limit functional is nothing but $-2C(\eta)$ in subsection 1.2.2. So we get a well-defined continuous map

$$\check{W} : \mathfrak{k} \times \mathbb{R} \to \mathbb{R} : (\eta, \kappa) \mapsto \check{W}(\eta, \kappa) = \check{W}^\kappa(\eta) := \kappa^{-1} W(\kappa\eta, \kappa^{-1}).$$

The limit functional $\check{W}^0 = -2C$ is proper, concave and its unique critical point gives the extremal vector. By a similar calculus, we can easily see that this map is at least $C^2$-smooth.

**Proposition 1.5.3.** There exists a constant $\lambda_0 \in \mathbb{R}$ such that $\mathrm{Vol}^\lambda$ has a unique critical point for every $\lambda < \lambda_0$.

*Proof.* The derivative of $\check{W}^\kappa$ at $\eta \in \mathfrak{k}$ is given by $\mathrm{Fut}^{\kappa^{-1}}_{\kappa\eta}$, so the critical points of $\check{W}^\kappa$ for $\kappa \neq 0$ are precisely $\kappa^{-1}$-times that of $\mathrm{Vol}^{\kappa^{-1}}$. It suffices to show that there exists some $\kappa_0 < 0$ such that $\check{W}^\kappa$ admits a unique critical point for each $\kappa \in (\kappa_0, 0)$. As we already see, the set $K := \{\lambda\xi \in \mathfrak{k} \mid \mathrm{Fut}^\lambda_\xi \equiv 0, \lambda \leq 0\}$ is compact (moreover, $\{\lambda\xi \mid \mathrm{Fut}^\lambda_\xi \equiv 0\}$ converges to $\xi_{\mathrm{ext}}$ as $\lambda \to -\infty$). Since $\check{W}^0 = -2C$ is strictly concave, a small $C^2$-perturbation of it is again strictly concave on $K$, so that there exists $\kappa_0 < 0$ such that $\check{W}^\kappa$ has a unique critical point on $K$ for every $\kappa \in (\kappa_0, -\kappa_0)$. Thus for $\kappa \in (\kappa_0, 0)$, $\check{W}^\kappa$ has a unique critical point on $\mathfrak{k}$ as there is no critical points outside $K$. $\square$

*Remark* 1.5.4. We saw in the above proof that $\check{W}^\kappa$ is strictly concave around $\xi_{\mathrm{ext}}$. On the other hand, we have proven in Proposition 1.3.14 that the slope at infinity $\lim_{t\to\infty} t^{-1}\check{W}^\kappa(t\eta) = \lim_{t\to\infty} \mathrm{sign}\kappa \cdot (t|\kappa|)^{-1} \log \mathrm{Vol}^{\kappa^{-1}}(t|\kappa|(\mathrm{sign}\kappa \cdot \eta))$ exists and its sign is that of $\kappa$ for each $\kappa \neq 0$. This in particular implies that for a positive $\kappa$ close to $0$ ($\lambda = \kappa^{-1} \gg 0$), the functional $\check{W}^\kappa$ is a 'mexican hat potential' on $\mathfrak{k}$, so that the critical points are not unique for these $\kappa > 0$ ($\lambda \gg 0$).

We can understand the relation of $W^\lambda$ and $\check{W}^\kappa$ as local indications of a map between Möbius bundles. Let $V$ be a vector space over $\mathbb{R}$. We construct a circle $S^1$ by gluing two copies of $\mathbb{R}$, which we distinguish as $\mathbb{R}_{(0)} = \mathbb{R}$ and $\mathbb{R}_{(\infty)} = \mathbb{R}$, by the diffeomorphism $\mathbb{R}_{(0)} \setminus \{0\} \xrightarrow{\sim} \mathbb{R}_{(\infty)} \setminus \{0\} : \lambda \mapsto \lambda^{-1}$ and denote by $\infty \in S^1$ the point corresponding to $0 \in \mathbb{R}_{(\infty)}$. We construct a vector bundle $\mathrm{M\ddot{o}b}(V)$ over $S^1$ by patching two copies of the trivial bundle $V \times \mathbb{R}_{(0/\infty)} \to \mathbb{R}_{(0/\infty)}$ over the charts by the isomorphism $V \times (\mathbb{R}_{(0)} \setminus \{0\}) \xrightarrow{\sim} V \times (\mathbb{R}_{(\infty)} \setminus \{0\}) : (\xi, \lambda) \mapsto (\lambda\xi, \lambda^{-1})$ of vector bundles.

We can construct a smooth map $\text{Möb}W : \text{Möb}(\mathfrak{k}) \to \text{Möb}(\mathbb{R})$ over $S^1$, which behaves non-linearly over the fibres, by patching the following two horizontal maps via the vertical gluing maps:

$$\begin{tikzcd}
\mathfrak{k} \times \mathbb{R}_{(0)} \arrow[r] \arrow[d, dashed, "(\xi,\lambda)\mapsto(\lambda\xi,\lambda^{-1})"] & \mathbb{R} \times \mathbb{R}_{(0)} \arrow[d, dashed, "(\rho,\lambda)\mapsto(\lambda\rho,\lambda^{-1})"] & (\xi, \lambda) \arrow[r, mapsto] & (W(\xi,\lambda), \lambda) \\
\mathfrak{k} \times \mathbb{R}_{(\infty)} \arrow[r] & \mathbb{R} \times \mathbb{R}_{(\infty)} & (\eta, \kappa) \arrow[r, mapsto] & (\kappa^{-1} W(\kappa\eta, \kappa^{-1}), \kappa)
\end{tikzcd}$$

The fibrewise derivative $D\text{Möb}W : \text{Möb}(\mathfrak{k}) \to \text{Hom}(\text{Möb}(\mathfrak{k}), \text{Möb}(\mathbb{R})) = \mathfrak{k}^\vee \times S^1$ of this map is given by

$$\begin{tikzcd}
\mathfrak{k} \times \mathbb{R}_{(0)} \arrow[r] \arrow[d, dashed, "(\xi,\lambda)\mapsto(\lambda\xi,\lambda^{-1})"] & \mathfrak{k}^\vee \times \mathbb{R}_{(0)} \arrow[d, dashed, "(\phi,\lambda)\mapsto(\phi,\lambda^{-1})"] & (\xi, \lambda) \arrow[r, mapsto] & (\text{Fut}^\lambda_\xi, \lambda) \\
\mathfrak{k} \times \mathbb{R}_{(\infty)} \arrow[r] & \mathfrak{k}^\vee \times \mathbb{R}_{(\infty)} & (\eta, \kappa) \arrow[r, mapsto] & (\text{Fut}^{\kappa^{-1}}_{\kappa\eta}, \kappa).
\end{tikzcd}$$

**From extremal metric to $\mu$-cscK metrics**

Consider the following for $\eta \in \mathfrak{k}$ and $\kappa \in \mathbb{R}$:

$$\check{s}^\kappa_\eta(\omega) := (s(\omega) + \bar{\Box}\theta_{\kappa\eta}) + (\bar{\Box}\theta_{\kappa\eta} - (\kappa\eta)^J\theta_{\kappa\eta}) - \theta_\eta. \tag{1.72}$$

When $\kappa = 0$, we have

$$\check{s}^0_\eta(\omega) = (s(\omega) - \theta_\eta),$$

so that $\check{s}^0_\eta(\omega)$ is constant if and only if $\omega$ is an extremal metric with respect to the vector field $\eta$. On the other hand, when $\kappa \neq 0$, we have $\check{s}^\kappa_\eta = s^{\kappa^{-1}}_{\kappa\eta}$, so that $\check{s}^\kappa_\eta(\omega)$ is constant if and only if $\omega$ is a $\mu^\lambda_\xi$-cscK metric with respect to $\lambda = \kappa^{-1}$ and $\xi = \kappa\eta$.

Let $\omega$ be an extremal metric on $X$ and $T \subset {}_{\mathrm{H}}\text{Isom}^0_\eta(X, \omega)$ be a maximal torus containing the extremal vector $\eta = \text{Im}\partial^\sharp s(\omega)$. Take an open set $\mathcal{U} \subset \mathcal{H}^{1,1}(X, \mathbb{R}) \times C^{k+4,\alpha}(X, \mathbb{R})^T$ as in section 1.5.1. We define a map $\check{\mathscr{S}}^0_\eta : \mathbb{R} \times \mathfrak{t} \times \mathcal{U} \to C^{k,\alpha}(X, \mathbb{R})^T$ by

$$\begin{aligned}
\check{\mathscr{S}}^0_\eta(\kappa, \chi, \alpha, \phi) &:= \check{s}^\kappa_{\eta+\chi}(\omega + \alpha + \sqrt{-1}\partial\bar{\partial}\phi) \\
&= (s(g_{\alpha,\phi}) + \bar{\Box}_{g_{\alpha,\phi}}\theta^{\alpha,\phi}_{\kappa(\eta+\chi)}) + (\bar{\Box}_{g_{\alpha,\phi}}\theta^{\alpha,\phi}_{\kappa(\eta+\chi)} - (\kappa(\eta+\chi))^J\theta^{\alpha,\phi}_{\kappa(\eta+\chi)}) - \theta^{\alpha,\phi}_{\eta+\chi},
\end{aligned}$$

where $(\alpha, \phi) \in \mathcal{U} \subset \mathcal{H}^{1,1}(X, \mathbb{R}) \times C^{k+4,\alpha}_\eta(X, \mathbb{R})^T$.

The linearization of this smooth map $\check{\mathscr{S}}^0_\eta$ is given by

$$(0, \chi, 0, 0) \mapsto -\theta_\chi \tag{1.73}$$

$$(0, 0, 0, \phi) \mapsto -\mathcal{D}^*\mathcal{D}\phi + (\bar{\partial}^\sharp \check{s}^0_\eta(\omega))(\phi) \tag{1.74}$$

By applying the implicit function theorem similarly to the proof of Theorem 1.5.1, we get the following theorem.

**Theorem 1.5.5.** *Let $\omega$ be an extremal metric on a compact Kähler manifold $X$ with the extremal vector $\eta$. There exists a neighbourhood $U$ of $[\omega]$ in the Kähler cone and constants $\lambda_-, \lambda_+ \in \mathbb{R}$ such that for each $\lambda \in (-\infty, \lambda_-) \cup (\lambda_+, \infty)$, every Kähler class $[\omega]$ in $U$ admits a $\mu$-cscK metric $\omega_\lambda$ with respect to some vector $\xi_\lambda \in \mathfrak{t}$ and the given $\lambda$. The vector $\xi_\lambda$ is uniquely determined when $\lambda \ll 0$.*

# 1.6 Examples

Here we observe explicit examples of Kähler classes admitting $\mu$-cscK metrics, using the method of Calabi ansatz. While we get some expected results for $\lambda \leq 0$, we also find some strange phenomenon when $\lambda \gg 0$.

## 1.6.1 Phase transition of $\mu^\lambda$-cscK metrics on $\mathbb{C}P^1$

### $\mu$-volume functional of $\mathbb{C}P^1$

We firstly compute the $\mu$-volume functional of $\mathbb{C}P^1$. Consider a $U(1)$-action on $\mathbb{C}P^1$ given by $(z : w).t = (zt : w)$. We denote by $\eta \in \mathfrak{u}(1)$ the positive generator of the $U(1)$-action.

Let us consider the following variant of $\mu$-volume functional:

$$\boldsymbol{\mu}^\lambda(-2\xi) := -\log \frac{\mathrm{Vol}^\lambda(\xi)}{(n!e^n)^\lambda}. \tag{1.75}$$

The critical points $\boldsymbol{\mu}^\lambda$ are precisely $(-2)$-times of the critical points of $\mathrm{Vol}^\lambda$. Then since $\theta_\xi = \mu_{-2\xi}$, the functional $\boldsymbol{\mu}^\lambda$ can be expressed as the integration of $U(1)$-equivariant closed forms:

$$\boldsymbol{\mu}^\lambda = -\frac{\int_X (\mathrm{Ric}_\omega + \bar{\square}\mu) e^{\omega+\mu}}{\int_X e^{\omega+\mu}} + \lambda \frac{\int_X (\omega+\mu) e^{\omega+\mu}}{\int_X e^{\omega+\mu}} - \lambda \log \int_X e^{\omega+\mu}.$$

When $\omega \in m\pi c_1(X)$, we can normalize the moment map $\mu$ so that $[\omega + \mu] = c_1^{U(1)}(X) = \frac{m}{2}[\mathrm{Ric}_\omega + \bar{\square}\mu]$ as equivariant cohomology classes. Under this normalization, we have

$$\int_X (\mathrm{Ric}_\omega + \bar{\square}\mu) e^{\omega+\mu} = \frac{2}{m} \int_X (\omega + \mu) e^{\omega+\mu}$$

since the integration of equivariant closed form depends only on its equivariant cohomology class. Thus the functional $\boldsymbol{\mu}^\lambda$ for $[\omega] = m\pi c_1(X)$ can be expressed as follows:

$$\begin{aligned}\boldsymbol{\mu}^\lambda &= (\lambda - \frac{2}{m}) \frac{\int_X (\omega + \mu) e^{\omega+\mu}}{\int_X e^{\omega+\mu}} - \lambda \log \int_X e^{\omega+\mu} \\ &= (\lambda - \frac{2}{m}) \frac{\int_X (n + \mu) e^{\mu} \frac{\omega^n}{n!}}{\int_X e^{\mu} \frac{\omega^n}{n!}} - \lambda \log \int_X e^{\mu} \frac{\omega^n}{n!}.\end{aligned}$$

We can compute these integrals using the Duistremaat–Heckman measure $\mathrm{DH} = \mu_*(\omega^n/n!)$ on $\mathfrak{u}(1)^\vee = \mathbb{R}.\eta^\vee$.

When $X = \mathbb{C}P^1$, the Duistremaat–Heckman measure is nothing but the Lebesgue measure restricted on $[-m\pi, m\pi] \subset \mathbb{R}$. So we explicitly compute $\boldsymbol{\mu}^\lambda$ for $\mathbb{C}P^1$ as

$$\begin{aligned}\boldsymbol{\mu}^\lambda(x.\eta) &= (\lambda - \frac{2}{m}) \frac{\int_{-m\pi}^{m\pi} (1 + xt) e^{xt} dt}{\int_{-m\pi}^{m\pi} e^{xt} dt} - \lambda \log \int_{-m\pi}^{m\pi} e^{xt} dt \\ &= (\lambda - \frac{2}{m}) (\frac{m\pi x}{\tanh(m\pi x)} - 1) - \lambda \log(\frac{\sinh(m\pi x)}{m\pi x}) - \lambda \log(2\pi m).\end{aligned}$$

Put $\chi = m\pi x$. Then the derivative is given by

$$\frac{d}{d\chi} \boldsymbol{\mu}^\lambda (\frac{\chi}{m\pi}.\eta) = \frac{1}{\chi (\sinh \chi)^2} \Big( \frac{2}{m} (\chi^2 - \chi \sinh \chi \cosh \chi) - \lambda (\chi^2 - (\sinh \chi)^2) \Big).$$

As long as $\lambda \leq 4/m$, $x = 0$ is the unique critical point of $\boldsymbol{\mu}^\lambda$. However, once $\lambda$ exceeds $4/m$, $\boldsymbol{\mu}^\lambda$ yields three distinct critical points. In this case, non-zero critical points of $\boldsymbol{\mu}^\lambda$ maximizes $\boldsymbol{\mu}^\lambda$ (minimizes $\mathrm{Vol}^\lambda$), while the critical point $\xi = 0$ turns into a 'metastable/supercooled' state.

### $\mu^\lambda$-cscK metrics on $\mathbb{C}P^1$ for $\lambda > \lambda_{\mathrm{freeze}}$ which are not cscK metrics

Any $U(1)$-invariant Kähler metric on $\mathbb{C}P^1$ can be written as

$$\omega = \frac{1}{2} u''(\rho) d\rho \wedge d\theta$$

on the open set $\mathbb{C}^* \subset \mathbb{C}P^1$ for some strictly positive smooth function $u$ on $\mathbb{R}$, using the coordinate $(\rho, \theta) \in \mathbb{R} \times S^1 \mapsto (e^{\rho+\sqrt{-1}\theta} : 1)$. For this metric and $\xi = x.\eta = 2\pi x \frac{\partial}{\partial\theta} \in \mathfrak{u}(1)$, we can compute the ingredients of $\mu$-scalar curvature as follows:

$$\mathrm{Ric}(\omega) = -\frac{1}{2}(\log u'')'' d\rho \wedge d\theta, \quad s(\omega) = -(u'')^{-1}(\log u'')'',$$
$$\theta_\xi = -(2\pi x)u' + \text{const.}, \quad \bar{\Box}\theta_\xi = (u'')^{-1}(2\pi x)u''', \quad \xi'\theta_\xi = (2\pi x)^2 u''.$$

Thus we get

$$s_\xi^\lambda(\omega) = -(u'')^{-1}(\log u'')'' + 2(u'')^{-1}(2\pi x)u''' - (2\pi x)^2 u'' + \lambda(2\pi x)u'. \quad (1.76)$$

We put $I := \mathrm{Im}(u')$ and $\chi = 2\pi x$. We denote by $\rho : I \to \mathbb{R}$ the inverse map of $\tau := u' : \mathbb{R} \to I$ and put $\varphi(\tau) := u''(\rho(\tau))$. Using

$$\frac{d}{d\rho} = \frac{d\tau}{d\rho}\frac{d}{d\tau} = \varphi\frac{d}{d\tau}, \quad \frac{d^2}{d\rho^2} = \varphi\varphi'\frac{d}{d\tau} + \varphi^2\frac{d^2}{d\tau^2},$$

we can reduce $s_\xi^\lambda(\omega)$ as follows:

$$s_\xi^\lambda(\omega) = -(\frac{d}{d\tau} - \chi)^2\varphi + \lambda\chi\tau.$$

We can recover $u$ (modulo linear function) from $\varphi$ since they are related by the Legendre transform $U : I \to \mathbb{R}$ of $u$: if we put

$$U(\tau) := \rho(\tau)\tau - u(\rho(\tau)),$$

then we have $\varphi(\tau) = 1/U''(\tau)$. Thus solving the equation of $\mu_\xi^\lambda$-cscK metric on $\mathbb{C}^*$ reduces to finding a positive function $\varphi$ on $I$ which solves the equation

$$-(\frac{d}{d\tau} - \chi)^2\varphi + \lambda\chi\tau = c$$

for a constant $c$. When $\chi \neq 0$, the equation is

$$(\frac{d}{d\tau} - \chi)^2(\varphi - \frac{\lambda}{\chi}\tau - \frac{2\lambda - c}{\chi^2})$$

we can see the solution is given by

$$\varphi_\chi^\lambda(\tau) = ae^{\chi\tau} + b\tau e^{\chi\tau} + \frac{\lambda}{\chi}\tau + \frac{2\lambda - c}{\chi^2} \quad (1.77)$$

for some $a, b, c \in \mathbb{R}$.

Now we impose boundary conditions on $\varphi$ to get a metric on $\mathbb{C}P^1$. We may assume $I = (0, 2m)$ for some $m$ by adding linear function to $u$. Since $\pi u'$ gives a moment map, we have $\int_{\mathbb{C}P^1} \omega = \pi \int_I d\tau = 2\pi m$. To get a solution with $\omega \in 2\pi c_1(\mathcal{O}(1))$, we assume $m = 1$. As usual Calabi ansatz (cf. [Sze-book, Section 4.4]), we can see that the following boundary conditions on $\varphi$ asserts that the metric $\frac{1}{2}u'' d\rho \wedge d\theta$ on $\mathbb{C}^*$ extends to $\mathbb{C}P^1$:

$$\begin{gathered} \varphi(0) = 0, \quad \varphi(2) = 0, \\ \varphi'(0) = -2, \quad \varphi(2) = 2. \end{gathered}$$

If we have a solution $\varphi$ satisfying this boundary condition, then $\varphi$ is automatically positive on $I$ since $\varphi$ has at most one inflection point:

$$\varphi''(\tau) = (b\chi^2\tau + a\chi^2 + b\chi)e^{\chi\tau}.$$

By the first three boundary conditions, we must have

$$\begin{gathered} a = \frac{2\lambda \sinh\chi - 2\chi e^\chi}{\chi(\sinh\chi - \chi e^\chi)}, \quad b = \frac{(2-2\lambda)\sinh\chi}{\sinh\chi - \chi e^\chi} - \frac{\lambda}{\chi} \\ c = \frac{2\lambda\chi\sinh\chi - 2\chi^2 e^\chi}{\sinh\chi - \chi e^\chi} + 2\lambda \end{gathered}$$

We can reduce the last boundary condition $\varphi'(1) = 2$ to the following equality on $\chi$:

$$\lambda(\chi^2 - (\sinh\chi)^2) - 2(\chi^2 - \chi\sinh\chi\cosh\chi) = 0, \tag{1.78}$$

which is equivalent to $\frac{d}{d\chi}\boldsymbol{\mu}^\lambda(\chi.\eta) = 0$. From the observation in 1.6.1, it has a solution $x \neq 0$ when $\lambda > 4$. Thus we get $\mu^\lambda_\xi$-cscK metrics in the Kähler class $c_1(X)$ for a non-zero $\xi$ when $\lambda > 4\pi$.

To see the limiting behavior of these $\mu^\lambda$-cscK metrics as $\lambda \to +\infty$, we see $\lambda$ as a function on $\chi$ and observe the limit $\varphi^{\lambda(\chi)}_\chi$ as $|\chi| \to \infty$. Using $\lambda(\chi) = 2\chi + O(\chi^2 e^{-2\chi})$ from (1.78), we get $ae^{\chi\tau}, be^{\chi\tau}, (2\lambda + c)/\chi^2 \to 0$ and $\lambda/\chi \to 2$ for each $\tau \in [0, 2)$. Thus we see $\varphi^{\lambda(\chi)}_\chi \to 2\tau$ on $[0, 2)$ as $\chi \to \infty$. The metric tensor $g$ corresponding to $\varphi$ is expressed as

$$g = \frac{1}{2}\varphi(\tau)^{-1} d\tau \otimes d\tau + \frac{1}{2}\varphi(\tau) d\theta \otimes d\theta$$

on $(0, 2) \times S^1$, which we identify a metric on $\mathbb{R} \times S^1 \cong \mathbb{C}^*$ via the diffeomorphism $U'(\tau) = \int_1^\tau \varphi(\tau)^{-1} d\tau : (0, 2) \to \mathbb{R}$. Thus the limit metric is expressed

as
$$g = \frac{1}{4\tau} d\tau \otimes d\tau + \tau d\theta \otimes d\theta = dr \otimes dr + r^2 d\theta \otimes d\theta$$
on $(r, \theta) = (\sqrt{\tau}, \theta) \in (0, \sqrt{2}) \times S^1$, which is the flat disk of radius $\sqrt{2}$. Since $\varphi_\chi^{\lambda(\chi)}$ converges locally uniformly on $[0, 2)$, the diffeomorphisms $U'_\chi : [0, 2) \to [-\infty, \infty)$ converges to a smooth map $U'_\infty(\tau) = \frac{1}{2} \log \tau$, which is not a diffeomorphism onto $[-\infty, \infty)$.

### 1.6.2 $\mu$-cscK metrics on $\mathbb{P}_\Sigma(L \oplus \mathcal{O})$

**The case $\lambda \geq 0$**

Let $L$ be an ample line bundle on a curve $\Sigma$ of degree $k \geq 1$. Let $F$ be a fibre of the ruled surface $X = \mathbb{P}_\Sigma(L \oplus \mathcal{O}) \to \Sigma$ and $B$ denote the section at infinity: $B := \{(x, (0 : 1)) \mid x \in \Sigma\}$. The second cohomology $H^2(X, \mathbb{R})$ is spanned by these divisors, whose intersections are given by
$$F \cdot F = 0, \quad F \cdot B = 1, \quad B \cdot B = k.$$
The Kähler cone is given by
$$\{aF + bB \mid b > 0, \ \frac{a}{b} > -\frac{k}{2}\}.$$
Now we show the following.

**Proposition 1.6.1.** Every Kähler class in the cone $\{aF + bB \mid a, b > 0\}$ admits a $\mu^\lambda$-cscK metric for every $\lambda \geq 0$.

Since the existence of $\mu^\lambda$-cscK metric depends only on the ray of Kähler class, we may assume the Kähler class is represented by $2\pi(F + mB)$ for some $m \in (0, \infty)$.

As in [Sze-book, Section 4.4], we consider metrics of the form
$$p^* \omega_\Sigma + \sqrt{-1} \partial \bar{\partial} u \circ s$$
for a function $u : \mathbb{R} \to \mathbb{R}$, where $\omega_\Sigma$ is the Kähler–Einstein metric on $\Sigma$ with $\int_\Sigma \omega_\Sigma = 1$ and $s$ is the function $s : L \setminus \Sigma \to \mathbb{R} : z \mapsto \log |z|_h^2$ defined by a metric $h$ on $L$ with curvature $k\omega_\Sigma$. Taking a local trivialization $w$ of $L$ around $z_0 \in \Sigma$ so that $(\partial h / \partial z)(z_0) = 0$, we have
$$\omega = (1 - ku')\omega_\Sigma + u'' \sqrt{-1} \frac{dw \wedge d\bar{w}}{|w|^2}$$

on the fibre $L_{z_0}$. Thus the metric is positive iff $1 - ku' > 0$ and $u'' > 0$. Since $\int_F \omega = 2\pi(u'(\infty) - u'(-\infty))$ and $\int_B \omega = 2\pi(1 - ku'(-\infty))$, we have

$$[\omega] = 2\pi((1 - ku'(\infty))F + (u'(\infty) - u'(-\infty))B).$$

When $u'(\infty) = 0$ and $u'(-\infty) = -m$, we have $[\omega] = 2\pi(F + mB)$.

We put $\tau := u' : \mathbb{R} \to (-m, 0)$ and denote by $s : (-m, 0) \to \mathbb{R}$ its inverse map. Using the function $\varphi(\tau) := u''(s(\tau))$ and $l_g := 2 - 2g$, we can express

$$\begin{aligned}
s(\omega) &= -\frac{1}{1-k\tau}((1-k\tau)\varphi)'' + \frac{l_g}{1-k\tau}, \\
\theta_\xi &= -4\pi x\tau, \\
\bar{\square}\theta_\xi &= -4\pi x\frac{k\varphi - (1-k\tau)\varphi'}{1-k\tau}, \\
\xi'\theta_\xi &= (-4\pi x)^2\varphi
\end{aligned}$$

for $\xi = 2\pi x\partial/\partial\theta$. Thus we have

$$s^\lambda_\xi(\omega) = -\frac{1}{1-k\tau}(\frac{d}{d\tau} - 4\pi x)^2((1-k\tau)\varphi) + 4\pi x\lambda\tau + \frac{l_g}{1-k\tau}.$$

Putting $\chi = 4\pi x$, the problem of the existence of $\mu^\lambda$-cscK metric under the Calabi ansatz reduces to solving the following equation

$$(\frac{d}{d\tau} - \chi)^2((1-k\tau)\varphi) = -\chi\lambda k\tau^2 + (\chi\lambda + kc)\tau + (l_g - c) \tag{1.79}$$

on $[-m, 0]$ together with the following boundary conditions (cf. [Sze-book, Section 4.4]):

$$\begin{aligned}
\varphi(0) = 0, &\quad \varphi(-m) = 0, \\
\varphi'(0) = -1, &\quad \varphi'(-m) = 1.
\end{aligned} \tag{1.80}$$

The solutions of (1.79) is given by

$$\varphi(\tau) = \frac{1}{1-k\tau}\Big(ae^{\chi\tau} + b\tau e^{\chi\tau} - \frac{\lambda k}{\chi}\tau^2 + \frac{\lambda\chi + kc - 4\lambda k}{x^2}\tau + \frac{(2\lambda + l_g - c)\chi + 2kc - 6\lambda k}{\chi^3}\Big)$$

Suppose we have a solution $\varphi$ with $\lambda \geq 0$, $x < 0$. Then since $((1 - k\tau)\varphi(\tau))'' = (b\chi^2\tau + a\chi^2 + 2b\chi)e^{\chi\tau} - \frac{2\lambda k}{\chi}$, $\varphi$ satisfies one of the following:

1. If $b \geq 0$, then $\varphi$ has at most inflection point.
2. If $b < 0$, then $\varphi$ may have two inflection points $a, b \in (-m, 0)$. However, $\varphi$ is convex on the intervals $(-m, a)$, $(b, 0)$ and is concave on $(a, b)$.

In both cases, $\psi(\tau) = (1 - k\tau)\varphi$ must be positive from the boundary conditions $\psi(0) = 0, \psi(-m) = 0$ and $\psi'(0) = -1, \psi(-m) = 1$.

From the first three boundary conditions, we get

$$a = c\frac{\chi - 2k}{\chi^3} + \frac{(-2\lambda - l_g)\chi + 6\lambda k}{\chi^3}, \quad b = c\frac{-\chi + k}{\chi^2} + \frac{-\chi^2 + (\lambda + l_g)\chi - 2\lambda k}{\chi^2}$$

and

$$\begin{aligned} c = &\frac{(-m\chi^3 + m(\lambda + l_g)\chi^2 + (2\lambda + l_g - 2\lambda km)\chi - 6\lambda k)e^{-m\chi}}{(m\chi^2 + (1 - mk)\chi - 2k)e^{-m\chi} - (1 + mk)\chi + 2k} \\ &+ \frac{(m^2\lambda + m\lambda)\chi^2 - (4m\lambda k + 2\lambda + l_g)\chi + 6\lambda k}{(m\chi^2 + (1 - mk)\chi - 2k)e^{-m\chi} - (1 + mk)\chi + 2k}. \end{aligned}$$

Regarding $a, b, c$ as a function on $x$, we can see that

$$\lim_{\chi \to 0} c(\chi) = \frac{6 + 3ml_g}{3m + m^2k},$$

$$\chi^3 a(\chi) = \Big(-2k\frac{6 + 3ml_g}{3m + m^2k} + 6\lambda k\Big) + \Big(-2\lambda - l_g + \frac{6 + 3ml_g}{3m + m^2k}\Big)\chi + O(\chi^2),$$

$$\chi^2 b(\chi) = \Big(k\frac{6 + 3ml_g}{3m + m^2k} - 2\lambda k\Big) + \Big(\lambda + l_g - \frac{6 + 3ml_g}{3m + m^2k}\Big)\chi + O(\chi^2)$$

around $\chi = 0$. Using this, we can see

$$\begin{aligned} \lim_{\chi \to 0} \varphi'_\chi(-m) &= \lim_{\chi \to 0} \frac{1}{1 + km}\Big(a\chi e^{-m\chi} + be^{-m\chi} - mb\chi e^{-m\chi} + \frac{2\lambda k}{\chi}m - a\chi - b - 1\Big) \\ &= \lim_{\chi \to 0} \frac{1}{1 + km}\Big(\frac{a\chi^3 + 2b\chi^2 - 2\lambda k}{\chi}\frac{e^{-m\chi} - 1}{\chi} - b\chi^2\frac{e^{-m\chi} + m\chi e^{-m\chi} - 1}{\chi^2} \\ &\qquad + 2\lambda k\frac{e^{-m\chi} + m\chi - 1}{\chi^2} - 1\Big) \\ &= \frac{1}{1 + km}\Big((l_g - \frac{6 + 3ml_g}{3m + m^2k})(-m) - (k\frac{6 + 3ml_g}{3m + m^2k} - 2\lambda k)(-\frac{m^2}{2}) \\ &\qquad + 2\lambda k(\frac{m^2}{2}) - 1\Big) \\ &= \frac{1}{2}\frac{kl_g m^2 + 4km + 6}{k^2m^2 + 4km + 3}. \end{aligned}$$

In particular, $\varphi'_\chi(-m)$ extends continuously across $\chi = 0$.

From the above explicit calculus, we can see that $\lim_{\chi\to 0}\varphi'_\chi(-m) < 1$, which is equivalent to $mk((2k - l_g)m + 4) > 0$, as we have $m > 0, k \geq 1, l_g \leq 2$. Thus if we have $\lim_{\chi\to-\infty}\varphi'_\chi(-m) = +\infty$, then there must be some $\chi \in (-\infty, 0)$ satisfying $\varphi'_\chi(-m) = 1$, which solves $\varphi$ satisfying all the boundary conditions.

As $\chi$ tends to $-\infty$, we can see the following:

$$
\begin{aligned}
c(\chi) = -\chi + &\frac{1 - mk + m\lambda + ml_g}{m} \\
&+ \frac{1}{m}\Big( - \frac{1 - mk + m\lambda + ml_g}{m}(1 - mk) + 2\lambda + l_g - 2\lambda km - 2k\Big)\chi^{-1} + O(\chi^{-2})
\end{aligned}
$$

and $\chi a(\chi) \to -1, \chi b(\chi) \to -\frac{1}{m}$ $(\chi a(\chi) - m\chi b(\chi))\chi \to 0$. Using this, we obtain

$$\varphi'_\chi(-m)\chi e^{m\chi} \to -\frac{1}{m}.$$

Thus $\varphi'_\chi(-m)$ tends to $+\infty$ as $\chi$ goes to $-\infty$. Similarly, we can also see that $\varphi'_\chi(-m) \to 0$ as $\chi \to -\infty$, but we do not use this fact as we already have $\lim_{\chi\to 0}\varphi'_\chi(-m) < 1$.

From the above observation, we get a positive solution $\varphi$ of (1.79) satisfying all the boundary conditions (1.80), which shows the existence of a $\mu^\lambda$-cscK metric in the Kähler class $2\pi(F + mB)$ for every $\lambda \geq 0$.

### Connecting Kähler–Ricci soliton and extremal metric via $\mu$-cscK metrics

Consider the case $X = \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O}(1) \oplus \mathcal{O}) = \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$. We have $K_X = \mathcal{O}_{X/\Sigma}(-2) \otimes \pi^*(K_\Sigma \otimes \det(\mathcal{O}(1) \oplus \mathcal{O})^\vee) = 2(kF - B) - l_gF - kF = -F - 2B$. It is known that there are both Kähler–Ricci soliton and extremal metric in the Kähler class $2\pi(F + 2B)$. Now we show that there exists $\mu^\lambda$-cscK metrics also for $(-\infty, 0)$ with $x < 0$, which converges to the extremal metric as $\lambda \to -\infty$ and to the $\mu^0$-cscK metric we constructed in section 1.6.1 as $\lambda \to 0$. Thus we get a continuity path of $\mu$-cscK metrics which connects the Kähler–Ricci soliton ($\mu^1$-cscK) and the extremal metric.

In this case, since $-\lambda/\chi < 0$, $(1 - \tau)\varphi(\tau)$ might have two inflection points $a, b \in (-2, 0)$ such that $(1 - \tau)\varphi(\tau)$ is concave on $(-2, a)$, $(a, 0)$ and is convex on $(a, b)$, which might make $\varphi$ negative at some point in $(-2, 0)$. If this happens, we must have $((1 - \tau)\varphi)'''(\tau_0) = 0$ at some point $\tau_0 \in (-2, 0)$.

Since $((1-\tau)\varphi)''' = (b\chi^3\tau + a\chi^3 + 3b\chi^2)e^{\chi\tau}$, we have $\tau_0 = -\frac{a(\chi)}{b(\chi)} - \frac{3}{\chi}$ for $\chi < 0$ solving $\varphi'_\chi(-2) = 1$. To see the positivity of $\varphi$ on $(-2, 0)$, it suffices to show $\tau_0 > 0$. To achieve this, we explicitly compute the following:

1. For $\chi < 0$ solving $\varphi'_\chi(-2) = 1$ with $\lambda \in (-\infty, 0)$, we have $\chi \in (-1, 0)$. Here the lower bound $-1$ is not effective. We actually have $\chi \in (-0.265..., 0)$.

2. The function $\tau_0(\chi) = -\frac{a(\chi)}{b(\chi)} - \frac{3}{\chi}$ is positive on $\chi \in (-1, 0)$ for every $\lambda < 0$.

For the second item, we explicitly write down as follows:

$$\frac{a(\chi)}{b(\chi)} + \frac{3}{\chi} = \frac{\alpha(\chi) + \lambda\beta(\chi)}{\chi(\gamma(\chi) + \lambda\delta(\chi))},$$

where

$$\begin{aligned}
\alpha(\chi) &= (-2\chi^4 + \chi^3 + 2\chi^2 - 6\chi)e^{-2\chi} + 9\chi^3 - 14\chi^2 + 6\chi, \\
\beta(\chi) &= (4\chi^2 + 6\chi - 6)e^{-2\chi} + 6\chi^3 + 5\chi^2 + 28\chi - 6, \\
\gamma(\chi) &= (-\chi^3 + 2\chi^2 - 2\chi)e^{-2\chi} + 3\chi^3 - 6\chi^2 + 2\chi, \\
\delta(\chi) &= (-\chi^2 + 4\chi - 2)e^{-2\chi} + 7\chi^2 + 8\chi - 2.
\end{aligned}$$

We can see that $\alpha(\chi), \gamma(\chi) > 0$ and $\beta(\chi), \delta(\chi) < 0$ for $\chi \in (-1, 0)$. So we have $\tau_0(\chi) > 0$ for $\chi \in (-1, 0)$. Since $\alpha(\chi)$ is not necessarily positive for more smaller $\chi < 0$, we must bound $\chi$. In fact, we have $\alpha(-1.5) < 0$.

To see that $\chi \in (-1, 0)$ for $\lambda < 0$, we observe $\lambda$ as a function on $\chi$. We can reduce $\varphi'_\chi(-m) = 1$ to the following equality:

$$\lambda = \chi\frac{(9\chi^2 - 6\chi - 2)e^{2\chi} + (-\chi^2 + 2\chi - 2)e^{-2\chi} + (-12\chi^3 + 16\chi^2 + 4\chi + 4)}{(9\chi^2 - 12\chi + 2)e^{2\chi} + (\chi^2 - 4\chi + 2)e^{-2\chi} + (-12\chi^4 + 16\chi^3 - 2\chi^2 + 16\chi - 4)}.$$

We can see that $\lambda$ is monotonically decreasing for $\chi \in (-\infty, 0)$, $\lambda(-1) > 0$ and $\lim_{\chi \to -0} \lambda(\chi) = -\infty$, thus we conclude that $\lambda(\chi) \leq 0$ implies $\chi \in (-1, 0)$.

# Chapter 2

# Equivariant calculus on $\mu$-character and $\mu$K-stability of polarized schemes

We introduce and study $\mu$K-stability of polarized schemes with respect to general test configurations as an algebro-geometric aspect of $\mu$-cscK metric we introduced in the last chapter. There are two fundamental ingredients. On one hand, we develop a framework on 'derivative of relative equivariant intersection', which we name *equivariant calculus*. As a part of equivariant calculus, we establish convergence results on some infinite series of equivariant cohomology classes given by *relative equivariant intersection*, based on a basic observation on the Cartan model of equivariant cohomology. On the other hand, we derive an equivariant character $\boldsymbol{\mu}^\lambda$ called *$\mu$-character* for equivariant family of polarized schemes from the equivariant intersection formula of the log of the $\mu$-volume functional. The derivative of the $\mu$-character not even yields $\mu$-Futaki invariant, but also produces an analogue of the equivariant first Chern class of CM line bundle, which is irrational in our general $\mu$K-stability setup. The product invites us to approach the compactification problem on the moduli space of Fano manifolds with Kähler–Ricci solitons, which we study in chapter 4 of part II.

The content is based on the article [Ino3].

## 2.1 Main results

The aim of this chapter is to justify the following definition of $\mu$-Futaki invariant by Theorem A and to unveil its GIT nature by Theorem G. We also propose an approach to the compactification and the algebraization problem of the moduli space of Fano manifolds with Kähler–Ricci solitons constructed in [Ino1], as an application of these results.

Throughout this chapter, we denote by $T$ an algebraic torus over $\mathbb{C}$ or its closed real torus and by $\mathfrak{t} = N(T) \otimes \mathbb{R}$ its real Lie algebra. All schemes and varieties are finite type over $\mathbb{C}$.

**Definition 2.1.1** ($\mu$-Futaki invariant)**.** Let $(X, L)$ be a $T$-polarized pure $n$-dimensional scheme. Fix parameters $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{t}$. For a $T$-equivariant test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, L)$, we define its $\mu_\xi^\lambda$-*Futaki invariant* by

$$\mathrm{Fut}_\xi^\lambda(\mathcal{X}, \mathcal{L}) = 4\pi \frac{\mathrm{Ev}_\xi\Big((\kappa^T_{\bar{\mathcal{X}}/\mathbb{C}P^1}.e^{\bar{\mathcal{L}}_T}) \cdot (e^{L_T}) - (\kappa^T_X.e^{L_T}) \cdot (e^{\bar{\mathcal{L}}_T})\Big)}{(\mathrm{Ev}_\xi(e^{L_T}))^2}$$
$$+ 2\lambda \left[ \frac{\mathrm{Ev}_\xi\Big((\bar{\mathcal{L}}_T.e^{\bar{\mathcal{L}}_T}) \cdot (e^{L_T}) - (L_T.e^{L_T}) \cdot (e^{\bar{\mathcal{L}}_T})\Big)}{(\mathrm{Ev}_\xi(e^{L_T}))^2} - \frac{\mathrm{Ev}_\xi(e^{\bar{\mathcal{L}}_T})}{\mathrm{Ev}_\xi(e^{L_T})} \right].$$

Here we denote

- by $\kappa^T_{\bar{\mathcal{X}}/\mathbb{C}P^1} \in H^{\mathrm{lf},T}_{2n}(\bar{\mathcal{X}}, \mathbb{Q})$ and $\kappa^T_X \in H^{\mathrm{lf},T}_{2n-2}(X, \mathbb{Q})$, the $T$-equivariant (relative) canonical classes derived from the equivariant homology todd class $\tau^T_X(\mathcal{O}_X)$. These are well-defined for general (non-reduced, non-irreducible, non-normal) schemes of finite type over $\mathbb{C}$ and are natural in view of the equivariant Grothendieck–Riemann–Roch theorem, while we do not have a general prescription associating a $T$-equivariant Weil divisor $K^T_X$ to the $T$-equivariant dualizing sheaf $\omega_X$ when $X$ is not Gorenstein in codimension one (or not normal). See section 2.4.2 for more information. We indeed deal with general schematic families in our application of Theorem G to the moduli problem.

- by $(e^{\bar{\mathcal{L}}_T}), (\bar{\mathcal{L}}_T.e^{\bar{\mathcal{L}}_T}), (\kappa^T_{\bar{\mathcal{X}}/\mathbb{C}P^1}.e^{\bar{\mathcal{L}}_T}), (e^{L_T}), (L_T.e^{L_T})$ and $(\kappa^T_X.e^{L_T})$, absolute equivariant intersections, which are a priori elements of the ring $\hat{S}\mathfrak{t}^\vee = \prod_{k=0}^\infty S^k\mathfrak{t}^\vee$ of formal power series. See the last paragraph in section 2.4.1 for the precise definition. We verify in section 2.3.1 that these elements are indeed Taylor expansions of real analytic functions on $\mathfrak{t}$, so we can regard these as elements of $C^\omega(\mathfrak{t})$.

- by $\mathrm{Ev}_\xi$, the evaluation map $\mathrm{Ev}_\xi : C^\omega(\mathfrak{t}) \to \mathbb{R}$ valued at $-2\xi \in \mathfrak{t}$. The factor $-2$ is essentially due to our convention on $\mu_\xi^\lambda$-cscK metric introduced in [Ino2]. It is the ratio of the $\bar{\partial}$-Hamiltonian potential $\theta$ to the moment map $\mu$. Namely, for $\mu_\xi$ and $\theta_\xi$ satisfying $-d\mu_\xi = i_\xi \omega$, $\sqrt{-1}\bar{\partial}\theta_\xi = i_{\xi^J}\omega$ for $\xi^J = J\xi + \sqrt{-1}\xi$, we have $\theta_\xi = -2\mu_\xi$ modulo constant.

Our $\mu$-Futaki invariant will be compared with the following established Futaki invariants (see Proposition 2.3.13):

- Donaldson–Futaki invariant: $\mathrm{Fut}_0^\lambda(\mathcal{X}/\mathbb{C}, \mathcal{L})$ is equivalent to the Donaldson–Futaki invariant $\mathrm{DF}(\mathcal{X}, \mathcal{L})$ for every test configuration $(\mathcal{X}, \mathcal{L})$ of a polarized scheme $(X, L)$.

- Modified Futaki invariant (cf. [Xio, BW]): Suppose $X$ is a $\mathbb{Q}$-Fano variety and $L = -\lambda^{-1}K_X$ for $\lambda > 0$. Then $\mathrm{Fut}_\xi^{2\pi\lambda}(\mathcal{X}, \mathcal{L})$ is equivalent to the modified Futaki invariant $\mathrm{Fut}_\xi(\mathcal{X}, \mathcal{L})$ for every test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, L)$ with $\mathcal{L} = -\lambda^{-1}K_{\mathcal{X}/\mathbb{C}}$.

Lahdili [Lah] proved the weighted K-semistability of weighted cscK manifolds with respect to smooth test configurations by establishing the slope formula and the boundedness for weighted Mabuchi functional . We also see in Proposition 2.3.13 that our definition of $\mu$-Futaki invariant is equivalent to Lahdili's definition of weighted Futaki invariant for smooth test configurations in our $\mu$-formalism.

We enhance his result to $\mu$K-semistability with respect to general test configurations.

**Theorem F.** If a smooth Kähler manifold $(X, L)$ admits a $\mu_\xi^\lambda$-cscK metric, then $(X, L)$ is $\mu_\xi^\lambda$K-semistable with respect to general test configurations. Namely, the $\mu_\xi^\lambda$-Futaki invariant is non-negative for every $T$-equivariant test configuration.

To reduce Theorem A to Lahdili's result, we establish basics on absolute equivariant intersection in section 2.3.1 and show the following fundamental lemma. The proof is reminiscent of arguments in [BHJ] and [DR] for the usual K-stability.

**Fundamental lemma** (Theorem 2.3.16)**.**

1. A $T$-polarized normal variety $(X, L)$ is $\mu^\lambda_\xi$K-semistable (resp. $\mu^\lambda_\xi$K-polystable, $\mu^\lambda_\xi$K-stable) with respect to general test configurations iff it is $\mu^\lambda_\xi$K-semistable (resp. $\mu^\lambda_\xi$K-polystable, $\mu^\lambda_\xi$K-stable) with respect to normal test configurations.

2. A $T$-polarized manifold $(X, L)$ is $\mu^\lambda_\xi$K-semistable with respect to general test configurations iff it is $\mu^\lambda_\xi$K-semistable with respect to smooth test configurations with reduced centrals fibres and ample $\mathcal{L}$.

Since the $\mu^\lambda_\xi$-Futaki invariant $\mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L})$ is continuous on $(\lambda, \xi) \in \mathbb{R} \times \mathfrak{t}$, Theorem D (1) of [Ino2] implies the following.

**Corollary.** If a smooth polarized manifold $(X, L)$ admits $\mu^\lambda$-cscK metrics for every $\lambda \ll 0$ in the Kähler class $c_1(L)$, then $(X, L)$ is relatively K-semistable with respect to general test configurations.

The Yau–Tian–Donaldson conjecture for extremal metric predicts that there exists an extremal metric in $c_1(L)$ when $(X, L)$ is relatively K-polystable. We proved in Theorem D (3) of [Ino2] that if there exists an extremal metric then there exists a $\mu^\lambda$-cscK metric for $\lambda \ll 0$ in the same Kähler class. It is natural to ask if one can show the estimate in Theorem D (2) under the relative K-polystablity, which must hold if the YTD conjecture is true.

Next, we consider a relative version. Throughout this chapter, $B$ denotes a connected smooth variety with an algebraic action of an algebraic group $G$ over $\mathbb{C}$. We always identify $H^0_G(B, \mathbb{R})$ with $\mathbb{R}$. We denote by $NS_G(B, \mathbb{R})$ the subspace of $H^2_G(B, \mathbb{R})$ spanned by *the $G$-equivariant Neron–Severi group* which consists of $G$-equivariant first Chern classes of $G$-equivariant algebraic line bundles.

**Theorem G.** Fix parameters $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{t}$. There exists an equivariant characteristic class

$$\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times G}(\mathcal{X}/B, \mathcal{L}) \in NS_G(B, \mathbb{R})$$

associated to a $T \times G$-equivariant family $(\mathcal{X}/B, \mathcal{L})$ of polarized schemes on a smooth $G$-variety $B$ with the trivial $T$-action such that it enjoys the following properties.

1. Naturality: Suppose we have a morphism $G' \to G$ of algebraic group and a $G'$-equivariant morphism $f : B' \to B$ from a smooth $G'$-variety $B'$. Let $(\mathcal{X}'/B', \mathcal{L}')$ be the $T \times G'$-equivaraint family given by the base change of $(\mathcal{X}/B, \mathcal{L})$ along $f$. Then we have $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times G'}(\mathcal{X}'/B', \mathcal{L}') = f^* \mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times G}(\mathcal{X}/B, \mathcal{L})$.

2. $\mu$-Futaki invariant: When the family $(\mathcal{X}/\mathbb{C}, \mathcal{L}) \circlearrowleft \mathbb{C}^*$ is a $T$-equivariant test configuration, then we have

$$\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times\mathbb{C}^*}(\mathcal{X}/\mathbb{C}, \mathcal{L}) = \mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}).\eta^\vee \in H^2_{\mathbb{C}^*}(\mathbb{C}, \mathbb{R}),$$

where $\eta^\vee$ denotes the *positive* generator of $H^2_{\mathbb{C}^*}(X, \mathbb{Z}) \cong \mathbb{Z}$.

3. CM line bundle: When $\xi = 0$ ($\mathfrak{t} = 0$), we have

$$\mathcal{D}_0 \boldsymbol{\mu}^\lambda_G(\mathcal{X}/B, \mathcal{L}) = -\frac{4\pi}{(L^{\cdot n})} c_1^G(\mathrm{CM}(\mathcal{X}/B, \mathcal{L}))$$

for the CM line bundle $\mathrm{CM}(\mathcal{X}/B, \mathcal{L})$, independent of $\lambda \in \mathbb{R}$. (cf. [PT])

4. Parameter: The function $\mathcal{D}_\cdot \boldsymbol{\mu}^\cdot_{T\times G}(\mathcal{X}/B, \mathcal{L}) : \mathfrak{t} \times \mathbb{R} \to H^2_G(B, \mathbb{R}) : (\xi, \lambda) \mapsto \mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times G}(\mathcal{X}/B, \mathcal{L})$ is real analytic. It is moreover affine with respect to $\lambda \in \mathbb{R}$ for each fixed $\xi \in \mathfrak{t}$.

We detect this characteristic class $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times G}(\mathcal{X}/B, \mathcal{L})$ based on the following step-wise observation:

- We observe in section 2.2.1 the localization formula on equivariant integration yields Odaka–Wang's intersection formula of Donaldson–Futaki invariant (for product configuration) directly from the differential geometric definition of Futaki invariant (for product configuration), which we can easily generalize to our $\mu$-cscK setup since the invariant comes out of Donaldson–Fujiki type moment map picture as explained in [Wang1]. We can also detect the equivariant first Chern class of the CM line bundle by the same idea.

  On the other hand, while we have a differential geometric definition of $\mu$-Futaki invariant fitting into the $\mu$-cscK setup, we can not readily derive an intersection formula of $\mu$-Futaki invariant, obstructed by the vector $\xi \in \mathfrak{t}$ turning up in the $\mu$-Futaki invariant $\mathrm{Fut}^\lambda_\xi(\eta)$ (see the equation (2.4)).

- There is a functional $\boldsymbol{\mu}^\lambda$ on $\mathfrak{t}$ (the log of the $\mu$-volume functional we introduced in [Ino2]) whose differential at $\xi \in \mathfrak{t}$ to the direction $\eta \in \mathfrak{t}$ gives the $\mu$-Futaki invariant $\mathrm{Fut}^\lambda_\xi(\eta)$. Contrast to the $\mu$-Futaki invariant, we easily find an equivariant cohomological expression of this functional. The expression enables us to interpret $\boldsymbol{\mu}^\lambda$ as a formal series of equivariant cohomology classes $\boldsymbol{\mu}^\lambda_T(X, L) \in \hat{H}^{\mathrm{even}}_T(\mathrm{pt}, \mathbb{R}) := \prod_{i=0}^\infty H^{2i}_T(\mathrm{pt}, \mathbb{R})$, which is nothing but the Taylor expansion of the functional under the identification $\hat{H}^{\mathrm{even}}_T(\mathrm{pt}, \mathbb{R}) \cong \hat{S}\mathfrak{t}^\vee := \prod_{i=0}^\infty \mathfrak{t}^\vee$. It is easy to generalize this to an equivariant characteristic class $\boldsymbol{\mu}^\lambda_G(\mathcal{X}/B, \mathcal{L}) \in \hat{H}^{\mathrm{even}}_G(B, \mathbb{R})$ for a $G$-equivariant family $(\mathcal{X}/B, \mathcal{L})$.
- We introduce 'the differential operation $\mathcal{D}$ on $\hat{H}_G(B, \mathbb{R})$' and study its basic properties. We design the concept so that the differential $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times\mathbb{C}^*}(X, L) \in H^2_{\mathbb{C}^*}(\mathrm{pt}, \mathbb{R})$ of $\boldsymbol{\mu}^\lambda_T(X, L)$ at $\xi \in \mathfrak{t}$ gives the $\mu$-Futaki invariant in the sense that we have $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times\mathbb{C}^*}(X, L) = \mathrm{Fut}^\lambda_\xi(\eta).\eta^\vee$ for the positive generator $\eta \in H^2_{\mathbb{C}^*}(\mathrm{pt}, \mathbb{Z}) \cong \mathbb{Z}$. We must show some convergence results in equivariant cohomology in order to justify the definition of the differential operation $\mathcal{D}_\xi : \hat{H}^{\mathrm{even}}_{T\times G}(B, \mathbb{R}) \to H^2_G(B, \mathbb{R})$.

By our construction of $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times G}(\mathcal{X}/B, \mathcal{L})$, the property (1) reduces to a problem on the base-change behavior of the eqiuvariant relative canonical class $\kappa_{\mathcal{X}/B}$, which is in general regarded as a problem related to singularities of families. Since we only need the base-change stability of the equivariant intersection of eqiuvariant relative canonical class with equivariant line bundle, it suffices to employ the equivariant Grothendieck–Riemann–Roch theorem by Edidin–Graham [EG2] to see the property (1).

Finally, we explain in section 4 an application of Theorem B the compactification problem on the moduli space of Fano manifolds with Kähler–Ricci solitons constructed in [Ino1].

## 2.2 Preliminaries

### 2.2.1 Brief review on $\mu$-cscK metric and $\mu$-volume functional

#### Weighted scalar curvature and $\mu$-cscK metric

We briefly explain some basic notions around $\mu$-cscK metric. The author introduced $\mu$-scalar curvature in the last chapter (cf. [Ino2]) to establish an inclusive framework of both cscK metric and Kähler–Ricci soliton based on the moment map picture observed in [Ino1]. On the other hand, Lahdili [Lah] also considered a generalization of Donaldson–Fujiki type moment map picture and introduced weighted scalar curvature as a far extensive framework, which includes $\mu$-scalar curvature. The moment map picture on weighted scalar curvature yields a version of Yau–Tian–Donaldson conjecture which states that the existence of weighted scalar curvature on a given manifold must be equivalent to a proper notion of 'weighted K-stability'. Lahdili introduced the weighted Futaki invariant for test configurations with smooth total spaces and proved that every weighted cscK manifold has non-negative weighted Futaki invariants for all smooth test configurations. Since we reduce our Theorem A to his result [Lah], we begin with his framework.

Let $X$ be a compact Kähler manifold with a Hamiltonian action of a torus $T$. Let $\omega$ be a $T$-invariant Kähler metric and $\mu : X \to \mathfrak{t}^\vee$ be a moment map. Since $X$ is compact, the moment polytope $P = \mu(X)$ (and even the measure $\mu_* \omega^n$ on $\mathfrak{t}^\vee$ supported on $P$) depends only on the equivariant cohomology class $[\omega + \mu] \in H^2_T(X, \mathbb{R})$ (cf. [GGK, Section 2.3–2.4]).

For a smooth positive function $v$ on $P$, Lahdili [Lah] defines the *weighted scalar curvature* $s_v(\omega)$ by

$$s_v(\omega) := s(\omega) \cdot (v \circ \mu^\omega) + \Delta_\omega (v \circ \mu^\omega) - \frac{1}{2} \sum_{1 \le i,j \le k} (J\xi_i) \mu^\omega_{\xi_j} \cdot (\frac{\partial^2 v}{\partial x^i \partial x^j} \circ \mu^\omega). \quad (2.1)$$

Note we follow Kählerian convention on the scalar curvature $s(\omega) = \mathrm{tr}_\omega(\mathrm{Ric}(\omega))$, so it is the half of the Riemannian scalar curvature.

When $v$ is of the form $v(x) = \tilde{v}(\langle x, \xi \rangle)$ with some smooth positive function $\tilde{v}$ on $\mathbb{R}$ and $\xi \in \mathfrak{t}$, we can simplify it as

$$\begin{aligned} s_v(\omega) &= s(\omega) \cdot (\tilde{v} \circ \mu^\omega_\xi) + \big( \Delta_\omega \mu^\omega_\xi \cdot (\tilde{v}' \circ \mu^\omega_\xi) - (\nabla \mu^\omega_\xi, \nabla \mu^\omega_\xi) \cdot (\tilde{v}'' \circ \mu^\omega_\xi) \big) - \frac{1}{2} (J\xi) \mu^\omega_\xi \cdot (\tilde{v}'' \circ \mu^\omega_\xi) \\ &= s(\omega) \cdot (\tilde{v} \circ \mu^\omega_\xi) + \Delta_\omega \mu^\omega_\xi \cdot (\tilde{v}' \circ \mu^\omega_\xi) + \frac{1}{2} (J\xi) \mu^\omega_\xi \cdot (\tilde{v}'' \circ \mu^\omega_\xi) \end{aligned}$$

In particular, when $v(x) = \tilde{v}(\langle x, -2\xi \rangle)$ with $\tilde{v} = e^t$, we derive the *$\mu$-scalar curvature* $s_\xi(\omega)$:

$$s_v(\omega) = \big( (s(\omega) + \bar{\Box} \theta_\xi) + (\bar{\Box} \theta_\xi - (J\xi) \theta_\xi) \big) e^{\theta_\xi} =: s_\xi(\omega) e^{\theta_\xi}$$

with $\theta_\xi = \mu_{-2\xi}$, which satisfies $\sqrt{-1} \bar{\partial} \theta_\xi = i_{\xi^J} \omega$ for the holomorphic vector field $\xi^J = J\xi + \sqrt{-1}\xi$.

For a real number $\lambda \in \mathbb{R}$, we put

$$s_\xi^\lambda(\omega) := s_\xi(\omega) - \lambda\theta_\xi = (s(\omega) + \bar{\Box}\theta_\xi) + (\bar{\Box}\theta_\xi - (J\xi)\theta_\xi) - \lambda\theta_\xi$$

and call it $\mu_\xi^\lambda$-scalar curvature. We call a Kähler metric $\omega$ has constant $\mu_\xi^\lambda$-scalar curvature if $s_\xi^\lambda(\omega)$ equals to a constant. When $\lambda = 0$, the constant

$$\bar{s}_\xi := \int_X s_\xi(\omega) e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n \tag{2.2}$$

depends only on the cohomology class $[\omega]$ and the parameters $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{t}$. When $\lambda \neq 0$, the constant

$$\bar{s}_\xi^\lambda := \bar{s}_\xi - \lambda \int_X \theta_\xi e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n \tag{2.3}$$

depends further on the normalization of $\mu$. Since moment maps are unique modulo constant, the notion of $\mu_\xi^\lambda$-cscK metric is independent of the choice of the moment maps.

The $\mu_\xi^\lambda$-cscK metric is equivalent to Kähler–Ricci soliton when $L = -K_X$ and $\lambda = 2\pi$ as observed in [Ino1]. The claim is as follows. Let $X$ be a Fano manifold and $\omega$ be a Kähler metric in the Kähler class $2\pi\lambda^{-1}c_1(X)$ for $\lambda > 0$. Then the Kähler metric $\omega$ satisfies the equation of Kähler–Ricci soliton $\mathrm{Ric}(\omega) - L_{J\xi}\omega = \lambda\omega$ iff $\omega$ has constant $\mu_\xi^\lambda$-scalar curvature.

The moment map picture on $\mu$-scalar curvature inspires us to introduce the following *$\mu$-Futaki invariant*

$$\mathrm{Fut}_\xi^\lambda(\eta) = -\int_X (s_\xi^\lambda(\omega) - \bar{s}_\xi^\lambda)\theta_\eta e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n. \tag{2.4}$$

for $\eta \in \mathfrak{t}$. It depends only on the cohomology class $[\omega] \in H^2(X, \mathbb{R})$ and hence vanishes if there exists a $\mu_\xi^\lambda$-cscK metric in the cohomology class $[\omega]$. The sign is reversed from the definition [Ino2, (49)] so that it coincides with the slope of the $\mu_\xi^\lambda$-Mabuchi functional along the geodesic $\phi_t = (\exp tJ\eta)^*\theta_\eta$.

**On the intersection formula of Donaldson–Futaki invariant**

Before studying $\mu$K-stability, we give a simple observation on Odaka–Wang's intersection formula. Donaldson [Don4] firstly introduced Futaki invariant for test configurations as a generalization of differential geometric Futaki invariant introduced in [Fut], using a polynomial expansion given by equivariant

Riemann–Roch theorem. Odaka [Oda1] and Wang [**?**] showed a cohomological expression of Donaldson–Futaki invariant via Donaldson's definition. Here we observe that Odaka–Wang's cohomological expression (for product configurations) directly follows from the differential geometric definition of Futaki invariant just by applying the localization formula in Example 2.4.6, which is essentially Stokes theorem. This observation even yields an equivariant cohomological expression for the equivariant first Chern class of CM line bundle for smooth family. The author believe that this observation will help the readers to understand the construction of the cohomological $\mu$-Futaki invariant $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}/B, \mathcal{L})$.

Let $X$ be a compact Kähler manifold and $\Lambda : \mathbb{C}^* \to \mathrm{Aut}(X)$ be a one parameter subgroup such that the $U(1)$-action on $X$ is Hamiltonian with respect to a Kähler metric $\omega$ in a cohomology class $L$. Let $\eta_X$ be the associated real holomorphic vector filed:

$$\eta_X(x) = \frac{d}{d\theta}\Big|_{\theta=0} x.\Lambda(e^{2\pi\sqrt{-1}\theta}).$$

Fix a moment map $\mu : X \to \mathfrak{u}(1)^\vee$. We denote the equivariant cohomology class $[\omega + \mu]$ by $L_{U(1)}$.

We denote by $\mathcal{X}^\Lambda$ the product $\mathbb{C} \times X$ endowed with the $\mathbb{C}^*$-action given by $(z, x).t = (zt, x.\Lambda(t))$. Let $\pi : \mathcal{X}^\Lambda \to \mathbb{C}$ and $p_X : \mathcal{X}^\Lambda \to X$ be the projections. The projections $\pi$ and $p_X$ are $\mathbb{C}^*$-equivariant. Consider the pulled-back 2-form $\Omega := p_X^* \omega$ on $\mathcal{X}^\Lambda$. Then the map $\mu_{\mathcal{X}} : \mathcal{X}^\Lambda \to \mathfrak{u}(1)^\vee$ defined by $\mu_{\mathcal{X}}(b, x) := \mu(x)$ is a moment map with respect to the restricted $U(1)$-action on $\mathcal{X}^\Lambda$ and $\Omega$.

As usual, we compactify $\mathcal{X}^\Lambda$ by gluing the product $\mathbb{C} \times X$ with a $\mathbb{C}^*$-action $(z, x).t = (t^{-1}z, x)$ via the $\mathbb{C}^*$-equivariant isomorphism $(\mathbb{C} \setminus \{0\}) \times X \to \mathcal{X}^\Lambda \setminus \pi^{-1}(0) : (u, x) \mapsto (u^{-1}, x.\Lambda(u^{-1}))$ and denote the compactification by $\bar{\mathcal{X}}^\Lambda$. We denote the glued morphism $\bar{\mathcal{X}}^\Lambda \to \mathbb{C}P^1$ by the same symbol $\pi$.

Let

$$j : \mathcal{X}^\Lambda \hookrightarrow \bar{\mathcal{X}}^\Lambda, \quad \check{j} : \mathbb{C} \times X \hookrightarrow \bar{\mathcal{X}}^\Lambda,$$
$$i : \mathbb{C} \hookrightarrow \mathbb{C}P^1 : z \mapsto (z : 1), \quad \check{i} : \mathbb{C} \hookrightarrow \mathbb{C}P^1 : w \mapsto (1 : w)$$

be the associated immersions and

$$j_0 = \check{j}_\infty : X \hookrightarrow \mathcal{X}^\Lambda \hookrightarrow \bar{\mathcal{X}}^\Lambda, \quad j_\infty = \check{j}_0 : X \hookrightarrow \mathbb{C} \times X \hookrightarrow \bar{\mathcal{X}}^\Lambda,$$
$$i_0 = \check{i}_\infty : \mathrm{pt} \to \mathbb{C}P^1 : i_0(\mathrm{pt}) = (0 : 1), \quad i_\infty = \check{i}_0 : \mathrm{pt} \to \mathbb{C}P^1 : \check{i}_0(\mathrm{pt}) = (1 : 0).$$

be the embeddings of the central fibres and the origins.

By the equivariant Mayer–Vietoris sequence for $\bar{\mathcal{X}}^\Lambda = \mathcal{X}^\Lambda \cup (\mathbb{C} \times X)$, we have the following exact sequence:

$$0 \to H^2_{U(1)}(\bar{\mathcal{X}}^\Lambda) \xrightarrow{j_0^* \oplus j_\infty^*} H^2_{U(1)}(X) \oplus (\mathfrak{u}(1)^\vee \oplus H^2(X)) \xrightarrow{[\alpha+\nu]\oplus(c,[\beta]) \mapsto [\alpha]-[\beta]} H^2(X) \to 0,$$

where we identified $H^2_{U(1)}(\mathcal{X}^\Lambda)$ with $H^2_{U(1)}(X)$ and $H^2_{U(1)}(\mathbb{C}\times X)$ with $H^2(\mathbb{C}P^\infty \times X) = \mathfrak{u}(1)^\vee \oplus H^2(X)$ in natural ways. (Note the $\mathbb{C}^*$-action on the central fibre of $\mathbb{C} \times X$ is trivial. ) In particular, we have the isomorphism

$$H^2_{U(1)}(\bar{\mathcal{X}}^\Lambda) \cong H^2_{U(1)}(X) \oplus \mathfrak{u}(1)^\vee : [\alpha + \nu] \mapsto j_0^*[\alpha + \nu] \oplus (\nu \circ j_\infty),$$

where $\nu \circ j_\infty$ is regarded as a constant. (Indeed, $\nu \circ j_\infty$ is a constant function on $X$ as $d(\nu_\eta \circ j_\infty) = j_\infty^* d\nu_\eta = -j_\infty^* i_{\eta_{\bar{\mathcal{X}}}} \nu = 0$ by $\eta_{\bar{\mathcal{X}}}|_{j_\infty(X)} = 0$. ) Let $\bar{\mathcal{L}}_{U(1)} \in H^2_{U(1)}(\bar{\mathcal{X}}^\Lambda, \mathbb{R})$ be the equivariant cohomology class corresponding to $[\omega + \mu] \oplus 0 \in H^2_{U(1)}(X) \oplus \mathfrak{u}(1)^\vee$ and $\bar{\mathcal{L}} \in H^2(\bar{\mathcal{X}}^\Lambda, \mathbb{R})$ be the associated cohomology class.

Now we see that the following.

**Lemma 2.2.1** (Intersection formula)**.** In the setting as above, the cohomology class $\mathrm{Fut}(\eta_X).\eta^\vee \in H^2_{U(1)}(\{0\})$ is given as the image of the following cohomology class along $i_0^* : H^2_{U(1)}(\mathbb{C}P^1) \to H^2_{U(1)}(\{0\}) \cong \mathbb{R}.\eta^\vee$:

$$-4\pi \cdot \bar{\pi}_* \Big( K^{U(1)}_{\bar{\mathcal{X}}/\mathbb{C}P^1} \smile \bar{\mathcal{L}}^{\smile n}_{U(1)} + \frac{n}{n+1} \frac{(-K_X.L^{\cdot(n-1)})}{(L^{\cdot n})} \bar{\mathcal{L}}^{\smile(n+1)}_{U(1)} \Big) \in H^2_{U(1)}(\mathbb{C}P^1).$$

As a consequence, we obtain the following well-known Odaka–Wang's intersection formula:

$$\mathrm{Fut}(\eta_X) = 4\pi \Big( (K_{\bar{\mathcal{X}}/\mathbb{C}P^1}.\bar{\mathcal{L}}^{\cdot n}) + \frac{n}{n+1} \frac{(-K_X.L^{\cdot(n-1)})}{(L^{\cdot n})} (\bar{\mathcal{L}}^{\cdot(n+1)}) \Big).$$

*Proof.* Since $j_0^* \bar{\mathcal{L}}_{U(1)} = L_{U(1)} = [\omega + \mu]$ and $j_0^* K^{U(1)}_{\bar{\mathcal{X}}} = K^{U(1)}_X = -\frac{1}{2\pi}[\mathrm{Ric}_\omega + \bar{\Box}\mu]$, we compute

$$\begin{aligned} -\mathrm{Fut}(\eta_X).\eta^\vee &= \int_X \theta_\eta \hat{s}_\omega \, \omega^n.\eta^\vee = \Big( n \int_X \mu_{-2\eta} \mathrm{Ric}_\omega \wedge \omega^{n-1} - \bar{s} \int_X \mu_{-2\eta} \omega^n \Big).\eta^\vee \\ &= -2 \int_X (\mathrm{Ric}_\omega + \bar{\Box}\mu) \wedge (\omega + \mu)^n + 2 \frac{\bar{s}}{n+1} \int_X (\omega + \mu)^{n+1} \\ &= 2 \cdot \int_X \Big( -[\mathrm{Ric}_\omega + \bar{\Box}\mu] \wedge [\omega + \mu]^n + \frac{\bar{s}}{n+1} [\omega + \mu]^{n+1} \Big) \\ &= 4\pi \cdot \int_X j_0^* \Big( K^{U(1)}_{\bar{\mathcal{X}}/\mathbb{C}P^1} \smile \bar{\mathcal{L}}^{\smile n}_{U(1)} + \frac{n}{n+1} \frac{-K_X.L^{\cdot(n-1)}}{L^{\cdot n}} \bar{\mathcal{L}}^{\smile(n+1)}_{U(1)} \Big). \end{aligned}$$

Then the first claim follows by the formula $\int_X j_0^* = i_0^* \pi_*$ for the submersion $\pi$.

For the second claim, we employ the localization formula. By the localization formula $\int_{\mathbb{C}P^1} u = -(i_0^* u - i_\infty^* u)/\eta^\vee$ on $u \in H^2_{U(1)}(\mathbb{C}P^1, \mathbb{R})$ in Example 2.4.6, we have

$$\int_{\bar{\mathcal{X}}} v = \int_{\mathbb{C}P^1} \pi_* v = -i_0^* \pi_* v/\eta^\vee$$

for $v \in H^{2n+2}_{U(1)}(\bar{\mathcal{X}}, \mathbb{R})$ with $j_\infty^* v = 0$. Since $j_\infty^* \bar{\mathcal{L}}_{U(1)} = [\tilde{\omega} + 0]$ and $j_\infty^* K^{U(1)}_{\mathcal{X}/\bar{\mathbb{C}}P^1} = [\mathrm{Ric}_{\tilde{\omega}} + 0]$, we have

$$i_\infty^* \pi_* \bar{\mathcal{L}}_{U(1)}^{\smile(n+1)} = \int_X (\tilde{\omega} + 0)^{n+1} = 0$$

$$i_\infty^* \pi_* (K^{U(1)}_{\bar{\mathcal{X}}/\mathbb{C}P^1} \smile \bar{\mathcal{L}}^n_{U(1)}) = \int_X (\mathrm{Ric}_{\tilde{\omega}} + 0)(\tilde{\omega} + 0)^n = 0,$$

so that we get

$$\begin{aligned} \mathrm{Fut}(\eta_X) &= -4\pi \cdot i_0^* \pi_* \Big( K^{U(1)}_{\bar{\mathcal{X}}/\mathbb{C}P^1} \smile \bar{\mathcal{L}}^n_{U(1)} + \frac{n}{n+1} \frac{-K_X.\omega^{\cdot(n-1)}}{\omega^{\cdot n}} \bar{\mathcal{L}}^{n+1}_{U(1)} \Big) \Big/ \eta^\vee \\ &= 4\pi \cdot \int_{\bar{\mathcal{X}}} \Big( K^{U(1)}_{\bar{\mathcal{X}}/\mathbb{C}P^1} \smile \bar{\mathcal{L}}^n_{U(1)} + \frac{n}{n+1} \frac{-K_X.\omega^{\cdot(n-1)}}{\omega^{\cdot n}} \bar{\mathcal{L}}^{n+1}_{U(1)} \Big) \\ &= 4\pi \Big( (K_{\bar{\mathcal{X}}/\mathbb{C}P^1}.\bar{\mathcal{L}}^{\cdot n}) + \frac{n}{n+1} \frac{-K_X.\omega^{\cdot(n-1)}}{\omega^{\cdot n}} (\bar{\mathcal{L}}^{\cdot(n+1)}) \Big). \end{aligned}$$

□

**Equivariant cohomological interpretation of $\mu$-Futaki invariant**

Here we give an essential observation for Theorem G.

The *$\mu$-volume functional* $\mathrm{Vol}^\lambda : \mathfrak{t} \to \mathbb{R}$

$$\mathrm{Vol}^\lambda(\xi) := e^{\bar{s}^\lambda_\xi} \left( \int_X e^{\theta_\xi} \omega^n \right)^\lambda \tag{2.5}$$

was introduced in [Ino2] so that the derivative satisfies $D_\xi(-\log \mathrm{Vol}^\lambda)(\eta) = \mathrm{Fut}^\lambda_\xi(\eta)$. The functional is designed so that it generalizes Tian–Zhu's volume functional $\int_X e^{\theta_\xi} \omega^n$ introduced in the study of Kähler–Ricci soliton [TZ2]. Indeed, we have $\bar{s}^{2\pi}_\xi = 2\pi n$ when $(X, L) = (X, -K_X)$ is an $n$-dimensional Fano manifold.

We proved in [Ino2] that the $\mu$-volume functional tends to $+\infty$ as $|\xi| \to \infty$ for each $\lambda \in \mathbb{R}$, so that the functional always admits a critical point and the critical points are unique for $\lambda \ll 0$, but always *not* unique for $\lambda \gg 0$. Moreover, the set of vectors associated to some $\mu$-cscK metric is (empty or) finite for each $\lambda \leq 0$ and is contained in the center of $\mathfrak{t}$. The author expects that the critical points are indeed unique for $\lambda \leq 0$.

Now we note the following equivariant intersection formulae

$$\int_X (s(\omega) + \bar{\square}\theta_\xi) e^{\theta_\xi}\omega^n = n!\mathrm{Ev}_\xi \int_X (\mathrm{Ric}_\omega + \bar{\square}\mu) e^{\omega+\mu} = -2\pi n! \mathrm{Ev}_\xi (\kappa_X^T . e^{L_T}),$$
$$\int_X (n + \theta_\xi) e^{\theta_\xi}\omega^n = n!\mathrm{Ev}_\xi \int_X (\omega + \mu) e^{\omega+\mu} = n!\mathrm{Ev}_\xi (L_T . e^{L_T}),$$
$$\int_X e^{\theta_\xi}\omega^n = n!\mathrm{Ev}_\xi \int_X e^{\omega+\mu} = n!\mathrm{Ev}_\xi (e^{L_T})$$

for $L_T := [\omega + \mu]$ and $\kappa_X^T := -c_1^T(X) = -\frac{1}{2\pi}[\mathrm{Ric}_\omega + \bar{\square}\mu]$. Here we identify the canonical class $\kappa_X^T$ with its equivariant Poincare dual $-c_1^T(X)$ just for simplicity.

See Appendix 2.4.1 and 2.4.1 for the precise definition of the notation in the last expressions. Here we just note that the equivariant push-forwards $(\kappa_X^T . e^{L_T}), (L_T . e^{L_T})$ and $(e^{L_T})$ along $p : X \to \mathrm{pt}$ are elements of $\hat{H}_T^{\mathrm{even}}(\mathrm{pt}, \mathbb{R})$ and the evaluation $\mathrm{Ev}_\xi = \mathrm{ev}_{-2\xi}$ of these elements are by definition the infinite series $\mathrm{Ev}_\xi(\alpha . e^{L_T}) := \sum_{k=0}^\infty \frac{1}{k!} \mathrm{Ev}_\xi (\alpha . L_T^{\smile k}) \in \mathbb{R}$ for $\alpha = \kappa_X^T, L_T, [X]^T$. The Cartan model of equivariant cohomology explains the equality $\mathrm{Ev}_\xi(\alpha . L_T^{\smile k}) = \int_X (\mathcal{A} + \nu_{-2\xi})(\omega + \mu_{-2\xi})^k$ with $[\mathcal{A} + \nu] = \alpha \in H^2_{\mathrm{dR},T}(X, \mathbb{R})$, so the the convergence of the infinite series follow by the convergence of the infinite series $\sum_{k=0}^\infty \frac{1}{k!} ((\mathcal{A} + \nu_{-2\xi})(\omega + \mu_{-2\xi})^k)^{\langle n \rangle}$ of $2n$-forms on $X$ to the above integrands. We discuss the convergence in more general setups in section 2.3.1 based on a preliminary in section 2.4.2.

Now we can express the following variant

$$\boldsymbol{\mu}^\lambda(\xi) := -\log \frac{\mathrm{Vol}^\lambda(\xi)}{(n!e^n)^\lambda} = -\bar{s}_\xi^\lambda + \lambda n - \lambda \log \Big( \frac{1}{n!} \int_X e^{\theta_\xi}\omega^n \Big) \tag{2.6}$$

as

$$2\pi \mathrm{Ev}_\xi(\kappa_X^T . e^{L_T}) \cdot (\mathrm{Ev}_\xi(e^{L_T}))^{-1} + \lambda \mathrm{Ev}_\xi(L_T . e^{L_T}) \cdot (\mathrm{Ev}_\xi(e^{L_T}))^{-1} - \lambda \log \mathrm{Ev}_\xi(e^{L_T}).$$

Since $\boldsymbol{\mu}^\lambda$ differs from $-\log \mathrm{Vol}^\lambda$ by a constant, we have

$$\mathrm{Fut}_\xi^\lambda(\eta) = \frac{d}{dt}\Big|_{t=0} \boldsymbol{\mu}^\lambda_{(X,\omega)}(\xi + t\eta) = D_\xi \boldsymbol{\mu}^\lambda_{(X,\omega)}(\eta). \tag{2.7}$$

On the other hand, since the degree zero part $(e^{L_T})^{\langle 0\rangle} = (L^{\cdot n})/n!$ of the equivariant cohomology class $(e^{L_T}) \in \hat{H}_T^{\mathrm{even}}(\mathrm{pt}, \mathbb{R})$ is positive, we can also regard $(e^{L_T})^{-1}, \log(e^{L_T})$ as equivariant cohomology classes in $\hat{H}_T^{\mathrm{even}}(\mathrm{pt}, \mathbb{R})$. So the following gives a well-defined equivariant cohomology class

$$\boldsymbol{\mu}_T^\lambda(X, L) := 2\pi(\kappa_X^T.e^{L_T}) \cdot (e^{L_T})^{-1} + \lambda(L_T.e^{L_T}) \cdot (e^{L_T})^{-1} - \lambda \log(e^{L_T}). \quad (2.8)$$

If we replace the equivariant lift $[\omega + \mu] \in H_T^2(X)$ of the cohomology class $[\omega] \in H^2(X)$ to another lift $L_T + c := [\omega + \mu + c]$ by a constant $c \in \mathfrak{t}^\vee$, then $(e^{L_T}), (L_T.e^{L_T}), (\kappa_X^T.e^{L_T})$ are replaced as

$$\begin{aligned} (e^{L_T+c}) &= e^c(e^{L_T}), \\ ((L_T + c).e^{L_T+c}) &= e^c(L_T.e^{L_T}) + ce^c(e^{L_T}), \\ (\kappa_X^T.e^{L_T+c}) &= e^c(\kappa_X^T.e^{L_T}), \end{aligned}$$

respectively. So $\boldsymbol{\mu}_T^\lambda(X, L)$ depends only on the cohomology class $L = [\omega]$ and the equivariant canonical class $\kappa_X^T$.

If we take a base $\{x_1, \ldots, x_k\}$ of $\mathfrak{t}^\vee$, we can identify $\hat{H}_T^{\mathrm{even}}(\mathrm{pt}, \mathbb{R}) = \hat{S}\mathfrak{t}^\vee$ with the ring of formal power series $\mathbb{R}[\![x_1, \ldots, x_k]\!]$. Then the element $\boldsymbol{\mu}_T^\lambda(X, L)$ is identified with the Taylor expansion of the functional $\boldsymbol{\mu}^\lambda(-\frac{1}{2}\cdot)$ on $\mathfrak{t}$ at the origin. Since the functional $\boldsymbol{\mu}^\lambda$ is real analytic, the differential $D_\xi\boldsymbol{\mu}^\lambda(\eta)$ must be recovered from the formal series $\boldsymbol{\mu}_T^\lambda(X, L)$ when $\xi$ is close to the origin. Moreover, the derivative $D\boldsymbol{\mu}^\lambda$ away from the origin must be recovered from $\boldsymbol{\mu}_T^\lambda(X, L)$, since $\boldsymbol{\mu}^\lambda$ is real analytic on the whole $\mathfrak{t}$. This is our strategy for detecting an equivariant cohomological expression of $\mu$-Futaki invariant. We will introduce a differential operation on equivariant cohomology to arrange this idea in section 2.2.2.

### 2.2.2 Differential operation on equivariant cohomology

Here we introduce a differential operation on equivariant cohomology. We use this notion to construct the cohomological $\mu$-Futaki invariant $\mathcal{D}_\xi^T\boldsymbol{\mu}_G^\lambda(\mathcal{X}/B, \mathcal{L})$.

Let $G$ be a topological group, $B$ be a connected (for simplicity) topological space with a $G$-action and $T$ be a (closed/algebraic) torus acting on $B$ trivially. (We can also deal with the case when $B$ is $T$-equivariantly homotopically equivalent to a space with the trivial $T$-action, such as $B = \mathbb{C}$ with a linear $\mathbb{C}^*$-action. ) From the assumption on the $T$-action, we have the following decomposition

$$\hat{H}^{\mathrm{even}}_{T\times G}(B,\mathbb{R}) = \prod_{k=0}^{\infty} \bigoplus_{i+j=k} S^i\mathfrak{t}^\vee \otimes H^{2j}_G(B,\mathbb{R}). \tag{2.9}$$

For an element $\alpha$ of $\hat{H}^{\mathrm{even}}_{T\times G}(B,\mathbb{R})$, we denote by $\alpha^{\langle i,j\rangle}$ the $S^i\mathfrak{t}^\vee \otimes H^{2j}_G(B,\mathbb{R})$-component of $\alpha$:

$$\alpha = \sum_{k=0}^{\infty} \alpha^{\langle k\rangle} = \sum_{k=0}^{\infty} \left( \sum_{i+j=k} \alpha^{\langle i,j\rangle} \right).$$

As our formulation around $\mu^\lambda_\xi$-cscK metric fits into the $\bar{\partial}$-Hamiltonian $\theta_\xi = \mu_{-2\xi}$ rather than the moment map $\mu_\xi$, we may better to prepare our notation by a factor of $-2$. First of all, we put

$$\mathrm{Ev}_\xi := \mathrm{ev}_{-2\xi} : H^k_{T\times G}(B,\mathbb{R}) \to \bigoplus_{0\le l\le k} H^l_G(B,\mathbb{R}). \tag{2.10}$$

**Definition 2.2.2** (Differential operation on equivariant cohomology)**.** For an equivariant cohomology class of even degree $\alpha \in \hat{H}^{\mathrm{even}}_{T\times G}(B,\mathbb{R})$, we define the *formal $k$-derivative* $\hat{\mathcal{D}}^k\alpha$ (on $T$ to the direction $G$) to be the element of $\hat{S}\mathfrak{t}^\vee \otimes H^{2k}_G(B,\mathbb{R}) := \prod_{i=0}^\infty S^i\mathfrak{t}^\vee \otimes H^{2k}_G(B,\mathbb{R})$ defined as:

$$\hat{\mathcal{D}}^k\alpha := (-2)^k k! \sum_{i=0}^{\infty} \alpha^{\langle i,k\rangle} \in \prod_{i=0}^{\infty} S^i\mathfrak{t}^\vee \otimes H^{2k}_G(B,\mathbb{R}).$$

For $k = 0, 1, \ldots, \infty$, we say that $\alpha$ is *of class $\varepsilon^k$ around* $\xi \in \mathfrak{t}$ if for each $l \in \mathbb{Z}_{\ge 0}$ with $l \le k$ (or $l < \infty$ when $k = \infty$) the sum $\sum_{i=0}^\infty \mathrm{Ev}_\xi(\alpha^{\langle i,l\rangle})$ is locally uniformly absolutely-convergent around $\xi \in \mathfrak{t}$ with respect to some (hence any) norm on $H^{2l}_G(B,\mathbb{R})$. In this case, the sum $(-2)^k k! \sum_{i=0}^\infty \mathrm{Ev}_\xi(\alpha^{\langle i,k\rangle})$ is unconditional convergent in $H^{2k}_G(B,\mathbb{R})$. We denote the limit by

$$\mathcal{D}^k_\xi\alpha := (-2)^k k! \sum_{i=0}^{\infty} \mathrm{Ev}_\xi(\alpha^{\langle i,k\rangle}) \in H^{2k}_G(B,\mathbb{R}) \tag{2.11}$$

and simply denote it by $\mathcal{D}_\xi\alpha$ for $k = 1$.

The following example illustrates the way to regard this operation as differential.

*Example* 2.2.3. Consider the case $G = \mathbb{C}^*$, $T = (\mathbb{C}^*)^{\times m}$ and $B = \mathrm{pt}$. In this case, an equivariant cohomology class $\alpha \in \hat{H}^{\mathrm{even}}_{T\times G}(B, \mathbb{R}) = \mathbb{R}[\![\nu_1^\vee, \ldots, \nu_m^\vee, \eta^\vee]\!]$ is identified with the formal power series

$$\alpha = \sum_{j=0}^{\infty} \frac{1}{j!}\Big( \sum_{\boldsymbol{i}=(i_1,\ldots,i_m)} \frac{1}{\boldsymbol{i}!} a_{\boldsymbol{i},j}.(\boldsymbol{\nu}^\vee)^{\boldsymbol{i}} \Big).(\eta^\vee)^j \in \mathbb{R}[\![\boldsymbol{\nu}^\vee]\!][\![\eta^\vee]\!]$$

with some $a_{\boldsymbol{i},j} \in \mathbb{R}$ for each $\boldsymbol{i} = (i_1, \ldots, i_m) \in \mathbb{Z}^m_{\geq 0}$. Here we put $\boldsymbol{i}! = i_1! \cdots i_m!$ and $(\boldsymbol{\nu}^\vee)^{\boldsymbol{i}} = (\nu_1^\vee)^{i_1} \cdots (\nu_m^\vee)^{i_m}$. Since

$$\alpha^{\langle i,j\rangle} = \frac{1}{j!}\Big( \sum_{|\boldsymbol{i}|=i} \frac{1}{\boldsymbol{i}!} a_{\boldsymbol{i},j}.(\boldsymbol{\nu}^\vee)^{\boldsymbol{i}} \Big).(\eta^\vee)^j,$$

we have

$$\hat{\mathcal{D}}^k \alpha = \Big( \sum_{\boldsymbol{i}} \frac{1}{\boldsymbol{i}!} a_{\boldsymbol{i},j}.(\boldsymbol{\nu}^\vee)^{\boldsymbol{i}} \Big).(-2\eta^\vee)^k.$$

So we formally get

$$\mathcal{D}^k_\xi \alpha = \Big( \sum_{\boldsymbol{i}} \frac{1}{\boldsymbol{i}!} a_{\boldsymbol{i},k}(-2\boldsymbol{x})^{\boldsymbol{i}} \Big).(-2\eta^\vee)^k$$

for $\xi = \boldsymbol{x}.\boldsymbol{\nu} = x_1\nu_1 + \cdots + x_m\nu_m \in \mathfrak{t}$.

Now we consider a group morphism $\Lambda : G \to T : t \mapsto (t^{\lambda_1}, \ldots, t^{\lambda_m})$ and treat the case when $\alpha$ is the pull-back $(\mathrm{id}_T \times \Lambda)^*\beta$ of some $\beta = \sum_{\boldsymbol{i}} \frac{1}{\boldsymbol{i}!} b_{\boldsymbol{i}}.(\boldsymbol{\nu}^\vee)^{\boldsymbol{i}} \in \hat{H}^{\mathrm{even}}_T(B, \mathbb{R}) = \mathbb{R}[\![\boldsymbol{\nu}^\vee]\!]$. The pull-back $\alpha = (\mathrm{id}_T \times \Lambda)^*\beta \in \hat{H}^{\mathrm{even}}_{T\times G}(B, \mathbb{R}) = \mathbb{R}[\![\boldsymbol{\nu}^\vee]\!][\![\eta^\vee]\!]$ is expressed as

$$\begin{aligned}
\alpha &= \sum_{\boldsymbol{i}=(i_1,\ldots,i_m)} \frac{1}{\boldsymbol{i}!} b_{\boldsymbol{i}}.(\boldsymbol{\nu}^\vee + (\lambda_1\eta^\vee, \ldots, \lambda_m\eta^\vee))^{\boldsymbol{i}} \\
&= \sum_{i_1,\ldots,i_m} \frac{1}{i_1! \cdots i_m!} b_{(i_1,\ldots,i_m)} (\nu_1 + \lambda_1\eta^\vee)^{i_1} \cdots (\nu_m + \lambda_m\eta^\vee)^{i_m} \\
&= \sum_{i_1,\ldots,i_m} \frac{1}{i_1! \cdots i_m!} b_{(i_1,\ldots,i_m)} \Big( \sum_{k_1=0}^{i_1} \binom{i_1}{k_1} (\nu_1^\vee)^{i_1-k_1} (\lambda_1\eta^\vee)^{k_1} \Big) \cdots \Big( \sum_{k_m=0}^{i_m} \binom{i_m}{k_m} (\nu_m^\vee)^{i_m-k_m} (\lambda_m\eta^\vee)^{k_m} \Big) \\
&= \sum_{\boldsymbol{i}} \sum_{\boldsymbol{k}\leq\boldsymbol{i}} \frac{1}{(\boldsymbol{i}-\boldsymbol{k})!\boldsymbol{k}!} b_{\boldsymbol{i}} (\boldsymbol{\nu}^\vee)^{\boldsymbol{i}-\boldsymbol{k}} \boldsymbol{\lambda}^{\boldsymbol{k}} (\eta^\vee)^{|\boldsymbol{k}|} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \Big( \sum_{|\boldsymbol{k}|=k} \sum_{\boldsymbol{l}} \frac{k!}{\boldsymbol{l}!\boldsymbol{k}!} b_{\boldsymbol{k}+\boldsymbol{l}} \boldsymbol{\lambda}^{\boldsymbol{k}} (\boldsymbol{\nu}^\vee)^{\boldsymbol{l}} \Big).(\eta^\vee)^k.
\end{aligned}$$

So we formally get

$$\mathcal{D}_\xi^k\alpha = \Big(\sum_{|\boldsymbol{k}|=k}\sum_{\boldsymbol{l}}\frac{k!}{\boldsymbol{l}!\boldsymbol{k}!}b_{\boldsymbol{k}+\boldsymbol{l}}(-2\boldsymbol{\lambda})^{\boldsymbol{k}}(-2\boldsymbol{x})^{\boldsymbol{l}}\Big).(\eta^\vee)^k.$$

The coefficient can be identified with the usual differential: suppose the power series $f_\beta(\boldsymbol{x}) = \sum_{\boldsymbol{i}}\frac{1}{\boldsymbol{i}!}b_{\boldsymbol{i}}\boldsymbol{x}^{\boldsymbol{i}}$ is locally uniformly absoulte-convergent on $\mathbb{R}^m$, then for $F_\beta(\boldsymbol{x}) := f_\beta(-2\boldsymbol{x})$ we have

$$\Big(\frac{d^k}{dt^k}\Big)\Big|_{t=0}F_\beta(\boldsymbol{x}+t\boldsymbol{\lambda}) = \sum_{|\boldsymbol{k}|=k}\sum_{\boldsymbol{l}}\frac{k!}{\boldsymbol{l}!\boldsymbol{k}!}b_{\boldsymbol{k}+\boldsymbol{l}}(-2\boldsymbol{\lambda})^{\boldsymbol{k}}(-2\boldsymbol{x})^{\boldsymbol{l}}.$$

In particular, the pull-back $\alpha = (\mathrm{id}_T\times\Lambda)^*\beta$ is of class $\varepsilon^\infty$. This is the reason we call $\hat{\mathcal{D}}^k\alpha$ the formal derivative *on $T$ to the direction $G$.*

**Proposition 2.2.4** (Leibniz rule)**.** We have the following formal Leibniz rule for $\alpha,\beta\in\hat{H}^{\mathrm{even}}_{T\times G}(B,\mathbb{R})$:

$$\hat{\mathcal{D}}^k(\alpha\smile\beta) = \sum_{l=0}^k\binom{k}{l}\hat{\mathcal{D}}^l\alpha\smile\hat{\mathcal{D}}^{k-l}\beta\in\prod_{i=0}^\infty S^i\mathfrak{t}^\vee\otimes H_G^{2k}(B,\mathbb{R}). \tag{2.12}$$

Suppose $\alpha$ and $\beta$ are of class $\varepsilon^k$ around $\xi\in\mathfrak{t}$, then so is $\alpha\smile\beta$ and we have the following Leibniz rule:

$$\mathcal{D}_\xi^k(\alpha\smile\beta) = \sum_{l=0}\binom{k}{l}\mathcal{D}_\xi^l\alpha\smile\mathcal{D}_\xi^{k-l}\beta\in H_G^{2k}(B,\mathbb{R}). \tag{2.13}$$

*Proof.* The first claim follows directly by

$$\begin{aligned}\hat{\mathcal{D}}^k(\alpha\smile\beta) &= k!\sum_{i=0}^\infty(\alpha\smile\beta)^{\langle i,k\rangle} = k!\sum_{i=0}^\infty\sum_{p+q=i}\sum_{l=0}^k\alpha^{\langle p,l\rangle}\smile\beta^{\langle q,k-l\rangle}\\ &= \sum_{l=0}^k k!\Big(\sum_{p=0}^\infty\alpha^{\langle p,l\rangle}\Big)\smile\Big(\sum_{q=0}^\infty\beta^{\langle q,k-l\rangle}\Big)\\ &= \sum_{l=0}^k\frac{k!}{l!(k-l)!}\Big(l!\sum_{p=0}^\infty\alpha^{\langle p,l\rangle}\Big)\smile\Big((k-l)!\sum_{q=0}^\infty\beta^{\langle q,k-l\rangle}\Big)\\ &= \sum_{l=0}^k\binom{k}{l}\hat{\mathcal{D}}^l\alpha\smile\hat{\mathcal{D}}^{k-l}\beta.\end{aligned}$$

The second claim on the absolute convergence follows by Tonelli's theorem. Indeed, we may take the norms $\{\|\cdot\|_l\}_{l\le k}$ on $\{H^{2l}_G(B,\mathbb{R})\}_{l\le k}$ so that $\|u \smile v\|_k \le \|u\|_l \cdot \|v\|_{k-l}$ for every $u \in H^{2l}_G(B,\mathbb{R})$ and $v \in H^{2(k-l)}_G(B,\mathbb{R})$. Then since

$$\|\mathrm{Ev}_\xi(\alpha^{\langle p,l\rangle} \smile \beta^{\langle q,k-l\rangle})\|_k \le \|\mathrm{Ev}_\xi\alpha^{\langle p,l\rangle}\|_l \cdot \|\mathrm{Ev}_\xi\beta^{\langle q,k-l\rangle}\|_{k-l},$$

we have

$$\begin{aligned}
\sum_{i=0}^{\infty}\|\mathrm{Ev}_\xi(\alpha\smile\beta)^{\langle i,k\rangle}\|_k &\le \sum_{i=0}^{\infty}\sum_{p+q=i}\sum_{l=0}^{k}\|\mathrm{Ev}_\xi(\alpha^{\langle p,l\rangle}\smile\beta^{\langle q,k-l\rangle})\|_k \\
&\le \sum_{l=0}^{k}\sum_{i=0}^{\infty}\sum_{p+q=i}\|\mathrm{Ev}_\xi\alpha^{\langle p,l\rangle}\|_l\cdot\|\mathrm{Ev}_\xi\beta^{\langle q,k-l\rangle}\|_{k-l} \\
&= \sum_{l=0}^{k}\Big(\sum_{p=0}^{\infty}\|\mathrm{Ev}_\xi\alpha^{\langle p,l\rangle}\|_l\Big)\Big(\sum_{q=0}^{\infty}\|\mathrm{Ev}_\xi\beta^{\langle q,k-l\rangle}\|_{k-l}\Big).
\end{aligned}$$

□

Suppose the degree zero part $\alpha^{\langle 0\rangle} \in H^0_{T\times G}(B,\mathbb{R}) \cong \mathbb{R}$ is not zero (resp. positive), then we can define an element $\alpha^{-1}$ (resp. $\log\alpha$) of $\hat{H}^{\mathrm{even}}_{T\times G}(B,\mathbb{R})$ by

$$\alpha^{-1} := \frac{1}{\alpha^{\langle 0\rangle}} + \frac{1}{\alpha^{\langle 0\rangle}}\sum_{k=1}^{\infty}\left(\sum_{l=1}^{k}\Big(\frac{-1}{\alpha^{\langle 0\rangle}}\Big)^l\sum_{\boldsymbol{k}\in\mathbb{N}^l,|\boldsymbol{k}|=k}\alpha^{\langle\boldsymbol{k}\rangle}\right) \in \hat{H}^{\mathrm{even}}_{T\times G}(B,\mathbb{R}) \quad (2.14)$$

and

$$\log\alpha := \log\alpha^{\langle 0\rangle} - \sum_{k=1}^{\infty}\left(\sum_{l=1}^{k}\frac{1}{l}\Big(\frac{-1}{\alpha^{\langle 0\rangle}}\Big)^l\sum_{\boldsymbol{k}\in\mathbb{N}^l,|\boldsymbol{k}|=k}\alpha^{\langle\boldsymbol{k}\rangle}\right) \in \hat{H}^{\mathrm{even}}_{T\times G}(B,\mathbb{R}). \quad (2.15)$$

Here we put $|\boldsymbol{k}| := k_1+\cdots+k_m$ and $\alpha^{\langle\boldsymbol{k}\rangle} := \alpha^{\langle k_1\rangle}\cdots\alpha^{\langle k_l\rangle} \in H^{2|\boldsymbol{k}|}_{T\times G}(B,\mathbb{R})$ for the set of $l$-tuples of positive integers and for $\boldsymbol{k} = (k_1,\dots,k_l) \in \mathbb{N}^l$. These definitions are based on the following expansion; for instance as for $\alpha^{-1}$,

$$\begin{aligned}
\alpha^{-1} &= \frac{1}{\alpha^{\langle 0\rangle}}\Big(1+(\frac{1}{\alpha^{\langle 0\rangle}}\alpha-1)\Big)^{-1} = \frac{1}{\alpha^{\langle 0\rangle}}\sum_{l=0}^{\infty}(-1)^l\Big(\frac{1}{\alpha^{\langle 0\rangle}}\alpha-1\Big)^l \\
&= \frac{1}{\alpha^{\langle 0\rangle}}\sum_{l=0}^{\infty}\Big(\frac{-1}{\alpha^{\langle 0\rangle}}\Big)^l\Big(\sum_{k=1}^{\infty}\alpha^{\langle k\rangle}\Big)^l
\end{aligned}$$

and for $l \geq 1$,

$$\Big(\sum_{k=1}^{\infty} \alpha^{\langle k \rangle}\Big)^{l} = \sum_{k=l}^{\infty} \sum_{\boldsymbol{k} \in \mathbb{N}^l, |\boldsymbol{k}|=k} \alpha^{\langle \boldsymbol{k} \rangle}$$

Note here for each $k \in \mathbb{N}$, the sum $\sum_{l=1}^{k} \left(\frac{-1}{\alpha^{\langle 0 \rangle}}\right)^{l} \sum_{\boldsymbol{k} \in \mathbb{N}^l, |\boldsymbol{k}|=k} \alpha^{\langle \boldsymbol{k} \rangle}$ in $H^{2k}_{T\times G}(B, \mathbb{R})$ is a finite sum.

**Proposition 2.2.5.** We have

$$\hat{\mathcal{D}}^0(\alpha^{-1}) = (\hat{\mathcal{D}}^0 \alpha)^{-1} \in \prod_{i=0}^{\infty} S^i \mathfrak{t}^\vee, \tag{2.16}$$

$$\hat{\mathcal{D}}(\alpha^{-1}) = -\hat{\mathcal{D}}\alpha \cdot (\hat{\mathcal{D}}^0 \alpha)^{-2} \in \prod_{i=0}^{\infty} S^i \mathfrak{t}^\vee \otimes H^2_G(B, \mathbb{R}) \tag{2.17}$$

for $\alpha \in \hat{H}^{\mathrm{even}}_{T\times G}(B, \mathbb{R})$ with $\alpha^{\langle 0 \rangle} \neq 0$ and have

$$\hat{\mathcal{D}}^0(\log \alpha) = \log(\hat{\mathcal{D}}\alpha) \in \prod_{i=0}^{\infty} S^i \mathfrak{t}^\vee, \tag{2.18}$$

$$\hat{\mathcal{D}}(\log \alpha) = \hat{\mathcal{D}}\alpha \cdot (\hat{\mathcal{D}}^0 \alpha)^{-1} \in \prod_{i=0}^{\infty} S^i \mathfrak{t}^\vee \otimes H^2_G(B, \mathbb{R}) \tag{2.19}$$

for $\alpha^{\langle 0 \rangle} > 0$.

If moreover $\alpha$ is of class $\varepsilon^0$ (resp. $\varepsilon^1$) around the origin, then $\alpha^{-1}$ and $\log \alpha$ are also $\varepsilon^0$ (resp. $\varepsilon^1$) around the origin (possibly on a smaller ball) and

$$\mathcal{D}^0_\xi(\alpha^{-1}) = (\mathcal{D}^0_\xi \alpha)^{-1} \in \mathbb{R}, \tag{2.20}$$

$$(\text{ resp. } \mathcal{D}_\xi(\alpha^{-1}) = -\mathcal{D}_\xi \alpha \cdot (\mathcal{D}^0_\xi \alpha)^{-2} \in H^2_G(B, \mathbb{R})), \tag{2.21}$$

$$\mathcal{D}^0_\xi(\log \alpha) = \log(\mathcal{D}^0_\xi \alpha) \in \mathbb{R}, \tag{2.22}$$

$$(\text{ resp. } \mathcal{D}_\xi(\log \alpha) = \mathcal{D}_\xi \alpha \cdot (\mathcal{D}^0_\xi \alpha)^{-1} \in H^2_G(B, \mathbb{R})). \tag{2.23}$$

around the origin.

*Proof.* First of all, we have $(\alpha^{-1})^{\langle 0,0 \rangle} = (\alpha^{\langle 0 \rangle})^{-1} \in H^0_G(B, \mathbb{R}) = \mathbb{R}$ and

$$(\alpha^{-1})^{\langle i,0 \rangle} = \frac{1}{\alpha^{\langle 0 \rangle}} \sum_{l=1}^{i} \left(\frac{-1}{\alpha^{\langle 0 \rangle}}\right)^{l} \sum_{\boldsymbol{i} \in \mathbb{N}^l, |\boldsymbol{i}|=i} \alpha^{\langle \boldsymbol{i},0 \rangle} \in S^i \mathfrak{t}^\vee \otimes H^0_G(B, \mathbb{R})$$

for $i \geq 1$, which is a finite sum. So we compute

$$\begin{aligned}\hat{\mathcal{D}}^0(\alpha^{-1}) &= \sum_{i=0}^{\infty}(\alpha^{-1})^{\langle i,0\rangle} = \frac{1}{\alpha^{\langle 0\rangle}} + \frac{1}{\alpha^{\langle 0\rangle}}\sum_{i=1}^{\infty}\sum_{l=1}^{i}\Big(\frac{-1}{\alpha^{\langle 0\rangle}}\Big)^l \sum_{\boldsymbol{i}\in\mathbb{N}^l, |\boldsymbol{i}|=i}\alpha^{\langle \boldsymbol{i},0\rangle} \\ &= \frac{1}{\alpha^{\langle 0\rangle}}\sum_{l=0}^{\infty}\Big(\frac{-1}{\alpha^{\langle 0\rangle}}\sum_{i=1}^{\infty}\alpha^{\langle i,0\rangle}\Big)^l = \frac{1}{\alpha^{\langle 0\rangle}}\Big(1+\frac{1}{\alpha^{\langle 0\rangle}}\sum_{i=1}^{\infty}\alpha^{\langle i,0\rangle}\Big)^{-1} \\ &= \Big(\sum_{i=0}^{\infty}\alpha^{\langle i,0\rangle}\Big)^{-1} = (\hat{\mathcal{D}}^0\alpha)^{-1}\end{aligned}$$

in the ring $\prod_{p=0}^{\infty} S^p\mathfrak{t}^\vee$.

Suppose $\alpha$ is of class $\varepsilon^0$. To show that $\alpha^{-1}$ is also of class $\varepsilon^0$, we must see that the infinite series

$$\sum_{i=1}^{\infty}\Big|\sum_{l=1}^{i}\Big(\frac{-1}{\alpha^{\langle 0\rangle}}\Big)^l \sum_{\boldsymbol{i}\in\mathbb{N}^l, |\boldsymbol{i}|=i}\mathrm{Ev}_\xi\alpha^{\langle \boldsymbol{i},0\rangle}\Big| \tag{2.24}$$

converges uniformly around the origin. We can bound this by

$$\sum_{i=1}^{\infty}\sum_{l=1}^{i}\Big(\frac{1}{|\alpha^{\langle 0\rangle}|}\Big)^l \sum_{\boldsymbol{i}\in\mathbb{N}^l, |\boldsymbol{i}|=i}|\mathrm{Ev}_\xi\alpha^{\langle \boldsymbol{i},0\rangle}| = \sum_{l=1}^{\infty}\Big(\frac{1}{|\alpha^{\langle 0\rangle}|}\sum_{i=1}^{\infty}|\mathrm{Ev}_\xi\alpha^{\langle i,0\rangle}|\Big)^l.$$

This converges when $\frac{1}{|\alpha^{\langle 0\rangle}|}\sum_{i=1}^{\infty}|\mathrm{Ev}_\xi\alpha^{\langle i,0\rangle}| < 1$. Since $\sum_{i=1}^{\infty}|\mathrm{Ev}_0\alpha^{\langle i,0\rangle}| = 0$ and $\sum_{i=1}^{\infty}|\mathrm{Ev}_\xi\alpha^{\langle i,0\rangle}|$ is continuous with respect to $\xi$ around the origin by its uniform convergence, we can achieve the convergence condition when $\xi$ is sufficiently close to the origin, so that we get the uniform convergence of (2.24).

To see $\mathcal{D}^0_\xi(\alpha^{-1}) = (\mathcal{D}^0_\xi\alpha)^{-1}$ for $\xi$ close to the origin, we put

$$f_\xi(i,l) := \begin{cases}\Big(\frac{-1}{\alpha^{\langle 0\rangle}}\Big)^l\sum_{\boldsymbol{i}\in\mathbb{N}^l, |\boldsymbol{i}|=i}\mathrm{Ev}_\xi\alpha^{\langle \boldsymbol{i},0\rangle} & l\leq i\\ 0 & l>i\end{cases}.$$

Since

$$\begin{aligned}\sum_{l=1}^{\infty}\sum_{i=1}^{\infty}|f_\xi(i,l)| &\leq \sum_{l=1}^{\infty}\Big(\frac{1}{|\alpha^{\langle 0\rangle}|}\Big)^l\sum_{i=1}^{\infty}\sum_{\boldsymbol{i}\in\mathbb{N}^l, |\boldsymbol{i}|=i}|\mathrm{Ev}_\xi\alpha^{\langle \boldsymbol{i},0\rangle}| \\ &= \sum_{l=1}^{\infty}\Big(\frac{1}{|\alpha^{\langle 0\rangle}|}\sum_{i=1}^{\infty}|\mathrm{Ev}_\xi\alpha^{\langle i,0\rangle}|\Big)^l,\end{aligned}$$

$f_\xi(i,l)$ is integrable on $\mathbb{N}^2$ by Tonelli's theorem. It follows that

$$\mathcal{D}^0_\xi(\alpha^{-1}) = \frac{1}{\alpha^{\langle 0\rangle}} + \sum_{i=1}^{\infty}\sum_{l=1}^{\infty} f_\xi(i,l) = \frac{1}{\alpha^{\langle 0\rangle}} + \sum_{l=1}^{\infty}\sum_{i=1}^{\infty} f_\xi(i,l) = (\mathcal{D}^0_\xi\alpha)^{-1}$$

by Fubini's theorem.

Similarly, we have $(\alpha^{-1})^{\langle 0,1\rangle} = -(\alpha^{\langle 0\rangle})^{-2}\alpha^{\langle 0,1\rangle}$ and

$$(\alpha^{-1})^{\langle i,1\rangle} = \frac{1}{\alpha^{\langle 0\rangle}}\sum_{l=1}^{i}\Big(\frac{-1}{\alpha^{\langle 0\rangle}}\Big)^l l \sum_{\boldsymbol{i}\in\mathbb{Z}_{\geq 0}\times\mathbb{N}^{l-1}, |\boldsymbol{i}|=i} \alpha^{\langle i_1,1\rangle}\cdot\prod_{p=2}^{l}\alpha^{\langle i_p,0\rangle} \in S^i\mathfrak{t}^\vee\otimes H^2_G(B,\mathbb{R})$$

for $i\geq 1$, where we put $\prod_{p=2}^{l}\alpha^{\langle i_p,0\rangle} := 1$ when $l=1$. We compute

$$\begin{aligned}
\hat{\mathcal{D}}(\alpha^{-1}) &= \sum_{i=0}^{\infty}(\alpha^{-1})^{\langle i,1\rangle} = -\frac{\alpha^{\langle 0,1\rangle}}{(\alpha^{\langle 0\rangle})^2} + \frac{1}{\alpha^{\langle 0\rangle}}\sum_{i=1}^{\infty}\sum_{l=1}^{i}\Big(\frac{-1}{\alpha^{\langle 0\rangle}}\Big)^l l \sum_{\boldsymbol{i}\in\mathbb{Z}_{\geq 0}\times\mathbb{N}^{l-1}, |\boldsymbol{i}|=i} \alpha^{\langle i_1,1\rangle}\cdot\prod_{p=2}^{l}\alpha^{\langle i_p,0\rangle} \\
&= -\Big(\sum_{i=0}^{\infty}\alpha^{\langle i,1\rangle}\Big)\frac{1}{(\alpha^{\langle 0\rangle})^2}\sum_{l=1}^{\infty} l\Big(\frac{-1}{\alpha^{\langle 0\rangle}}\sum_{i=1}^{\infty}\alpha^{\langle i,0\rangle}\Big)^{l-1} \\
&= -\hat{\mathcal{D}}\alpha\cdot\frac{1}{(\alpha^{\langle 0\rangle})^2}\Big(1+\frac{1}{\alpha^{\langle 0\rangle}}\sum_{i=1}^{\infty}\alpha^{\langle i,0\rangle}\Big)^{-2} \\
&= -\hat{\mathcal{D}}\alpha\cdot(\hat{\mathcal{D}}^0\alpha)^{-2}
\end{aligned}$$

in $\hat{S}\mathfrak{t}^\vee\otimes H^2_G(B,\mathbb{R})$. By the same argument as above, we see that $\alpha^{-1}$ is of class $\varepsilon^1$ around the origin when $\alpha$ is so. There is no essential difference for $\log\alpha$. So we obtain the claim.

□

## 2.3 Equivariant calculus and $\mu$-character

### 2.3.1 Equivariant calculus

#### On the absolute equivariant intersection $(\alpha.e^L)$

Let $X$ be a pure $n$-dimensional scheme with a $T$-action. Here we consider the differential $\mathcal{D}^0_\xi(\alpha.e^L)$ of the absolute equivariant intersection $(\alpha.e^L)\in \hat{H}^{\mathrm{even}}_{T\times\{1\}}(\mathrm{pt},\mathbb{R})$ of a second equivariant locally finite homology class $\alpha\in H^{\mathrm{lf},T}_{2p}(X,\mathbb{R})$

and a second equivariant cohomology class $L \in H^2_T(X,\mathbb{R})$. Since the decomposition (2.9) gives the identification $\hat{H}^{\mathrm{even}}_{T\times\{1\}}(\mathrm{pt},\mathbb{R}) = \prod_{i=0}^{\infty} S^i\mathfrak{t}^\vee \otimes H^0(\mathrm{pt},\mathbb{R})$, we have

$$(\alpha.e^L)^{\langle i,k\rangle} = \begin{cases} \frac{1}{(i+(n-p))!}(\alpha.L^{\smile(i+(n-p))}) & k=0 \\ 0 & k\geq 1 \end{cases},$$

so that $\hat{\mathcal{D}}^k(\alpha.e^L)=0$ for $k\geq 1$ and

$$\hat{\mathcal{D}}^0(\alpha.e^L) = \sum_{j=0}^{\infty}\frac{1}{j!}(\alpha.L^{\smile j}).$$

For a pure $n$-dimensional $G$-scheme $X$, we denote by $H^{\mathrm{alg},G}_{2n-2}(X,\mathbb{R})$ the subspace of $H^{\mathrm{lf},G}_{2n-2}(X,\mathbb{R}) = H^{\mathrm{lf}}_{2n-2+2\dim_{\mathbb{C}} B_2G}(E_2G\times_G X,\mathbb{R})$ spanned by divisors on $E_2G\times_G X$. Here the scheme $E_2G$ is as in Appendix, i.e. a $G$-invariant Zariski open set of the subset $\{v\in V \mid v.g = v \iff g=1\}$ of a $G$-representation $V$ with $\dim_{\mathbb{C}}(V\setminus E_2G)\geq 2$. This is a well-defined subspace of $H^{\mathrm{lf},G}_{2n-2}(X,\mathbb{R})$ by [EG1]. Note that $H^{\mathrm{alg},G}_{2n-2}(X,\mathbb{R})$ is larger than the subspace $\{\sum_i a_i[E_2G\times_G Z_i] \mid Z_i\subset X : G\text{-invariant divisor}\}$ spanned by $G$-equivariant fundamental classes of $G$-invariant divisors. (See the case $G=\mathbb{C}^*$ and $X$ is a point for example. ) We firstly note the following lemma.

**Lemma 2.3.1.** Let $X$ and $Z$ be pure $n$-dimensional $G$-schemes and $f: Z\to X$ be a proper surjective $G$-equivariant morphism. Then the push-forward maps $f_*: H^{\mathrm{alg},G}_{2n-2}(Z,\mathbb{R})\to H^{\mathrm{alg},G}_{2n-2}(X,\mathbb{R})$ and $f_*: H^{\mathrm{lf},G}_{2n}(Z,\mathbb{R})\to H^{\mathrm{lf},G}_{2n}(X,\mathbb{R})$ are surjective.

*Proof.* Since the induced morphism $\mathrm{id}\times_G f : E_2G\times_G Z\to E_2G\times_G X$ is also proper surjective, the claim reduces to the trivial case $G=\{1\}$. By the reduction and the irreducible decomposition, we may assume that $X$ and $Z$ is irreducible. (For each irreducible component $X_i$ of $X$, there is an irreducible component $Z_i$ of $Z$ with $f(Z_i)=X_i$. ) For any prime divisor $Y\subset X$, $f^{-1}(Y)\subset Z$ contains a prime divisor of $Z$. Indeed, if not, then $f^{-1}(Y)$ is $(n-2)$-dimensional, so that $f(f^{-1}(Y))\neq Y$ as $Y$ is $(n-1)$-dimensional, which contradicts to the surjectivity. $\square$

Now we prove the following fundamental on absolute equivariant intersection.

**Proposition 2.3.2.** Let $X$ be a pure $n$-dimensional proper scheme and $L$ be a second equivariant cohomology class on $X$. If $\alpha \in H^{\mathrm{lf},T}_{2n}(X,\mathbb{R})$ or $\alpha \in H^{\mathrm{alg},T}_{2n-2}(X,\mathbb{R})$ the infinite series

$$\mathcal{D}^0_\xi(\alpha.e^L) = \sum_{j=0}^{\infty} \frac{1}{j!}\mathrm{Ev}_\xi(\alpha.L^{\smile j}) =: \mathrm{Ev}_\xi(\alpha.e^L)$$

is locally uniformly absolutely-convergent on $\mathfrak{t}$. When $L$ is semi-ample and big, $\mathrm{Ev}_\xi(e^L) = \mathrm{Ev}_\xi([X].e^L)$ is positive.

*Proof.* Take the irreducible decomposition of the reduction $X^{\mathrm{irr}} \to X^{\mathrm{red}} \to X$ and a $T$-equivariant resolution of singularities $\tilde{X} \to X^{\mathrm{irr}}$. Let $f : \tilde{X} \to X$ be the composition of these morphisms. By Lemma 2.3.1, we can pick an element $\tilde{\alpha} \in H^{\mathrm{alg},T}_{2n-2}(\tilde{X},\mathbb{R})$ (resp. $\tilde{\alpha} \in H^{\mathrm{lf},T}_{2n}(\tilde{X},\mathbb{R})$) with $f_*\tilde{\alpha} = \alpha$ for $\alpha \in H^{\mathrm{alg},T}_{2n-2}(X,\mathbb{R})$ (resp. $\alpha \in H^{\mathrm{lf},T}_{2n}(X,\mathbb{R})$). Since we have $(\alpha.e^L) = (\tilde{\alpha}.e^{f^*L})$ by the projection formula, we may assume $X$ is smooth.

Suppose $\alpha \in H^{\mathrm{lf},T}_{2n-2}(X,\mathbb{R})$. Pick equivariant 2-forms $A+\nu$ in the equivariant Poincare dual $\mathrm{PD}_T(\alpha)$ and $\Omega+\mu$ in $L$. We have

$$\begin{aligned}\mathrm{Ev}_\xi(\alpha.L^{\smile j}) &= \mathrm{Ev}_\xi \int_X (A+\nu)(\Omega+\mu)^j \\ &= \int_X \Big(\binom{j}{n-1}\mu_{-2\xi}^{j-(n-1)} A\wedge\Omega^{n-1} + \binom{j}{n}\nu_{-2\xi}\mu_{-2\xi}^{j-n}\Omega^n\Big).\end{aligned}$$

Now the infinite series of $2n$-forms

$$\sum_{j=0}^{\infty}\frac{1}{j!}\Big(\binom{j}{n-1}\mu_{-2\xi}^{j-(n-1)} A\wedge\Omega^{n-1} + \binom{j}{n}\nu_{-2\xi}\mu_{-2\xi}^{j-n}\Omega^n\Big)$$

is locally uniformly absolutely-convergent on $\mathfrak{t}$ with respect to the $C^l$-norm for every $l \in \mathbb{Z}_{\geq 0}$, to the limit $2n$-form

$$\frac{1}{(n-1)!}e^{\mu_{-2\xi}} A\wedge\Omega^{n-1} + \frac{1}{n!}\nu_{-2\xi}e^{\mu_{-2\xi}}\Omega^n.$$

It follows that by Corollary 2.4.12 and Lemma 2.4.13, the infinite series $\sum_{j=0}^{\infty}\frac{1}{j!}\mathrm{Ev}_\xi(\alpha.L^{\smile j})$ is locally uniformly absolutely-convergent to the integral

$$\int_X\Big(\frac{1}{(n-1)!}e^{\mu_{-2\xi}} A\wedge\Omega^{n-1} + \frac{1}{n!}\nu_{-2\xi}e^{\mu_{-2\xi}}\Omega^n\Big) =: \int_X (A+\nu_{-2\xi})e^{\Omega+\mu_{-2\xi}}.$$

(In the absolute case, we may apply the usual harmonic analysis to see the continuity of the form-to-cohomology proper push forward (2.49) of $p : X \to \mathrm{pt.}$ )

We similarly obtain the claim for $\alpha \in H_{2n}^{\mathrm{lf},T}(X, \mathbb{R})$. If $L$ is semi-ample and big, then we can pick a 2-form $\Omega$ so that it is semi-positive and is strictly positive on some open set of $X$. Since $e^{\mu - 2\xi}$ is positive,

$$\mathrm{Ev}_\xi(\alpha.e^L) = \sum_i \frac{\mathrm{PD}_T(\alpha|_{X_i})}{n!} \int_{X_i} e^{\mu - 2\xi} \Omega|_{X_i}^n$$

is positive if the Poincare duals $\mathrm{PD}_T(\alpha|_{X_i}) \in \mathbb{R} \cong H_T^0(X_i, \mathbb{R})$ are positive for each irreducible component. Here $X_i$ are the connected components of the smooth $X$. This proves the claim on the positivity as the Poincare dual $([X^{\mathrm{red}}]^T \frown)^{-1}([X]^T)$ of the fundamental class of an irreducible scheme $X$ is the length of $\mathcal{O}_X/\mathcal{O}_{X^{\mathrm{red}}}$ at the generic point, which is positive. □

The following is a key proposition to prove Theorem A.

**Proposition 2.3.3.** Let $\beta : X' \to X$ be a $T$-equivariant morphism of pure $n$-dimensional projective schemes and $L$ be a $T$-equivariant $\mathbb{Q}$-line bundle on $X$.

1. If $\beta$ is an isomorphism away from a codimension one subscheme of the target $X$, then we have

$$\mathrm{Ev}_\xi(e^{\beta^* L}) = \mathrm{Ev}_\xi(e^L), \quad \mathrm{Ev}_\xi(\beta^* L.e^{\beta^* L}) = \mathrm{Ev}_\xi(L.e^L)$$

2. If moreover $\beta$ is finite away from a codimension two subscheme of the target $X$ & $L$ is semi-ample and big (resp. if moreover $\beta$ is an isomorphism away from a codimension two subscheme of the target $X$), then we have

$$\mathrm{Ev}_\xi(\kappa_{X'}.e^{\beta^* L}) \leq \mathrm{Ev}_\xi(\kappa_X.e^L) \quad (\text{resp. } \mathrm{Ev}_\xi(\kappa_{X'}.e^{\beta^* L}) = \mathrm{Ev}_\xi(\kappa_X.e^L))$$

*Proof.* The first claim follows by $\beta_*[X']^T = [X]^T$ and the equivariant projection formula

$$\int_{X'} [X']^T \frown (\beta^* L)^{\smile(n+k)} = \int_X \beta_*[X']^T \frown L^{\smile(n+k)} = \int_X [X]^T \frown L^{\smile(n+k)} \in S^k \mathfrak{t}^\vee.$$

By the equivariant Grothendieck–Riemann–Roch theorem, we have $\beta_* \tau^T_{X'}(\mathcal{O}_{X'}) = \tau^T_X(\beta_! \mathcal{O}_{X'})$. Since $\beta$ is finite away from a codimension two subscheme of $X$, the supports of the higher direct images of $\mathcal{O}_{X'}$ is contained in the codimension two subscheme. So we have

$$(\tau^T_X(\beta_! \mathcal{O}_{X'}))_{\langle n-1\rangle} = (\tau^T_X(\beta_* \mathcal{O}_{X'}))_{\langle n-1\rangle} = (\tau^T_X(\beta_* \mathcal{O}_{X'}/\mathcal{O}_X) + \tau^T_X(\mathcal{O}_X))_{\langle n-1\rangle}.$$

Since $\beta$ is isomorphism away from a codimension one subscheme of $X$, we have

$$(\tau^T_X(\beta_* \mathcal{O}_{X'}/\mathcal{O}_X))_{\langle n-1\rangle} = \sum_i m_i [D_i]^T,$$

where $m_i \geq 0$ are the multiplicities of $\beta_* \mathcal{O}_{X'}/\mathcal{O}_X$ along $T$-invariant prime divisors $D_i$ contained in the codimension one subscheme. It follows that we have

$$\beta_* \kappa^T_{X'} = \kappa^T_X - 2\sum_i m_i [D_i]^T.$$

By the equivariant projection formula, we have

$$\begin{aligned} \int_{X'} \kappa^T_{X'} \frown (\beta^* L)^{\smile(n+k-1)} &= \int_X \beta_* \kappa^T_{X'} \frown L^{\smile(n+k-1)} \\ &= \int_X \kappa^T_X \frown L^{\smile(n+k-1)} - 2\sum_i m_i \int_X [D_i]^T \frown L^{\smile(n+k-1)} \in S^k \mathfrak{t}^\vee. \end{aligned}$$

So we have

$$\mathrm{Ev}_\xi(\kappa_{X'}.e^{\beta^* L}) = \mathrm{Ev}_\xi(\kappa_X.e^L) - 2\sum_i m_i \mathrm{Ev}_\xi(e^{L|_{D_i}}).$$

Take a resolution of singularities $\delta_i : \tilde{D}_i \to D_i$. Since $\delta_i^* L|_{D_i}$ is semi-ample and big, we can pick an equivariant 2-form $\omega + \mu$ so that $\omega^{n-1} \geq 0$ with $\omega^{n-1}(p) > 0$ at some point $p \in D_i$. Applying the first claim, we obtain

$$\mathrm{Ev}_\xi \int_{D_i} [D_i]^T \frown e^L = \frac{1}{(n-1)!} \int_{\tilde{D}_i} e^{\mu - 2\xi} \omega^{n-1} > 0$$

as the function $e^{\mu-2\xi}$ is positive. $\square$

**The relative equivariant intersection $(\alpha.e^{\mathcal{L}})_B$ is of class $\varepsilon^\infty$**

Now we study the relative case. Let $G$ be an algebraic group, $T$ be an algebraic torus. Let $\mathcal{X}$ be a pure dimensional $T\times G$-scheme, $B$ be a smooth $G$-variety with the trivial $T$-action, $\pi : \mathcal{X} \to B$ be a $T\times G$-equivariant proper morphism. Let $\mathcal{L}$ be a $T\times G$-equivariant second cohomology class on $\mathcal{X}$.

**Theorem 2.3.4.** If $\alpha \in H^{\mathrm{alg}, T\times G}_{2\dim \mathcal{X}}(\mathcal{X}, \mathbb{R})$ or $\alpha \in H^{\mathrm{alg}, T\times G}_{2\dim \mathcal{X}-2}(\mathcal{X}, \mathbb{R})$, then $(\alpha.e^{\mathcal{L}})_B \in \hat{H}^{\mathrm{lf}, T\times G}(B, \mathbb{R})$ is of class $\varepsilon^\infty$ on $\mathfrak{t}$.

*Proof.* We put $n := \dim \mathcal{X} - \dim B$. As in the proof of Proposition 2.3.2, we may assume that $\mathcal{X}$ is smooth. Let $K$ be a maximal compact subgroup of $G$. Pick an equivariant 2-from $\Omega + \mu = \Omega + (\mu^T + \mu^K)$ in $\mathcal{L}$. We firstly compute $(\mathcal{L}^{\smile j})_B^{\langle i,k\rangle}$. Comparing the degree, we have $(\mathcal{L}^{\smile j})_B^{\langle i,k\rangle} = 0$ when $j - n \neq i + k$. For $j = n + i + k$, we compute

$$
\begin{aligned}
(\mathcal{L}^{\smile n+i+k})_B^{\langle i,k\rangle} &= (\mathrm{PD}_{T\times K}[\pi_*(\Omega+\mu)^{n+i+k}])^{\langle i,k\rangle} \\
&= \sum_{p=n}^{n+i+k} \binom{n+i+k}{p} (\mathrm{PD}_{T\times K}[\pi_*(\mu^{n+i+k-p}\Omega^p)])^{\langle i,k\rangle} \\
&= \sum_{p=n}^{n+k} \binom{n+i+k}{n+i+k-p} \mathrm{PD}_{T\times K}\left[\binom{n+i+k-p}{i} \pi_*((\mu^T)^i (\mu^K)^{n+k-p}\Omega^p)\right] \\
&= \mathrm{PD}_{T\times K}\left[\binom{n+i+k}{n+k} \pi_*((\mu^T)^i \sum_{p=n}^{n+k} \binom{n+k}{p} (\mu^K)^{n+k-p}\Omega^p)\right] \\
&= \binom{n+i+k}{n+k} \mathrm{PD}_{T\times K}[\pi_*((\mu^T)^i (\Omega+\mu^K)^{n+k})],
\end{aligned}
$$

so that we obtain

$$
\mathrm{Ev}_\xi(e^{\mathcal{L}})_B^{\langle i,k\rangle} = \frac{1}{(n+i+k)!}\mathrm{Ev}_\xi(\mathcal{L}^{\smile n+i+k})_B^{\langle i,k\rangle} = \frac{1}{(n+k)!}\mathrm{PD}_K[\pi_*(\frac{1}{i!}(\mu^T_{-2\xi})^i(\Omega+\mu^K)^{n+k})].
$$

We pick a collection of semi-norms $\{\|\cdot\|_l\}_{l\in\mathbb{Z}_{\geq 0}}$ on $\Omega^{2(n+k)}_K(\mathcal{X})$ so that it defines the Fréchet structure of $\Omega^{2(n+k)}_K(\mathcal{X})$. For instance, we may put $\|\varphi\|_l := \|\varphi|_{D_l}\|_{C^l}$ using an exhaustion $\{D_l \subset \mathcal{X}\}_{l\in\mathbb{Z}_{\geq 0}}$ by compact sets. Now we easily see the infinite series of $K$-equivariant forms

$$
\sum_{i=0}^{\infty} \frac{1}{i!}(\mu^T_{-2\xi})^i(\Omega+\mu^K)^{n+k}
$$

is locally uniformly absolutely-convergent on $\mathfrak{t}$ with respect to the semi-norm $\|\cdot\|_l$ for each $l \in \mathbb{Z}_{\geq 0}$, to the limit $K$-equivariant form

$$e^{\mu^T_{-2\xi}}(\Omega+\mu^K)^{n+k}.$$

Since $\pi_*(\frac{1}{i!}(\mu^T_{-2\xi})^i(\Omega+\mu^K)^{n+k})$ are $K$-equivariantly closed forms, the infinite series of $2k$-th $K$-equivariant cohomology classes on $B$

$$\sum_{i=0}^{\infty} \mathrm{Ev}_\xi(e^{\mathcal{L}})_B^{\langle i,k\rangle} = \frac{1}{(n+k)!}\sum_{i=0}^{\infty} \mathrm{PD}_K[\pi_*(\frac{1}{i!}(\mu^T_{-2\xi})^i(\Omega+\mu^K)^{n+k})]$$

is locally uniformly absolutely-convergent on $\mathfrak{t}$ by Corollary 2.4.12 and Lemma 2.4.13. The limit is given by

$$\frac{1}{(n+k)!}\mathrm{PD}_K[\pi_*(e^{\mu^T_{-2\xi}}(\Omega+\mu^K)^{n+k})].$$

Similarly, for $\alpha \in H^{\mathrm{lf},T\times G}_{2\dim\mathcal{X}-2}(\mathcal{X},\mathbb{R})$, we pick an equivariant 2-form $A+\nu = A+(\nu^T+\nu^K)$ in the second equivariant cohomology class $\mathrm{PD}_{T\times K}(\alpha)$. We compute

$$\begin{aligned}(\alpha.e^{\mathcal{L}})_B^{\langle i,k\rangle} &= \frac{1}{(n+i+k-1)!}(\mathrm{PD}_{T\times K}\left[\pi_*\big((A+\nu)(\Omega+\mu)^{n+i+k-1}\big)\right])^{\langle i,k\rangle}\\ &= \begin{cases} \frac{1}{(n+k)!}\mathrm{PD}_K[\pi_*\big((n+k)(A+\nu^K)(\Omega+\mu^K)^{n+k-1}\big)] & i=0\\ \frac{1}{(n+k)!}\mathrm{PD}_K[\pi_*\big(\nu^T\frac{1}{(i-1)!}(\mu^T)^{i-1}(\Omega+\mu^K)^{n+k} & \\ \qquad\qquad +(n+k)\frac{1}{i!}(\mu^T)^i(A+\nu^K)(\Omega+\mu^K)^{n+k-1}\big)] & i\geq 1.\end{cases}\end{aligned}$$

By the same argument as above, we obtain that the infinite series

$$\begin{aligned}\sum_{i=0}^{\infty}\mathrm{Ev}_\xi(\alpha.e^{\mathcal{L}_{T\times G}})_B^{i,k} = \frac{\mathrm{PD}_K}{(n+k)!}\sum_{i=0}^{\infty}[\pi_*(\nu^T_{-2\xi}\frac{1}{i!}(\mu^T_{-2\xi})^i(\Omega+\mu^K)^{n+k}\\ +(n+k)\frac{1}{i!}(\mu^T_{-2\xi})^i(A+\nu^K)(\Omega+\mu^K)^{n+k-1})]\end{aligned}$$

is locally uniformly absolutely-convergent on $\mathfrak{t}$ to

$$\frac{1}{(n+k)!}\mathrm{PD}_K[\pi_*(\nu^T_{-2\xi}e^{\mu^T_{-2\xi}}(\Omega+\mu^K)^{n+k}+(n+k)e^{\mu^T_{-2\xi}}(A+\nu^K)(\Omega+\mu^K)^{n+k-1})].$$

Thus we obtain

$$\mathcal{D}^k_\xi(e^{\mathcal{L}}) = \frac{(-2)^k}{(n+k)!}\mathrm{PD}_K[\pi_*(e^{\mu^T_{-2\xi}}(\Omega+\mu^K)^{n+k})] \tag{2.25}$$

$$\mathcal{D}^k_\xi(\alpha.e^{\mathcal{L}}) = \frac{(-2)^k}{(n+k)!}\mathrm{PD}_K[\pi_*(\nu^T_{-2\xi}e^{\mu^T_{-2\xi}}(\Omega+\mu^K)^{n+k}) \tag{2.26}$$
$$+ (n+k)e^{\mu^T_{-2\xi}}(A+\nu^K)(\Omega+\mu^K)^{n+k-1}]$$

□

**Lemma 2.3.5.** Suppose $\pi : \mathcal{X} \to B$ is moreover flat and $\mathcal{L}$ is in the equivariant Neron–Severi group $NS_{T\times G}(\mathcal{X},\mathbb{R})$. Let $\mathcal{L}_b \in H^2_T(\mathcal{X}_b,\mathbb{R})$ be the restriction to the fibre of a point $b \in B$. Then we have

$$\mathcal{D}^0_\xi(e^{\mathcal{L}})_B = \mathrm{Ev}_\xi(e^{\mathcal{L}_b}) \in \mathbb{R}, \tag{2.27}$$

$$\mathcal{D}^0_\xi(\kappa_{\mathcal{X}/B}.e^{\mathcal{L}})_B = \mathrm{Ev}_\xi(\kappa_{\mathcal{X}_b}.e^{\mathcal{L}_b}) \in \mathbb{R}. \tag{2.28}$$

In particular, $\mathcal{D}^0_\xi(e^{\mathcal{L}})_B$ is positive when $\mathcal{L}$ is relatively semi-ample and big.

*Proof.* The claim on $\mathcal{D}^0_\xi(\kappa_{\mathcal{X}/B}.e^{\mathcal{L}})_B$ is nothing but Corollary 2.4.16. The case $\mathcal{D}^0_\xi(e^{\mathcal{L}})_B$ follows similarly from the equivariant Grothendieck–Riemann–Roch theorem. □

### 2.3.2 $\mu$-character, $\mu$-Futaki invariant and $\mu$K-stability

Now, our main theorems are just the application of the equivariant calculus we developed in the previous sections.

#### $\mu$-character and its derivative

We introduce a relative version of the $\mu$-entropy (2.8), imitating its equivariant intersection formula.

**Definition 2.3.6** ($\mu$-character)**.** When $\mathcal{L} \in NS_G(\mathcal{X},\mathbb{R})$ is relatively semi-ample and big, $(e^{\mathcal{L}})^{\langle 0\rangle}_B = (\mathcal{L}^{\cdot n}_b) \in \mathbb{R}$ is positive. In this case, we can define the following equivariant cohomology class in $\hat{H}^{\mathrm{even}}_G(B,\mathbb{R})$:

$$\boldsymbol{\mu}^\lambda_G(\mathcal{X}/B,\mathcal{L}) := 2\pi(\kappa^G_{\mathcal{X}/B}.e^{\mathcal{L}})_B\cdot(e^{\mathcal{L}})^{-1}_B + \lambda(\mathcal{L}.e^{\mathcal{L}})_B\cdot(e^{\mathcal{L}})^{-1}_B - \lambda\log(e^{\mathcal{L}})_B. \tag{2.29}$$

We call this the *$\mu$-character* of the $G$-equivariant family $(\mathcal{X}/B,\mathcal{L})$.

By Theorem 2.3.4 and the basic properties of the differential operation, we obtain the following.

**Theorem 2.3.7.** The element $\boldsymbol{\mu}^{\lambda}_{T\times G}(\mathcal{X}/B,\mathcal{L})$ is of class $\varepsilon^{\infty}$ around the origin. The derivative $\mathcal{D}_{\xi}\boldsymbol{\mu}^{\lambda}_{T\times G}(\mathcal{X}/B,\mathcal{L})$ extends to the following analytic prolongation on $\mathfrak{t}$:

$$2\pi\frac{\mathcal{D}_{\xi}(\kappa^{T\times G}_{\mathcal{X}/B}.e^{\mathcal{L}_{T\times G}})_B\cdot\mathrm{Ev}_{\xi}(e^{\mathcal{L}_b})-\mathrm{Ev}_{\xi}(\kappa^{T}_{\mathcal{X}_b}.e^{\mathcal{L}_b})\cdot\mathcal{D}_{\xi}(e^{\mathcal{L}_{T\times G}})_B}{(\mathrm{Ev}_{\xi}(e^{\mathcal{L}_b}))^2}$$
$$+\lambda\left[\frac{\mathcal{D}_{\xi}(\mathcal{L}_{T\times G}.e^{\mathcal{L}_{T\times G}})_B\cdot\mathrm{Ev}_{\xi}(e^{\mathcal{L}_b})-\mathrm{Ev}_{\xi}(\mathcal{L}_b.e^{\mathcal{L}_b})\cdot\mathcal{D}_{\xi}(e^{\mathcal{L}_{T\times G}})_B}{(\mathrm{Ev}_{\xi}(e^{\mathcal{L}_b}))^2}-\frac{\mathcal{D}_{\xi}(e^{\mathcal{L}_{T\times G}})_B}{\mathrm{Ev}_{\xi}(e^{\mathcal{L}_b})}\right]$$

for $\xi$ close to 0.

We define the second cohomology class $\mathcal{D}_{\xi}\boldsymbol{\mu}^{\lambda}_{T\times G}(\mathcal{X}/B,\mathcal{L})$ for $\xi\in\mathfrak{t}$ away from the origin by the above analytic prolongation.

*Remark* 2.3.8. In the construction of the characteristic class $\mathcal{D}_{\xi}\boldsymbol{\mu}^{\lambda}_{T\times G}(\mathcal{X}/B,\mathcal{L})$, we assume the smoothness of the base $B$ in order to ensure the Poincaré duality between the equivariant cohomology $H^2_G(B,\mathbb{R})$ and the equivariant locally finite homology $H^{\mathrm{lf},G}_{2\dim B-2}(B,\mathbb{R})$. So there is a room for extending our result to some singular bases, or perhaps to general singular bases by using other cohomology theory, if we can establish equivariant calculus for such cohomology theory as we did it for singular/deRham equivariant cohomology.

As another viewpoint, our construction works even for families of almost complex manifolds. (In this case, $\mathcal{L}$ is just a $T\times G$-equivariant cohomology class. ) As a consequence, we can define $\mathcal{D}_{\xi}\boldsymbol{\mu}^{\lambda}_{T\times G}(\mathcal{X}/B,\mathcal{L})$ for a Kuranishi family of $T$-polarized manifold with a singular base $B$ by pulling back the equivariant cohomology class $\mathcal{D}_{\xi}\boldsymbol{\mu}^{\lambda}_{T\times G}(\tilde{\mathcal{X}}/\tilde{B},\tilde{\mathcal{L}})\in H^2(\tilde{B},\mathbb{R})$ associated to the Kuranishi slice $\tilde{B}\to\mathcal{J}$ which we take when constructing the Kuranishi family. The author suspects this idea allows us to construct $\mathcal{D}_{\xi}\boldsymbol{\mu}^{\lambda}_{T\times G}(\mathcal{X}/B,\mathcal{L})$ for a global base $B$ by gluing these characteristic classes in some canonical way.

It is preferable for gluing that we realize the characteristic class $\mathcal{D}_{\xi}\boldsymbol{\mu}^{\lambda}_{T\times G}(\mathcal{X}/B,\mathcal{L})$ as a geometric object whose category forms a stack (namely, has a natural criterion for the descent of objects), as an analogy of the CM line bundle. Such a geometric realization is also important when descending it to a cohomology class on the moduli space. Indeed, since in general we a priori know the moduli space does not admit a universal family, our Theorem B constructs nothing on the moduli space, at present. Since the cohomology class $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times G}(\mathcal{X}/B,\mathcal{L})$ continuously deforms in $H^2_G(B,\mathbb{R})$ as $\xi$ varies, it does not make sense to realize it as a complex line bundle on $B$, imitating the CM line bundle. On the other hand, real line bundle on $B$ endowed with pluri-harmonic transition functions may serve as such geometric object, but the actual construction is out of the author's consideration at the moment.

**Test configuration**

By a *$G$-equivariant $\mathbb{Q}$-line bundle* on a scheme $X$, we mean a $G$-equivariant Neron–Severi class $L \in NS_G(X,\mathbb{Q}) \subset H^2_G(X,\mathbb{Q})$. We call a $G$-equivariant $\mathbb{Q}$-line bundle $L$ *ample (resp. semi-ample, big)* if some multiple of the corresponding Neron–Severi class $L \in NS(X,\mathbb{Q})$ is the first Chern class of some ample (resp. semi-ample, big) line bundle. Here the bigness means that the volume $(L|_Z)^{\cdot \dim Z}$ is strictly positive on each irreducible component $Z$ of $X$. We call a pair $(X,L)$ of pure dimensional projective $G$-scheme (resp. variety, normal variety) $X$ and a *semi-ample* and big $G$-equivariant $\mathbb{Q}$-line bundle $L$ a *semi-polarized $G$-scheme (resp. variety, normal variety)*. When $L$ is ample, we call it polarized $G$-scheme.

A ($T$-equivariant) *test configuration* $(\mathcal{X},\mathcal{L})$ of a semi-polarized $T$-scheme $(X,L)$ consists of the following data:

- A $T\times\mathbb{C}^*$-scheme $\mathcal{X}$ with a $T\times\mathbb{C}^*$-equivariant projective flat morphism $\pi:\mathcal{X}\to\mathbb{C}$, where we define the $T\times\mathbb{C}^*$-action on the base $\mathbb{C}$ by $z.(t,u)=zu$ for $z\in\mathbb{C}$ and $(t,u)\in T\times\mathbb{C}^*$.

- A $T\times\mathbb{C}^*$-equivariant $\mathbb{Q}$-line bundle $\mathcal{L}\in NS_{T\times\mathbb{C}^*}(\mathcal{X},\mathbb{Q})$ on $\mathcal{X}$ which is (relatively) semi-ample and relatively big.

- A $T\times\mathbb{C}^*$-equivariant isomorphism $\varphi : X\times(\mathbb{C}\setminus\{0\}) \xrightarrow{\sim} \mathcal{X}\setminus\mathcal{X}_0$ over the base with $\varphi^*\mathcal{L}=p_X^*L\in NS_{T\times\mathbb{C}^*}(X\times(\mathbb{C}\setminus\{0\}),\mathbb{Q})$, which we often reduce from our notation.

We call a test configuration $(\mathcal{X},\mathcal{L})$ *ample* if $\mathcal{L}$ is (relatively) ample. In this case, the compactification $(\bar{\mathcal{X}},\bar{\mathcal{L}})$ possesses relatively ample $\bar{\mathcal{L}}$.

As in section 2.2.1, we can construct a test configuration $\mathcal{X}^\Lambda$ from a one parameter subgroup $\Lambda:\mathbb{C}^*\to\mathrm{Aut}_T(X)$.

When $\mathcal{X}$ is normal (hence $X$ is normal) and is isomorphic in codimension one to some product configuration $\mathcal{X}^\Lambda$, then the isomorphism automatically extends to the whole space.

**$\mu$-Futaki invariant**

To compare $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times\mathbb{C}^*}(\mathcal{X}/\mathbb{C},\mathcal{L})$ of $T$-equivariant test configuration with the $\mu$-Futaki invariant $\mathrm{Fut}^\lambda_\xi(\mathcal{L},\mathcal{L})$, we show the following.

**Proposition 2.3.9.** For any test configuration $(\mathcal{X}/\mathbb{C},\mathcal{L})$, we have

$$\begin{aligned}
\mathcal{D}_\xi(e^{\mathcal{L}_{T\times\mathbb{C}^*}})_{\mathbb{C}} &= 2\mathrm{Ev}_\xi(e^{\bar{\mathcal{L}}_T}).\eta^\vee,\\
\mathcal{D}_\xi(\mathcal{L}_{T\times\mathbb{C}^*}.e^{\mathcal{L}_{T\times\mathbb{C}^*}})_{\mathbb{C}} &= 2\mathrm{Ev}_\xi(\bar{\mathcal{L}}_T.e^{\bar{\mathcal{L}}_T}).\eta^\vee,\\
\mathcal{D}_\xi(\kappa^{T\times\mathbb{C}^*}_{\mathcal{X}/\mathbb{C}}.e^{\mathcal{L}_{T\times\mathbb{C}^*}})_{\mathbb{C}} &= 2\mathrm{Ev}_\xi(\kappa^T_{\bar{\mathcal{X}}/\mathbb{C}P^1}.e^{\bar{\mathcal{L}}_T}).\eta^\vee,
\end{aligned}$$

where we identify $\hat{H}_{\mathbb{C}^*}(\mathbb{C},\mathbb{R})$ with $\mathbb{R}[\![\eta^\vee]\!]$.

*Proof.* Since $j_\infty^*\bar{\mathcal{L}}_{T\times\mathbb{C}^*} = L_T \in NS_{T\times\mathbb{C}^*}(X,\mathbb{R})$, we have

$$i_\infty^*\pi_*\bar{\mathcal{L}}^{\smile(n+i+1)}_{T\times\mathbb{C}^*} = (\pi_\infty)_* j_\infty^*\bar{\mathcal{L}}^{\smile(n+i+1)}_{T\times\mathbb{C}^*} = (L_T^{\smile(n+i+1)}) \in H^{2i+2}_{T\times\mathbb{C}^*}(\mathrm{pt},\mathbb{R})$$

and

$$i_\infty^*\pi_\star(\kappa^{T\times\mathbb{C}^*}_{\mathcal{X}/\mathbb{C}} \frown \bar{\mathcal{L}}^{\smile(n+i)}_{T\times\mathbb{C}^*}) = (\kappa^T_X.L_T^{\smile(n+i)}) \in H^{2i+2}_{T\times\mathbb{C}^*}(\mathrm{pt},\mathbb{R})$$

by Corollary 2.4.16. In particular, we have $(i_\infty^*\pi_*\bar{\mathcal{L}}^{\smile(n+i+1)}_{T\times\mathbb{C}^*})^{\langle i,1\rangle} = 0$ and $(i_\infty^*\pi_\star(\kappa^{T\times\mathbb{C}^*}_{\mathcal{X}/\mathbb{C}} \frown \bar{\mathcal{L}}^{\smile(n+i)}_{T\times\mathbb{C}^*}))^{\langle i,1\rangle} = 0$. So we compute

$$\begin{aligned}
(\pi_*\mathcal{L}^{\smile(n+i+1)}_{T\times\mathbb{C}^*})^{\langle i,1\rangle} &= (i_0^*\pi_*\mathcal{L}^{\smile(n+i+1)}_{T\times\mathbb{C}^*})^{\langle i,1\rangle} = -(i_\infty^*\pi_*\bar{\mathcal{L}}^{\smile(n+i+1)}_{T\times\mathbb{C}^*} - i_0^*\pi_*\bar{\mathcal{L}}^{\smile(n+i+1)}_{T\times\mathbb{C}^*})^{\langle i,1\rangle}\\
&= -((\bar{\mathcal{L}}^{\smile(n+i+1)}_{T\times\mathbb{C}^*}).\eta^\vee)^{\langle i,1\rangle} = -(\bar{\mathcal{L}}_T^{\smile(n+i+1)})^{\langle i,0\rangle}.\eta^\vee
\end{aligned}$$

and

$$\begin{aligned}
(\pi_\star(\kappa^{T\times\mathbb{C}^*}_{\mathcal{X}/\mathbb{C}} \frown \mathcal{L}^{\smile(n+i)}_{T\times\mathbb{C}^*}))^{\langle i,1\rangle} &= (i_0^*\pi_\star(\kappa^{T\times\mathbb{C}^*}_{\mathcal{X}/\mathbb{C}} \frown \mathcal{L}^{\smile(n+i)}_{T\times\mathbb{C}^*}))^{\langle i,1\rangle}\\
&= -(i_\infty^*\pi_\star(\kappa^{T\times\mathbb{C}^*}_{\mathcal{X}/\mathbb{C}} \frown \bar{\mathcal{L}}^{\smile(n+i)}_{T\times\mathbb{C}^*}) - i_0^*\pi_\star(\kappa^{T\times\mathbb{C}^*}_{\mathcal{X}/\mathbb{C}} \frown \bar{\mathcal{L}}^{\smile(n+i)}_{T\times\mathbb{C}^*}))^{\langle i,1\rangle}\\
&= -((\kappa^{T\times\mathbb{C}^*}_{\mathcal{X}/\mathbb{C}}.\bar{\mathcal{L}}^{\smile(n+i)}_{T\times\mathbb{C}^*}).\eta^\vee)^{\langle i,1\rangle} = -(\kappa^T_{\mathcal{X}/\mathbb{C}}.\bar{\mathcal{L}}_T^{\smile(n+i)})^{\langle i,0\rangle}.\eta^\vee
\end{aligned}$$

by the localization formula. Thus we obtain

$$\mathcal{D}_\xi(\mathcal{L}^{\smile(n+i+1)}_{T\times\mathbb{C}^*})_{\mathbb{C}} = 2\mathrm{Ev}_\xi(\bar{\mathcal{L}}_T^{\smile(n+i+1)})$$

and

$$\mathcal{D}_\xi(\kappa^{T\times\mathbb{C}^*}_{\mathcal{X}/\mathbb{C}}.\mathcal{L}^{\smile(n+i+1)}_{T\times\mathbb{C}^*})_{\mathbb{C}} = 2\mathrm{Ev}_\xi(\kappa^T_{\mathcal{X}/\mathbb{C}}.\bar{\mathcal{L}}_T^{\smile(n+i)}),$$

which proves the claim as all the cohomology classes we treat here are of class $\varepsilon^\infty$. $\square$

As corollaries, we get the following.

**Corollary 2.3.10.**

$$\mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}/\mathbb{C}, \mathcal{L}) = \mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}).\eta^\vee$$

**Corollary 2.3.11.** For the product configuration $(\mathcal{X}^\Lambda, \mathcal{L}^\Lambda)$ associated to a one parameter subgroup $\Lambda : \mathbb{C}^* \to \mathrm{Aut}(X, L)$, we have

$$\mathrm{Fut}^\lambda_\xi(\mathcal{X}^\Lambda, \mathcal{L}^\Lambda) = \mathrm{Fut}^\lambda_\xi(\eta_X).$$

*Proof.* As both $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}^\Lambda/\mathbb{C}, \mathcal{L}^\Lambda)$ and $\mathrm{Fut}^\lambda_\xi(\eta_X)$ are real analytic with respect to $\xi$, the claim follows from the construction of $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda$. (See also seciton 2.1 and 2.3. ) $\square$

Now we prove the rest claim of Theorem G.

*Proof of Theorem G.* The property (1) is a consequence of Corollary 2.4.16 and our construction of $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times G}(\mathcal{X}/B, \mathcal{L})$. From the expression (2.25) and (2.26), the maps $\mathcal{D}.(e^{\mathcal{L}}), \mathcal{D}.(\alpha.e^{\mathcal{L}}) : \mathfrak{t} \to H^2_G(B, \mathbb{R})$ extend to holomorphic maps between the complexification $\mathfrak{t} \otimes \mathbb{C}$ and $H^2_G(B, \mathbb{C})$. Hence the map $\mathcal{D}.\boldsymbol{\mu}^\lambda_{T\times G}(\mathcal{X}/B, \mathcal{L}) : \mathfrak{t} \to H^2_G(B, \mathbb{R})$ is real analytic.

Recall the definition of the CM line bundle $\mathrm{CM}(\mathcal{X}/B, \mathcal{L})$: we put

$$\mathrm{CM}(\mathcal{X}/B, \mathcal{L}) := \lambda_{n+1}^{\otimes \frac{n}{n+1} \frac{(-K_X.L^{\cdot(n-1)})}{(L^{\cdot n})}} \otimes (\lambda_{n+1}^{\otimes n} \otimes \lambda_n^{\otimes(-2)})$$

where $\lambda_i$ are line bundles on $B$ in the Knudsen–Mumford expansion

$$\det(\pi_*(L^{\otimes k})) \cong \lambda_{n+1}^{\binom{k}{n+1}} \otimes \lambda_n^{\binom{k}{n}} \otimes \cdots \otimes \lambda_0$$

for $k \gg 0$. So we have

$$\begin{aligned} c_1^G(\det(\pi_*(L^{\otimes k}))) &= \binom{k}{n+1} c_1^G(\lambda_{n+1}) + \binom{k}{n} c_1^G(\lambda_n) + \cdots + c_1^G(\lambda_0) \\ &= \frac{k^{n+1}}{(n+1)!} c_1^G(\lambda_{n+1}) - \frac{1}{2}\frac{k^n}{n!}(n c_1^G(\lambda_{n+1}) - 2 c_1^G(\lambda_n)) + \cdots . \end{aligned}$$

On the other hand, the equivariant Grothendieck–Riemann–Roch shows

$$\begin{aligned} c_1^G(\det(\pi_*(L^{\otimes k}))) &= k^{n+1}\big(\frac{1}{-2}\mathcal{D}_0(e^{\mathcal{L}})_B\big) - \frac{1}{2}k^n\big(\frac{1}{-2}\mathcal{D}_0(\kappa_{\mathcal{X}/B}.e^{\mathcal{L}})_B\big) + \cdots \\ &= \frac{k^{n+1}}{(n+1)!}\big(\frac{1}{-2}\mathcal{D}_0(\mathcal{L}^{\cdot(n+1)})_B\big) - \frac{1}{2}\frac{k^n}{n!}\big(\frac{1}{-2}\mathcal{D}_0(\kappa_{\mathcal{X}/B}.\mathcal{L}^{\cdot n})_B\big) + \cdots . \end{aligned}$$

Thus we get

$$\begin{aligned}
c_1^G(\mathrm{CM}(\mathcal{X}/B,\mathcal{L})) &= \frac{n}{n+1}\frac{(-K_X.L^{\cdot(n-1)})}{(L^{\cdot n})}(n+1)!\big(\frac{1}{-2}\mathcal{D}_0(e^{\mathcal{L}})_B\big)+n!\big(\frac{1}{-2}\mathcal{D}_0(\kappa_{\mathcal{X}/B}.e^{\mathcal{L}})_B\big)\\
&= -\frac{1}{2}(L^{\cdot n})\frac{\mathcal{D}_0(\kappa_{\mathcal{X}/B}.e^{\mathcal{L}})_B\cdot \mathrm{Ev}_0(e^{\mathcal{L}_b})-\mathrm{Ev}_0(\kappa^T_{\mathcal{X}_b}.e^{\mathcal{L}_b})\cdot\mathcal{D}_0(e^{\mathcal{L}})_B}{(\mathrm{Ev}_0(e^{\mathcal{L}_b}))^2}\\
&= -\frac{(L^{\cdot n})}{4\pi}\mathcal{D}_0\boldsymbol{\mu}^0_G(\mathcal{X}/B,\mathcal{L})
\end{aligned}$$

by $\mathrm{Ev}_0(e^{\mathcal{L}_b})=\frac{1}{n!}(L^{\cdot n})$ and $\mathrm{Ev}_0(\kappa^T_{\mathcal{X}_b}.e^{\mathcal{L}_b})=\frac{1}{(n-1)!}(K_X.L^{\cdot(n-1)})$. The independence of $\lambda$ follows from $\mathcal{D}_0(\mathcal{L}_G.e^{\mathcal{L}_G})_B=\frac{1}{n!}\mathcal{D}_0(\mathcal{L}_G^{\smile(n+1)})_B$, $\mathcal{D}_0(e^{\mathcal{L}_G})_B=\frac{1}{(n+1)!}\mathcal{D}_0(\mathcal{L}_G^{\smile(n+1)})_B$ and $\mathrm{Ev}_0(\mathcal{L}_b.e^{\mathcal{L}_b})=\frac{1}{(n-1)!}(L^{\cdot n})$:

$$\begin{aligned}
\mathcal{D}_0\boldsymbol{\mu}^\lambda-\mathcal{D}_0\boldsymbol{\mu}^0 &= \lambda\left[\frac{\mathcal{D}_0(\mathcal{L}_{T\times G}.e^{\mathcal{L}_{T\times G}})_B\cdot\mathrm{Ev}_0(e^{\mathcal{L}_b})-\mathrm{Ev}_0(\mathcal{L}_b.e^{\mathcal{L}_b})\cdot\mathcal{D}_0(e^{\mathcal{L}_{T\times G}})_B}{(\mathrm{Ev}_0(e^{\mathcal{L}_b}))^2}-\frac{\mathcal{D}_0(e^{\mathcal{L}_{T\times G}})_B}{\mathrm{Ev}_0(e^{\mathcal{L}_b})}\right]\\
&= \frac{\lambda}{(L^{\cdot n})}\Big(\mathcal{D}_0(\mathcal{L}_G^{\smile(n+1)})_B-\frac{n}{n+1}\mathcal{D}_0(\mathcal{L}_G^{\smile(n+1)})-\frac{1}{n+1}\mathcal{D}_0(\mathcal{L}_G^{\smile(n+1)})\Big)\\
&= 0.
\end{aligned}$$

□

**Relation with established Futaki invariants**

Here we check that our definition of $\mu$-Futaki invariant is compatible with the following established notions.

**Definition 2.3.12.** Let $(X,L)$ be a (semi-)polarized scheme and $(\mathcal{X},\mathcal{L})$ be a test configuration of $(X,L)$. The following Futaki invariants are studied in the literatures.

1. *Donaldson–Futaki invariant* for the usual K-stability (cf. ): We put

$$\mathrm{DF}(\mathcal{X},\mathcal{L}):=(K_{\bar{\mathcal{X}}/\mathbb{C}P^1}.\bar{\mathcal{L}}^{\cdot n})-\frac{n}{n+1}\frac{(K_X.L^{\cdot(n-1)})}{(L^{\cdot n})}(\bar{\mathcal{L}}^{\cdot(n+1)}). \qquad (2.30)$$

2. *Modified Futaki invariant* for the modified K-stability of $\mathbb{Q}$-Fano variety (cf. [Xio, BW]): Let $(X,L)=(X,-K_X)$ be a $\mathbb{Q}$-Fano variety with a

torus $T$ action and $(\mathcal{X}, \mathcal{L})$ be a $T$-equivariant test configuration with $\mathbb{Q}$-Gorenstein $\mathcal{X}$ and $\mathcal{L} = -K_{\mathcal{X}/\mathbb{C}}$. For $\xi \in \mathfrak{t}$, we put

$$\mathrm{Fut}_\xi(\mathcal{X}, \mathcal{L}) := -\int_{\mathfrak{t}^\vee \times \mathbb{R}} (-2t) e^{\langle x, -2\xi \rangle} \mathrm{DH}_{(\mathcal{X}, \mathcal{L})}(x, t), \tag{2.31}$$

where the Duistermaat–Heckman measure $\mathrm{DH}_{(\mathcal{X}, \mathcal{L})}$ on $\mathfrak{t}^\vee \times \mathbb{R}$ is given by

$$\mathrm{DH}_{(\mathcal{X}, \mathcal{L})} := \lim_{k \to \infty} n! k^{-n} \sum_{(m,l) \in M \times \mathbb{Z}} \dim H^0(\mathcal{X}_0, \mathcal{L}|_{\mathcal{X}_0})_{(m,l)} \delta_{k^{-1}(m,l)}.$$

When the central fibre $\mathcal{X}_0$ is a $\mathbb{Q}$-Fano variety, we may express it as

$$\mathrm{Fut}_\xi(\mathcal{X}, \mathcal{L}) = -\int_{\mathcal{X}_0} \theta_\eta e^{\theta_\xi} \omega^n = \int_{\mathcal{X}_0} \eta^J (h - \theta_\xi) e^{\theta_\xi} \omega^n,$$

which is the expression in [Xio, BW]. Here the $\bar{\partial}$-Hamiltonian potential $\theta$ is normalized as $[\beta^* \omega - \frac{1}{2} \beta^* \theta] = -\beta^* K_{\mathcal{X}_0}^{T \times U(1)} \in H^2_{T \times U(1)}(\tilde{\mathcal{X}}_0, \mathbb{R})$ for some/any resolution $\beta : \tilde{\mathcal{X}}_0 \to \mathcal{X}_0$. This normalization is equivalent to the equation $\bar{\square} \theta_\eta - \eta^J h = \theta_\eta$ for the Ricci potential $h$: $\sqrt{-1} \partial \bar{\partial} h = \mathrm{Ric} \omega - \omega$.

3. *Weighted Futaki invariant* of smooth test configuration (cf. [Lah]) for weighted K-stability: Let $v$ and $w$ be smooth positive functions on the moment polytope $P := \mu^\omega(X) \subset \mathfrak{t}^\vee$. For a $T$-equivariant smooth test configuration $(\mathcal{X}, \mathcal{L})$ with ample $\bar{\mathcal{L}}$, we pick a Kähler form $\Omega$ in $\bar{\mathcal{L}}$ and the moment map $\mu^\Omega$ with $[\Omega + \mu^\Omega] = \bar{\mathcal{L}}_T$ and put

$$\mathcal{F}_{v,w}(\mathcal{X}, \mathcal{L}) = -\frac{2}{n+1} \int_{\bar{\mathcal{X}}} \Big( s_v(\Omega) - \frac{\int_X s_v(\omega) \omega^n}{\int_X (w \circ \mu^\omega) \omega^n} (w \circ \mu^\Omega) \Big) \Omega^{n+1} \tag{2.32}$$
$$+ 8\pi \int_X (v \circ \mu^\omega) \omega^n.$$

We consider the following variant of weighted Futaki invariant for $T$-equivariant smooth test configuration with ample $\bar{\mathcal{L}}$:

$$\mathcal{F}^\lambda_{e^{\langle x, -2\xi \rangle}}(\mathcal{X}, \mathcal{L}) := \mathcal{F}_{e^{\langle x, -2\xi \rangle}, e^{\langle x, -2\xi \rangle}}(\mathcal{X}, \mathcal{L}) + \frac{2\lambda}{n+1} \int_{\bar{\mathcal{X}}} \big( \mu^\Omega_{-2\xi} - \frac{\int_X \mu^\omega_{-2\xi} e^{\mu^\omega_{-2\xi}} \omega^n}{\int_X e^{\mu^\omega_{-2\xi}} \omega^n} \big) e^{\mu^\Omega_{-2\xi}} \Omega^{n+1} \tag{2.33}$$
$$= -\frac{2}{n+1} \int_{\bar{\mathcal{X}}} \big( s^\lambda_\xi(\Omega) - \bar{s}^\lambda_\xi(\omega) \big) e^{\theta^\Omega_\xi} \Omega^{n+1} + 8\pi \int_X e^{\theta^\omega_\xi} \omega^n.$$

Though it seems not explicitly claimed in [Lah] for this variant case, we easily see that the above invariant is the slope of $\mu_\xi^\lambda$-Mabuchi functional along a smooth subgeodesic of a test configuration, using computations in [Lah]. As a $\mu_\xi^\lambda$-cscK metric is a (proportionally) extremal weighted cscK metric, we have the boundedness of $\mu_\xi^\lambda$-Mabuchi functional if there exists a $\mu_\xi^\lambda$-cscK metric by [Lah]. As a consequence, a polarized manifold is $\mu_\xi^\lambda$K-semistable with resepct to smooth test configurations with ample $\mathcal{L}$ if there exists a $\mu_\xi^\lambda$-cscK metric.

Now we compare them with our $\mu$-Futaki invariant.

**Proposition 2.3.13.** We can compare the $\mu$-Futaki invariant with these established Futaki invariants as follows.

1. When $(X, L)$ is a normal polarized variety, $\mathrm{Fut}_0^\lambda(\mathcal{X}/\mathbb{C}, \mathcal{L}) = 4\pi (L^{\cdot n})^{-1} \mathrm{DF}(\mathcal{X}, \mathcal{L})$ for every normal test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, L)$.
2. When $X$ is a $\mathbb{Q}$-Fano variety and $L = -\lambda^{-1} K_X$ for $\lambda > 0$, we have $\mathrm{Fut}_\xi^{2\pi\lambda}(\mathcal{X}, \mathcal{L}) = 2\pi\lambda n! (\mathrm{Ev}_\xi(e^{\bar{\mathcal{L}}}))^{-1} \mathrm{Fut}_\xi(\mathcal{X}, \mathcal{L})$ for every test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, L)$ with $\mathcal{L} = -\lambda^{-1} K_{\mathcal{X}/\mathbb{C}}$.
3. Let $(\mathcal{X}, \mathcal{L})$ be a $T$-equivariant smooth test configuration with ample $\bar{\mathcal{L}}$. We pick a Kähler form $\Omega$ in $\bar{\mathcal{L}}$. Then we have $\mathrm{Fut}_\xi^\lambda(\mathcal{X}, \mathcal{L}) = (\int_X e^{\theta_\xi^\omega} \omega^n)^{-1} \mathcal{F}^\lambda_{e^{\langle x, -2\xi \rangle}}(\mathcal{X}, \mathcal{L})$.

*Proof.*

(1) We compute $\mathrm{Fut}_0^\lambda(\mathcal{X}/\mathbb{C}, \mathcal{L})$ as

$$4\pi \frac{\mathrm{Ev}_0\Big((\kappa_{\bar{\mathcal{X}}/\mathbb{C}P^1}.e^{\bar{\mathcal{L}}}) \cdot (e^L) - (\kappa_X.e^L) \cdot (e^{\bar{\mathcal{L}}})\Big)}{(\mathrm{Ev}_0(e^L))^2} + 2\lambda \left[ \frac{\mathrm{Ev}_0\Big((\bar{\mathcal{L}}.e^{\bar{\mathcal{L}}}) \cdot (e^L) - (L.e^L) \cdot (e^{\bar{\mathcal{L}}})\Big)}{(\mathrm{Ev}_0(e^L))^2} - \frac{\mathrm{Ev}_0(e^{\bar{\mathcal{L}}})}{\mathrm{Ev}_0(e^L)} \right]$$

$$\begin{aligned}
&= 4\pi \left(\frac{n!}{(L^{\cdot n})}\right)^2 \left( \frac{1}{n!} (K_{\bar{\mathcal{X}}/\mathbb{C}P^1}.\bar{\mathcal{L}}^{\cdot n}) \cdot \frac{1}{n!} (L^{\cdot n}) - \frac{1}{(n-1)!} (K_X.L^{\cdot (n-1)}) \cdot \frac{1}{(n+1)!} (\bar{\mathcal{L}}^{\cdot (n+1)}) \right) \\
&\qquad + 2\lambda \left(\frac{n!}{(L^{\cdot n})}\right)^2 \left( \frac{(\bar{\mathcal{L}}^{\cdot (n+1)})}{n!} \cdot \frac{(L^{\cdot n})}{n!} - \frac{(L^{\cdot n})}{(n-1)!} \cdot \frac{(\bar{\mathcal{L}}^{\cdot (n+1)})}{(n+1)!} - \frac{(L^{\cdot n})}{n!} \frac{(\bar{\mathcal{L}}^{\cdot (n+1)})}{(n+1)!} \right) \\
&= \frac{4\pi}{(L^{\cdot n})} \left( (K_{\bar{\mathcal{X}}/\mathbb{C}P^1}.\bar{\mathcal{L}}^{\cdot n}) - \frac{n}{n+1} \frac{(K_X.L^{\cdot (n-1)})}{(L^{\cdot n})} (\bar{\mathcal{L}}^{\cdot (n+1)}) \right).
\end{aligned}$$

(2) We compute $\mathrm{Fut}^{2\pi\lambda}_\xi(\mathcal{X}/\mathbb{C},\mathcal{L})$ as

$$4\pi\frac{\mathrm{Ev}_\xi\Big((\kappa_{\bar{\mathcal{X}}/\mathbb{C}P^1}.e^{\bar{\mathcal{L}}})\cdot(e^L)-(\kappa_X.e^L)\cdot(e^{\bar{\mathcal{L}}})\Big)}{(\mathrm{Ev}_\xi(e^L))^2}+4\pi\lambda\left[\frac{\mathrm{Ev}_\xi\Big((\bar{\mathcal{L}}.e^{\bar{\mathcal{L}}})\cdot(e^L)-(L.e^L)\cdot(e^{\bar{\mathcal{L}}})\Big)}{(\mathrm{Ev}_\xi(e^L))^2}-\frac{\mathrm{Ev}_\xi(e^{\bar{\mathcal{L}}})}{\mathrm{Ev}_\xi(e^L)}\right]$$

$$=-4\pi\lambda\frac{\mathrm{Ev}_\xi(e^{\bar{\mathcal{L}}})}{\mathrm{Ev}_\xi(e^L)}=4\pi\lambda\int_{\mathfrak{t}^\vee\times\mathbb{R}}te^{\langle x,-2\xi\rangle}\mathrm{DH}_{(\mathcal{X},\mathcal{L})}(x,t)\Big/\int_{\mathfrak{t}^\vee}e^{\langle x,-2\xi\rangle}\mathrm{DH}_{(X,L)}(x).$$

(3) Since

$$\frac{\mathrm{Ev}_\xi\big(2\pi(\kappa_X^T.e^{L_T})+\lambda(L_T.e^{L_T})\big)}{\mathrm{Ev}_\xi(e^{L_T})}=-\bar{s}^\lambda_\xi(\omega)+\lambda n,$$

we can express $\mathrm{Fut}^\lambda_\xi(\mathcal{X},\mathcal{L})$ as

$$\frac{2\mathrm{Ev}_\xi\big(2\pi(\kappa^T_{\bar{\mathcal{X}}}.e^{\bar{\mathcal{L}}_T})+\lambda(\bar{\mathcal{L}}.e^{\bar{\mathcal{L}}})\big)+2\big(\bar{s}^\lambda_\xi(\omega)-\lambda(n+1)\big)\mathrm{Ev}_\xi(e^{\bar{\mathcal{L}}_T})-4\pi\mathrm{Ev}_\xi(\pi^*K^T_{\mathbb{C}P^1}.e^{\bar{\mathcal{L}}_T})}{\mathrm{Ev}_\xi(e^{L_T})}.$$

Then the claim is a consequence of the following calculations:

$$\begin{aligned}\mathrm{Ev}_\xi\Big(2\pi(\kappa^T_{\bar{\mathcal{X}}}.e^{\bar{\mathcal{L}}_T})+\lambda(\bar{\mathcal{L}}.e^{\bar{\mathcal{L}}})\Big)&=-\frac{1}{(n+1)!}\int_{\bar{\mathcal{X}}}s^\lambda_\xi(\Omega)e^{\theta^\Omega_\xi}\Omega^{n+1}+\lambda(n+1)\mathrm{Ev}_\xi(e^{\bar{\mathcal{L}}_T}),\\ \mathrm{Ev}_\xi(e^{\bar{\mathcal{L}}})&=\frac{1}{(n+1)!}\int_{\bar{\mathcal{X}}}e^{\theta^\Omega_\xi}\Omega^{n+1},\\ 4\pi\mathrm{Ev}_\xi(\pi^*K_{\mathbb{C}P^1}.e^{\bar{\mathcal{L}}_T})&=4\pi\int_{\mathbb{C}P^1}K_{\mathbb{C}P^1}\cdot\mathcal{D}^0_\xi(e^{\bar{\mathcal{L}}_T})_{\mathbb{C}P^1}\\ &=-8\pi\mathrm{Ev}_\xi(e^{L_T})=-\frac{8\pi}{n!}\int_X e^{\theta^\omega_\xi}\omega^n.\end{aligned}$$

□

### $\mu$K-stability

Now we define the $\mu$K-stability of a $T$-polarized scheme in the usual way.

**Definition 2.3.14** ($\mu$K-stability)**.** We call a $T$-(semi-)polarized scheme $(X,L)$

- $\mu^\lambda_\xi$*K-semistable* if $\mathrm{Fut}^\lambda_\xi(\mathcal{X},\mathcal{L})\geq 0$ for every test configuration $(\mathcal{X},\mathcal{L})$ of $(X,L)$.

- $\mu^\lambda_\xi$*K-polystable* if it is $\mu^\lambda_\xi$K-semistable and we have $\mathrm{Fut}^\lambda_\xi(\mathcal{X},\mathcal{L})=0$ for big test configurations $(\mathcal{X},\mathcal{L})$ only when $(\mathcal{X},\mathcal{L})$ is $\mathbb{C}^*\times T$-equivariantly isomorphic in codimension one to some product configuration over the base $\mathbb{C}$. Namely, there is a one parameter subgroup $\Lambda:\mathbb{C}^*\to\mathrm{Aut}_T(X)$ such that the $\mathbb{C}^*$-equivariant isomorphism $\mathcal{X}\setminus\mathcal{X}_0\to\mathcal{X}^\Lambda\setminus\mathcal{X}^\Lambda_0$ away from the central fibre extends to an isomorphism $\mathcal{X}\setminus Z\to\mathcal{X}^\Lambda\setminus Z^\Lambda$ away from $T$-invariant subschemes $Z\subset\mathcal{X}$ and $Z^\Lambda\subset\mathcal{X}^\Lambda$ of codimension $\geq 2$, respectively.

- $\mu^\lambda_\xi$*K-stable* if it is $\mu^\lambda_\xi$K-polystable and $\mathrm{Aut}^0(X/\mathrm{Alb})=T$.

*Remark* 2.3.15. The $\mu$-Futaki invariant $\mathrm{Fut}^\lambda_\xi(\mathcal{X},\mathcal{L})$ is invariant under the addition of a $T\times\mathbb{C}^*$-equivariant cohomology class $c\in H^2_{T\times\mathbb{C}^*}(\mathbb{C},\mathbb{R})$ of the base: $\mathrm{Fut}^\lambda_\xi(\mathcal{X},\mathcal{L}+\pi^*c)=\mathrm{Fut}^\lambda_\xi(\mathcal{X},\mathcal{L})$.

Now it is the turn to apply the results in section 2.3.1.

**Theorem 2.3.16.**

1. A $T$-polarized normal variety $(X,L)$ is $\mu^\lambda_\xi$K-semistable (resp. $\mu^\lambda_\xi$K-polystable, $\mu^\lambda_\xi$K-stable) with respect to general test configurations iff it is $\mu^\lambda_\xi$K-semistable (resp. $\mu^\lambda_\xi$K-polystable, $\mu^\lambda_\xi$K-stable) with respect to normal test configurations.

2. A $T$-polarized manifold $(X,L)$ is $\mu^\lambda_\xi$K-semistable with respect to general test configurations iff it is $\mu^\lambda_\xi$K-semistable with respect to smooth test configurations with reduced centrals fibre and ample $\mathcal{L}$.

*Proof.* Pick a semi-ample test configuration $(\mathcal{X},\mathcal{L})$ of $(X,L)$. As $L$ is ample and $\mathcal{L}$ is relatively semi-ample, we have a unique ample test configuration $(\mathcal{X}^{\mathrm{amp}},\mathcal{L}^{\mathrm{amp}})$ of the same $(X,L)$ associated to $(\mathcal{X},\mathcal{L})$ as in [**?**, Definition 2.16]. The associated morphism $\mu:\mathcal{X}\to\mathcal{X}^{\mathrm{amp}}$ is an isomorphism away from a codimension one subscheme of the central fibre, which is a codimension two subscheme of the total space. It follows that $\mathrm{Fut}^\lambda_\xi(\mathcal{X}^{\mathrm{amp}},\mathcal{L}^{\mathrm{amp}})=\mathrm{Fut}^\lambda_\xi(\mathcal{X},\mathcal{L})$ by Proposition 2.3.3. Thus we may assume $\mathcal{L}$ is relatively ample. By the above remark, we may further assume that $\mathcal{L}$ is ample. We apply Proposition 2.3.3 (2) to the normalization $\nu:\tilde{\mathcal{X}}\to\mathcal{X}$ and obtain the first claim.

Now, we may assume $(\mathcal{X},\mathcal{L})$ is a normal ample test configuration to prove the second claim. Since $\mathcal{X}$ is normal, there is a $T\times\mathbb{C}^*$-equivariant resolution $\beta:\tilde{\mathcal{X}}\to\mathcal{X}$ of singularities which is isomorphism away from a codimension two subscheme in $\mathcal{X}$. Then we have $\mathrm{Fut}^\lambda_\xi(\tilde{\mathcal{X}}, \beta^*\mathcal{L}) = \mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L})$ by Proposition 2.3.3.

By the reduced fibre theorem, there is a positive integer $d$ such that the normalized base-change $\hat{f}_\nu : \tilde{\mathcal{X}}^\nu_d \to \tilde{\mathcal{X}}$ along the morphism $f(z) = z^d : \mathbb{C} \to \mathbb{C}$ has the reduced central fibre. Let $\hat{f} : \tilde{\mathcal{X}}_d \to \tilde{\mathcal{X}}$ be the base change morphism along $f$. Since $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times\mathbb{C}^*}(\mathcal{X}_d/\mathbb{C}, \hat{f}^*\beta^*\mathcal{L}) = f^*\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times\mathbb{C}^*}(\mathcal{X}/\mathbb{C}, \beta^*\mathcal{L})$ and $f^*\eta^\vee = d.\eta^\vee$, we get

$$\begin{aligned}\mathrm{Fut}^\lambda_\xi(\mathcal{X}_d, \hat{f}^*\beta^*\mathcal{L}) &= \mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times\mathbb{C}^*}(\mathcal{X}_d/\mathbb{C}, \hat{f}^*\beta^*\mathcal{L})/\eta^\vee \\ &= d.f^*(\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{T\times\mathbb{C}^*}(\mathcal{X}/\mathbb{C}, \beta^*\mathcal{L})/\eta^\vee) = d.\mathrm{Fut}^\lambda_\xi(\mathcal{X}, \beta^*\mathcal{L}) \in H^0_{\mathbb{C}^*}(\mathbb{C}, \mathbb{R}).\end{aligned}$$

Thus we get $\mathrm{Fut}^\lambda_\xi(\tilde{\mathcal{X}}^\nu_d, \hat{f}^*_\nu\beta^*\mathcal{L}) \leq d.\mathrm{Fut}^\lambda_\xi(\tilde{\mathcal{X}}, \beta^*\mathcal{L})$ by Proposition 2.3.3 (2).

Put $\tilde{\mathcal{L}}_\epsilon := \hat{f}^*_\nu\beta^*\mathcal{L} - \epsilon \sum_{E\in\mathrm{Exc}(\beta)}[E]^{T\times\mathbb{C}^*}$ for $\epsilon > 0$. Then $(\tilde{\mathcal{X}}_d, \tilde{\mathcal{L}}_\epsilon)$ is a smooth test configuration with reduced central fibre and ample $\tilde{\mathcal{L}}_\epsilon$. We have $\mathrm{Fut}^\lambda_\xi(\tilde{\mathcal{X}}, \tilde{\mathcal{L}}_\epsilon) \to \mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L})$ as $\epsilon \to 0$ from the proof of Proposition 2.3.3. Now suppose $(X, L)$ is $\mu$K-semistable with respect to smooth test configurations with reduced central fibre and ample $\mathcal{L}$. Then since $\mathrm{Fut}^\lambda_\xi(\tilde{\mathcal{X}}, \tilde{\mathcal{L}}_\epsilon) \geq 0$, we get $\mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}) \geq 0$. Thus $(X, L)$ is $\mu$K-semistable with respect to general test configurations. The converse claim is obvious. □

The above theorem reduces Theorem A to Lahdili's result [Lah, Theorem 2]. Thus we obtain the following.

**Corollary 2.3.17.** If a smooth Kähler manifold $(X, L)$ admits a $\mu^\lambda_\xi$-cscK metric, then $(X, L)$ is $\mu^\lambda_\xi$K-semistable.

*Remark* 2.3.18. By [Lah, Proposition 4], we can express $\mu^\lambda_\xi$-Futaki invariant of toric test configurations of a toric polarized manifold as an integration on polytope, similarly to the usual Futaki invariant. For the toric test configuration $(\mathcal{X}_f, \mathcal{L}_f)$ associated to a convex piecewise linear function $f$ on the polytope $P$, we have

$$c.\mathrm{Fut}^\lambda_\xi(\mathcal{X}_f, \mathcal{L}_f) = \int_{\partial P} f e^{\langle x, -2\xi\rangle} d\sigma - \int_P (\lambda\langle x, -2\xi\rangle + \bar{s}^\lambda_\xi) f e^{\langle x, -2\xi\rangle} d\mu$$

for some uniform positive constant $c$. We may also have such expression even for singular $(X, L)$ as $\mu^\lambda_\xi$-Futaki invariant is expressed via an equivariant intersection formula.

## 2.4 Appendix: Preliminaries for equivariant calculus

Here we (re)arrange background materials on equivariant cohomology and equivariant locally finite homology to fix our notations and sign conventions in equivariant cohomology. The sign arrangement is crucial when computing the right sign of ($\mu$-)Futaki invariant via equivariant cohomology.

We also briefly explain some advantage of Cartan model, which employs differential forms as its chains. While there is an analogous equivariant theory for Chow group, which works also for schemes not even over arbitrary characteristic field but also over $\mathbb{Z}$, we prefer to use the singular/de Rham cohomology with $\mathbb{R}$-coefficient to benefit from the Cartan model when proving the convergence of some sequences in equivariant cohomology.

### 2.4.1 Equivariant cohomology & locally finite homology

#### Singular equivariant cohomology & locally finite homology

We firstly review *singular* equivariant cohomology and locally finite homology as these work also for singular spaces. Let $X$ be a topological space with a continuous action of a topological group $G$ from the right. Using a classifying bundle $EG \to BG$ of $G$, which can be constructed for instance by Milnor construction, the singular *$G$-equivariant cohomology* $H^*_G(X, \mathbb{Z})$ is defined to be the singular cohomology of the Borel space $EG \times_G X := (EG \times X)/G = \{[p, x] \mid [p, x] = [pg, xg],\ \forall g \in G\}$ :

$$H^*_G(X, \mathbb{Z}) := H^*(EG \times_G X, \mathbb{Z}). \tag{2.34}$$

Let $Y$ be another topological space with a continuous action of a topological group $H$, $\varphi : G \to H$ be a topological group morphism and $f : X \to Y$ be a continuous map satisfying $f(x.g) = f(x).\varphi(g)$ for $g \in G$. Then we have a pulling-back map $f^* : H^p_H(Y, \mathbb{Z}) \to H^p_G(X, \mathbb{Z})$. We usually define the equivariant cohomology in this way among another possible candidate named as 'equivariant cohomology' so that it enjoys the homotopy invariance. Considering the case $H = G$ and $Y = X$ with different choices of classifying bundles $EG \to BG$ and $E'G \to B'G$, we find that (2.34) gives a well-defined contravariant functor independent of the choice of the classifying bundle $EG \to BG$.

Since $EG$ is infinite dimensional in general, it is possible that $H^p_G(X,\mathbb{Z}) \neq 0$ for infinitely many $p \geq 0$ even if $X$ is finite dimensional. We put $\hat{H}^{\mathrm{even}}_G(X,\mathbb{Z}) := \prod_{p=0}^{\infty} H^{2p}_G(X,\mathbb{Z})$ and denote by $\alpha^{\langle p \rangle} \in H^{2p}_G(X,\mathbb{Z})$ the degree $2p$-part for an element $\alpha \in \hat{H}^{\mathrm{even}}_G(X,\mathbb{Z})$. When the action is free, we have a natural isomorphism $H^*_G(X,\mathbb{Z}) \cong H^*(X/G,\mathbb{Z})$, so that $H^*_G(X,\mathbb{Z}) = 0$ for $* \geq \dim(X/G)$ in this case.

For an almost connected locally compact group $G$ (i.e. the quotient $G/G^0$ by the identity component is compact), we have a maximal compact subgroup $\varphi : K \hookrightarrow G$ by Iwasawa's theorem. It admits a $K$-equivariant deformation retract $H_t : G \to K$. Thus the induced natural map $\varphi_\# : EK \to EK \times_K G \to EG$ coming from the homotopical universality of $BG$ is a $K$-equivariant homotopy equivalence, so that we obtain the induced isomorphism $H^*_G(X,\mathbb{Z}) \xrightarrow{\sim} H^*_K(X,\mathbb{Z})$. Since $\varphi_\# : EK \to EG$ is a $K$-equivariant homotopy equivalence for any inclusion $\varphi : K \hookrightarrow G$, the above isomorphism is independent of the choice of the inclusion $\varphi$. We often identify these two equivariant cohomologies.

For a $G$-equivariant complex vector bundle $E \to X$, we have the associated vector bundle $EG \times_G E \to EG \times_G X$. We define the *equivariant Chern class* $c^G_*(E) \in H^*_G(X,\mathbb{Z})$ to be the Chern class of the associated bundle $c_*(EG \times_G E) \in H^*(EG \times_G X,\mathbb{Z}) = H^*_G(X,\mathbb{Z})$. The *equivariant Chern character* $\mathrm{ch}_G(L) \in \hat{H}^{\mathrm{even}}_G(X,\mathbb{Z}) := \prod_{p=0}^{\infty} H^{2p}_G(X,\mathbb{Z})$ of a $G$-equivariant complex line bundle $L$ is defined as

$$\mathrm{ch}_G(L) := e^{c^G_1(L)} = \sum_{p=0}^{\infty} \frac{1}{p!}(c^G_1(L))^{\smile p} \in \hat{H}^{\mathrm{even}}_G(X,\mathbb{Z}) := \prod_{p=0}^{\infty} H^{2p}_G(X,\mathbb{Z}).$$

Note the highest degree of $\mathrm{ch}_G(L)$ is not bounded in general as it is possible that $H^{2p}_G(X,\mathbb{Z})$ are non-zero for infinitely many $p$.

We denote by the element $\eta^\vee \in H^2_{\mathbb{C}^*}(\mathrm{pt},\mathbb{Z})$ corresponding to the class $c_1(\mathcal{O}(1)) \in H^2(\mathbb{C}P^\infty,\mathbb{Z})$ via the canonical isomorphism $H^2_{\mathbb{C}^*}(\mathrm{pt},\mathbb{Z}) \cong H^2(\mathbb{C}P^\infty,\mathbb{Z})$ and call it *the positive generator*.

*Example* 2.4.1 (Weight and positive generator). Here we compare the sign of the weight of $\mathbb{C}^*$-action and the generator of $H^2_{\mathbb{C}^*}(\mathrm{pt},\mathbb{Z})$. Let $X$ be a point and denote by $L_1$ the trivial line bundle $\mathbb{C}$ on $X$ endowed with the nontrivial $\mathbb{C}^*$-action $z.t = zt$. The classifying bundle $E\mathbb{C}^* \to B\mathbb{C}^*$ is nothing but $\mathbb{C}^\infty \setminus \{0\} \to \mathbb{C}P^\infty$. In this case, the associated bundle $E\mathbb{C}^* \times_{\mathbb{C}^*} L_1$ on $B\mathbb{C}^* \times X = \mathbb{C}P^\infty$ is the tautological line bundle $\mathcal{O}(-1)$, so that $c^{\mathbb{C}^*}_1(L_1) \in H^2_{\mathbb{C}^*}(X)$ is identified with the *negative* generator $c_1(\mathcal{O}(-1)) \in H^2(\mathbb{C}P^\infty)$.

For $m \in \mathbb{Z}$, the trivial line bundle $L_m = \mathbb{C}$ with the $\mathbb{C}^*$-action $z.t = zt^m$ is the $m$-tensor product of $L_1$, so that we have

$$c_1^{\mathbb{C}^*}(L_m) = -m\eta^\vee \in H^2_{\mathbb{C}^*}(\mathrm{pt}, \mathbb{Z}). \tag{2.35}$$

If we employ the *left* $G$-action on $EG$ in the definition of the equivariant cohomology, the sign reverses. Indeed, the Borel space is given by $X \times^G EG = \{[x,p] \mid [x,p] = [xg, g^{-1}p],\ \forall g \in G\}$, so $L_{-1} \times^{\mathbb{C}^*} E\mathbb{C}^* = \mathcal{O}(-1)$ on $E\mathbb{C}^* = \mathbb{C}P^\infty$.

When $T$ acts on $X$ trivially, we have a canonical isomorphism $H^k_{T\times G}(X,\mathbb{R}) \cong \bigoplus_{p+q=k} S^p\mathfrak{t}^\vee \otimes H_G^{q-p}(X,\mathbb{R})$. For $\xi \in \mathfrak{t}$, we define the *evaluation map*

$$\mathrm{ev}_\xi : H^k_{T\times G}(X,\mathbb{R}) \to \bigoplus_{0\le l\le k} H^l_G(X,\mathbb{R}) \tag{2.36}$$

via this isomorphism as

$$\mathrm{ev}_\xi : S^p\mathfrak{t}^\vee \otimes H_G^{q-p}(X,\mathbb{R}) \to H_G^{q-p}(X,\mathbb{R}) : \rho \otimes c \mapsto \rho(\xi).c. \tag{2.37}$$

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For a locally compact Hausdorff space $X$, the *locally finite homology* $H^{\mathrm{lf}}_*(X,\mathbb{Z})$ is defined to be the homology of the chain complex $C^{\mathrm{lf}}_*$ of the locally finite chains, i.e.

$$C_p^{\mathrm{lf}} := \left\{ \sigma : \mathrm{Map}(\Delta^p, X) \to \mathbb{Z} \;\middle|\; \begin{array}{c} \forall K \subset X:\ \text{compact set} \\ \#\{c \in \sigma^{-1}(\mathbb{Z}\setminus\{0\}) \mid c^{-1}(K) \neq \emptyset\} < \infty \end{array} \right\},$$

where $\mathrm{Map}(\Delta^p, X)$ denotes the set of continuous maps. We usually denote its chain by a formal expression $\sum_{c\in\mathrm{Map}(\Delta^p,X)} \sigma(c).c$. The boundary map $\partial : C_p^{\mathrm{lf}} \to C_{p-1}^{\mathrm{lf}}$ is given similarly as the usual homology. The locally finite homology $H^{\mathrm{lf}}_*(\cdot,\mathbb{Z})$ gives a covariant functor from the category of locally compact Hausdorff spaces with *proper continuous maps* to the category of $\mathbb{Z}$-modules. The functor is not a homotopy functor, but only invariant under *proper homotopy*. For example, $H_p^{\mathrm{lf}}(X,\mathbb{Z}) \not\cong H_p^{\mathrm{lf}}(X\times\mathbb{R}^q,\mathbb{Z})$ while we have $H_p^{\mathrm{lf}}(X,\mathbb{Z}) \cong H_{p+q}^{\mathrm{lf}}(X\times\mathbb{R}^q,\mathbb{Z})$.

We have the following cap product:

$$\frown : H_p^{\mathrm{lf}}(X,\mathbb{Z}) \otimes H^q(X,\mathbb{Z}) \to H_{p-q}^{\mathrm{lf}}(X,\mathbb{Z}), \tag{2.38}$$

which makes the anti-graded module $H^{\mathrm{lf}}_{-*}(X,\mathbb{Z})$ into a $(H^*(X,\mathbb{Z}),\smile)$-module and enjoys the projection formula:

$$f_*(\sigma \frown f^*\phi) = f_*\sigma \frown \phi \tag{2.39}$$

for every proper continuous map $f : X \to Y$ and $\sigma \in H^{\mathrm{lf}}_p(X,\mathbb{Z})$, $\phi \in H^q(Y,\mathbb{Z})$.

When $X$ is a connected $n$-dimensional oriented manifold, we have an orientation preserving triangulation $\sum_{\alpha\in A}\Delta^n_\alpha$ of $X$ and obtain a generator $[X] \in H^{\mathrm{lf}}_n(X,\mathbb{Z})$ called the *fundamental class* of $X$, independent of the choice of the triangulation. The map $([X]\frown\cdot) : H^q(X,\mathbb{Z}) \to H^{\mathrm{lf}}_{n-q}(X,\mathbb{Z})$ gives an isomorphism of $\mathbb{Z}$-modules for each $q\in\mathbb{Z}$. We denote its inverse $([X]\frown\cdot)^{-1}$ by $\mathrm{PD} : H^{\mathrm{lf}}_p(X,\mathbb{Z}) \to H^{n-p}(X,\mathbb{Z})$.

The following are key properties for the well-definedness of the equivariant version of locally finte homology:

- For any closed subset $Y \subset X$, we have a long exact sequence

$$\cdots \to H^{\mathrm{lf}}_p(Y,\mathbb{Z}) \to H^{\mathrm{lf}}_p(X,\mathbb{Z}) \to H^{\mathrm{lf}}_p(X\setminus Y,\mathbb{Z}) \to H^{\mathrm{lf}}_{p-1}(Y,\mathbb{Z}) \to \cdots . \tag{2.40}$$

  In particular, when $\dim Y < l$, we have the isomorphism $H^{\mathrm{lf}}_p(X,\mathbb{Z}) \cong H^{\mathrm{lf}}_p(X\setminus Y,\mathbb{Z})$ for $p > l$.

- For a vector bundle $\pi : E \to X$ of rank $r$, we have an isomorphism

$$\pi^* : H^{\mathrm{lf}}_p(X,\mathbb{Z}) \xrightarrow{\sim} H^{\mathrm{lf}}_{p+r}(E,\mathbb{Z}) \tag{2.41}$$

  for each $p\in\mathbb{Z}$.

Now we explain the equivariant version of locally finite homology. Let $X$ be an $n$-dimensional locally compact space with a continuous action of an almost connected Lie group $G$. For an almost connected Lie group $G$, we have a 'finite dimensional approximation' $\{E_lG \to B_lG\}_{l\in\mathbb{N}}$ of classifying bundle $EG \to BG$ of $G$ which enjoys the following properties:

1. For each $l\in\mathbb{N}$, $E_lG$ is a $G$-invariant Zariski open set of the subset $\{v \in V_l \mid v.g = v \iff g = 1\}$ of a (complex) $G$-representation $V_l$ with $\dim_{\mathbb{R}}(V \setminus E_lG) > l+1$.

2. $G$ acts on $E_lG$ freely and $E_lG \to B_lG$ is the quotient.

For example when $G = \mathbb{C}^*$, $E_l\mathbb{C}^* := \mathbb{C}^{l+1} \setminus \{0\}$ with the diagonal $\mathbb{C}^*$-action on $\mathbb{C}^{l+1}$ gives such a finite dimensional approximation. In this case, we have $B_lG = \mathbb{C}P^l$.

Using such a finite dimensional approximation, we define the *$G$-equivariant locally finite homology* $H_p^{\mathrm{lf},G}(X, \mathbb{Z})$ of degree $p \in \mathbb{Z}$ (negative degree allowed) by

$$H_p^{\mathrm{lf},G}(X, \mathbb{Z}) := H^{\mathrm{lf}}_{p+\dim_{\mathbb{R}} B_{n-p}G}(E_{n-p}G \times_G X, \mathbb{Z}). \tag{2.42}$$

For example, we have

$$H_p^{\mathrm{lf},\mathbb{C}^*}(\mathrm{pt}, \mathbb{Z}) = H^{\mathrm{lf}}_{p+2(-p)}(\mathbb{C}P^{-p}, \mathbb{Z}) = \begin{cases} 0 & p > 0 \text{ or } p \text{ odd} \\ \mathbb{Z} & p \leq 0 \text{ and } p \text{ even} \end{cases}.$$

By the key properties of locally finite homology, we can show the above construction is independent of the choice of a finite dimensional approximation of classifying space by the similar argument as in [EG1] for equivariant Chow group. For a $G$-equivariant proper continuous map $f : X \to Y$, we have the proper push-forward $f_* : H_p^{\mathrm{lf},G}(X) \to H_p^{\mathrm{lf},G}(Y)$ induced from the map $f : E_{n-p}G \times_G X \to E_{n-p}G \times_G Y$ with $n = \max\{\dim X, \dim Y\}$.

When $X$ is a smooth oriented manifold and the action of $G$ on $X$ is orientation preserving, $X \times_G E_{n-p}G$ is again a smooth oriented manifold and its fundamental class of $X \times_G E_{n-p}G$ defines a homology class $[X \times_G E_{n-p}G] \in H_n^{\mathrm{lf},G}(X, \mathbb{Z})$. Along the above proof, we can easily check that this homology class is independent of the choice of the finite dimensional approximation, thus we get the *equivariant fundamental class* $[X]^G \in H_n^{\mathrm{lf},G}(X, \mathbb{Z})$.

**Definition 2.4.2** (Equivariant fundamental class of complex analytic space and the equivariant cycle map)**.** Let $X$ be a pure $n$-dimensional complex analytic space with an orientation preserving action of an almost connected Lie group $G$ (not necessarily holomorphic). Along the irreducible decomposition $X = \bigcup_{i \in I} X_i$, the exact sequence (2.40) induces the canonical isomorphism $H_{2n}^{\mathrm{lf},G}(X, \mathbb{Z}) \cong H_{2n}^{\mathrm{lf},G}(X \setminus X^{\mathrm{red,sing}}, \mathbb{Z}) \cong \bigoplus_{i \in I} H_{2n}^{\mathrm{lf},G}(X_i \setminus X_i^{\mathrm{red,sing}}, \mathbb{Z})$. We define the *equivariant fundamental class* $[X]^G \in H_{2n}^{\mathrm{lf},G}(X, \mathbb{Z})$ by

$$[X]^G := \sum_{i \in I} m_i [X_i^{\mathrm{red,reg}}]^G \in H_{2n}^{\mathrm{lf},G}(X, \mathbb{Z}), \tag{2.43}$$

where $m_i$ is the length of $\mathcal{O}_{X_i}/\mathcal{O}_{X_i^{\mathrm{red}}}$ at a general point and $[X_i^{\mathrm{red,reg}}]^G$ denotes the equivariant fundamental class of the oriented manifold $X_i$.

Since we have $H^q_G(X,\mathbb{Z}) \cong H^q(E_lG \times_G X, \mathbb{Z})$ for $l \geq q$, we have the equivariant cap product:

$$\frown : H^{\mathrm{lf},G}_p(X,\mathbb{Z}) \otimes H^q_G(X,\mathbb{Z}) \to H^{\mathrm{lf},G}_{p-q}(X,\mathbb{Z}), \tag{2.44}$$

which is also independent of the choice of the finite dimensional approximation. When $X$ is an oriented manifold with an orientation preserving action of an almost connected Lie group $G$, the map $([X]^G \frown \cdot) : H^q_G(X,\mathbb{Z}) \to H^{\mathrm{lf},G}_{n-q}(X,\mathbb{Z})$ gives an isomorphism for each $q \in \mathbb{Z}$ (as we can take $E_lG$ as a manifold). We denote its inverse $([X]^G \frown \cdot)^{-1}$ by $\mathrm{PD}_G : H^{\mathrm{lf},G}_p(X,\mathbb{Z}) \to H^{n-p}_G(X,\mathbb{Z})$.

Using the equivariant fundamental class, we can define the homology-to-cohomology push-forward map $f_\star : H^{\mathrm{lf},G}_p(X,\mathbb{Z}) \to H^{\dim Y - p}_G(Y,\mathbb{Z})$ (resp. the cohomology-to-homology push-forward map $f_\star : H^p_G(X,\mathbb{Z}) \to H^{\mathrm{lf},G}_{p-\dim X}(Y,\mathbb{Z})$, the cohomology-to-cohomology push-forward map $f_* : H^p_G(X,\mathbb{Z}) \to H^{p-\dim(X/Y)}_G(Y,\mathbb{Z})$) for a $G$-equivariant proper continuous map $f : X \to Y$ to an oriented manifold $Y$ (resp. from a pure dimensional complex analytic space $X$, from a pure dimensional complex analytic space $X$ to an oriented manifold $Y$). When $X$ is compact, we denote by $\int_X$ the (co)homology-to-cohomology push-forward map to the point, using the equivariant fundamental class $[X]^G$ defined in (2.43).

Let $\pi : \mathcal{X} \to B$ be a $G$-equivariant proper continuous map to a manifold $B$ and $\mathcal{L} \in H^2_G(\mathcal{X},\mathbb{R})$ be a $G$-equivariant cohomology class on $\mathcal{X}$. For a $G$-equivariant locally finite cohomology class $\alpha \in H^{\mathrm{lf},G}_{\mathrm{even}}(\mathcal{X},\mathbb{R})$, we denote by $(\alpha.e^{\mathcal{L}})_B$ the $G$-equivariant cohomological formal series

$$\sum_{k=0}^{\infty} \frac{1}{k!} \pi_\star(\alpha \frown \mathcal{L}^{\smile k}) \in \hat{H}^{\mathrm{even}}_G(B,\mathbb{R})$$

on $B$, which we call *relative equivariant intersection*. We abbreviate $([\mathcal{X}]^G.e^{\mathcal{L}})_B$ as $(e^{\mathcal{L}})_B$ and $([\mathcal{X}]^G.\mathcal{L} \smile e^{\mathcal{L}})_B = ([\mathcal{X}]^G \frown \mathcal{L}.e^{\mathcal{L}})_B$ as $(\mathcal{L}.e^{\mathcal{L}})_B$. When $B$ is a point, we usually abbreviate $(\alpha.e^{\mathcal{L}})_B$ as $(\alpha.e^{\mathcal{L}})$ and $(e^{\mathcal{L}})_B, (\mathcal{L}.e^{\mathcal{L}})_B$ as $(e^{\mathcal{L}}), (\mathcal{L}.e^{\mathcal{L}})$ respectively, which we usually identify as elements of $\hat{S}(\mathfrak{g}^\vee)^G$ and call *absolute equivariant intersection*. We also abbreviate $(\alpha.e^{c_1^G(\mathcal{L})})_B$ as $(\alpha.e^{\mathcal{L}})_B$ for a $G$-equivariant line bundle $\mathcal{L}$ on $\mathcal{X}$.

### Cartan model of equivariant cohomology & locally finite homology

Now we turn to the Cartan model. The Cartan model of equivariant cohomology behaves well when the action is proper. Let $X$ be a smooth manifold with a smooth action of a *compact* Lie group $K$ and $\mathfrak{k}$ be the Lie algebra of $K$. Put $C^{p,q} := S^p\mathfrak{k}^\vee \otimes \Omega^{q-p}(X)$. Identifying elements of the symmetric product $S^p\mathfrak{k}^\vee$ with the degree $p$-homogeneous polynomial maps on $\mathfrak{k}$, we regard $C^{p,q}$ the space of $p$-homogeneous polynomial maps from $\mathfrak{k}$ to $\Omega^{q-p}(X)$. Consider the subspace of $K$-equivariant maps:

$$C_K^{p,q} := (S^p\mathfrak{k}^\vee \otimes \Omega^{q-p}(X))^K. \tag{2.45}$$

Then $C_K^{p,q}$ becomes a double complex by giving the differentials $d : C^{p,q} \to C^{p,q+1}$, $\delta : C^{p,q} \to C^{p+1,q}$ by $(d\phi_\rho)(\xi) = d(\phi_{\rho(\xi)})$ and

$$(\delta\phi_\rho)(\xi) = i_\xi(\phi_{\rho(\xi)})$$

for $\phi_\rho = \rho\otimes\phi \in C_K^{p,q}$ regarded as a map $\phi_\rho : \mathfrak{k} \to \Omega^{q-p}(X)$ and $\xi \in \mathfrak{k}$. Indeed, we have $(d\delta + \delta d)(\phi_\rho)(\xi) = L_\xi\phi_{\rho(\xi)} = \phi_{\rho([\xi,\xi])} = 0$ by the $K$-equivariance. The *Cartan model* $H^*_{\mathrm{dR},K}(X,\mathbb{R})$ *of equivariant cohomology* is defined to be the cohomology of the total complex $(\Omega_K^*(X), d_K) := (\bigoplus_{p+q=*} C_K^{p,q}, d+\delta) = \bigoplus_{2i+j=*}(S^i\mathfrak{k}^\vee \otimes \Omega^j(X))^K$ of the double complex $C_K^{p,q}$. We call elements of $\Omega_K^k(X)$ *$K$-equivariant $k$-forms.*

This cohomology $H^*_{\mathrm{dR},K}(X)$ is known to be naturally isomorphic to the equivariant cohomology $H_K^*(X,\mathbb{R})$ for any (non-compact) $X$ and compact Lie group $K$ (cf. [GS, Section 2.5 and 4.2]).

We have a chain-level pulling-back map $f^* : \Omega_K^k(Y) \to \Omega_K^k(X)$ along any $K$-equivariant smooth map $f : X \to Y$ which induces the pulling-back map $f^* : H_K^k(Y) \to H_K^k(X)$. We also have a chain-level cup product $\wedge :$ $\Omega_K^k(X) \otimes \Omega_K^l(X) \to \Omega_K^{k+l}(X) : (\rho_1 \otimes \phi_1) \otimes (\rho_2 \otimes \phi_2) \mapsto (\rho_1 \cdot \rho_2) \otimes (\phi_1 \wedge \phi_2)$ which induces the cup product $\wedge : H_K^k(X) \otimes H_K^l(X) \to H_K^{k+l}(X)$.

When $X$ is a smooth $n$-dimensional oriented compact manifold, we have an integration map $\int_X : \Omega_K^p(X) \to S^{(p-n)/2}\mathfrak{k}^\vee$ for $p \geq n$ with even $p-n$ given by the integration of the component in $S^{(p-n)/2}\mathfrak{k}^\vee \otimes \Omega^n(X)$.

*Example* 2.4.3. A $d_K$-closed equivariant 2-form is given by a pair $(\omega,\mu) = \omega+\mu$ of a $K$-invariant 2-form $\omega$ and a $K$-equivariant smooth map $\mu : X \to \mathfrak{k}^\vee$ satisfying the 'moment identity' $-d\langle\mu,\xi\rangle = i_\xi\omega$ for every $\xi \in \mathfrak{k}$. On a $2n$-dimensional $X$, the integration $\int_X(\omega+\mu)^{n+k}$ of equivariant $2(n+k)$-form $(\omega+\mu)^{n+k}$ is then expressed as $\binom{n+k}{k}\int_X \mu^k\omega^n \in S^k\mathfrak{k}^\vee$. Conversely, for instance we can regard the map $\mathfrak{k} \to \mathbb{R} : \xi \mapsto \int_X e^{\langle\mu,\xi\rangle}\omega^n$ as an element of some 'completion' of the ring of polynomials $S\mathfrak{k}^\vee = \bigoplus_{k=0}^\infty S^k\mathfrak{k}^\vee$.

For a $K$-equivariant complex line bundle $L$ on $X$, a representative $\omega+\mu$ in the equivariant first Chern class $c_1^K(L) \in H^2_{\mathrm{dR},K}(X,\mathbb{R})$ is given using a connection form $\theta \in \Omega^1(P, \sqrt{-1}\mathbb{R})$ as $\sqrt{-1}\pi^*\omega = d\theta$ and $\langle \mu, \xi \rangle = \sqrt{-1} i_{\xi_P}\theta$: the equivariant cohomology class $[\omega+\mu]$ is independent of the choice of connection and is identified with $c_1^K(L) \in H^2_K(X,\mathbb{R})$ in the singuar equivariant cohomology via the canonical isomorphism $H^2_{\mathrm{dR},K}(X,\mathbb{R}) \cong H^2_K(X,\mathbb{R})$ to the singular equivariant cohomology. Note that adding a constant $c \in (\mathfrak{k}^\vee)^K$ gives another moment map $\mu + c$ for the same $\omega$, but its equivariant cohomology class differs from the original one. The moment map $\mu$ is normalized appropriately by this construction so that we have $[\omega+\mu] \in c_1^K(L)$.

*Example* 2.4.4 (Weight and the value of moment map). Consider the $U(2)$-equivariant line bundle $\mathcal{O}(-1) = \mathrm{Bl}_0\mathbb{C}^2 \to \mathbb{C}P^1$ with the $U(2)$-action induced from the right action on $\mathbb{C}^2$ by the matrix product. We have a connection form $\theta = \frac{1}{2\pi}\partial_{\mathrm{Bl}_0\mathbb{C}^2}\log(|z|^2+|w|^2) = \frac{1}{2\pi}\frac{\bar{z}dz+\bar{w}dw}{|z|^2+|w|^2}$ with the curvature $\omega_- = \frac{\sqrt{-1}}{2\pi}\bar{\partial}_{\mathbb{C}P^1}\partial_{\mathbb{C}P^1}\log(|z|^2+|w|^2)$.

For $\Lambda : U(1) \to U(2) : u \mapsto \mathrm{diag}(0, u^{-1})$, the associated $U(1)$-action on $\mathcal{O}(-1)$ is given by $(z,w).u = (z, wu^{-1})$ and the fundamental vector field $\eta_{\mathrm{Bl}_0\mathbb{C}^2} = \frac{d}{dt}\Big|_{t=0}(z,w)e^{t\eta}$ for $\eta = 2\pi\sqrt{-1} \in \mathfrak{u}(1) = \sqrt{-1}\mathbb{R}$ is given by $2\pi(\mathrm{Im}w)\partial_{\mathrm{Re}w} - 2\pi(\mathrm{Re}w)\partial_{\mathrm{Im}w}$. Then the moment map $\mu : \mathbb{C}P^1 \to \mathfrak{u}(1)^\vee$ with $\omega_- + \mu \in c_1^{U(1)}(\mathcal{O}(-1))$ is given by $\mu(z:w) = \frac{|w|^2}{|z|^2+|w|^2}\eta^\vee$ with the dual basis $\eta^\vee \in \mathfrak{u}(1)^\vee$ of $\eta \in \mathfrak{u}(1)$.

Pulling it back along the map $i_0 : \{0\} \to \mathbb{C}P^1 : i_0(0) = (0:1)$, we obtain $i_0^*(\omega+\mu) = \mu(0:1) = \eta^\vee$, which represents the equivariant Chern class $c_1^{U(1)}(i_0^*\mathcal{O}(-1)) \in H^2_{\mathrm{dR},U(1)}(\{0\},\mathbb{R})$. On the other hand, the pulled back equivariant line bundle $i_0^*\mathcal{O}(-1)$ is nothing but $L_{-1}$ in Example 2.4.1, so that we have $c_1^{U(1)}(i_0^*\mathcal{O}(-1)) = \eta^\vee \in H^2_{U(1)}(\{0\},\mathbb{Z})$, which is the positive generator. Our conventions for $\eta^\vee$ are compatible in this sense.

We in particular obtain that the value of the moment map $\mu_\eta$ associated to $L_m$ is the minus of the weight $m$.

*Example* 2.4.5 (Equivariant Chern class of canonical bundle). Let $\omega$ be a $K$-invariant Kähler metric on a Kähler manifold $X$ and $\mu : X \to \mathfrak{k}^\vee$ be a moment map with respect to $\omega$. Then the equivariant first Chern class of the canonical bundle $K_X$ is represented by $-\frac{1}{2\pi}(\mathrm{Ric}(\omega) + \bar{\square}\mu)$.

*Example* 2.4.6 (Localization formula). Consider the $U(1)$-action on $\mathbb{C}P^1$ defined by $(z:w).t = (z.t:w)$. There are two fixed points: $i_0(0) = (0:$

1), $\check{i}_0(0) = (1:0) \in \mathbb{C}P^1$. For every $u \in H^2_{\mathrm{dR},U(1)}(\mathbb{C}P^1, \mathbb{R})$, we have the following localization formula:

$$\int_{\mathbb{C}P^1} u = (\check{i}_0^* u - i_0^* u)/\eta^\vee. \tag{2.46}$$

Here the division $/\eta^\vee$ shifts the degree of equivariant cohomology:

$$/\eta^\vee : H^2_{U(1)}(\mathrm{pt}, \mathbb{R}) \xrightarrow{\sim} H^0_{U(1)}(\mathrm{pt}, \mathbb{R}) = \mathbb{R}.$$

The localization formula is just a paraphrase of Stokes theorem in this setup. Pick an equivariant 2-form $\omega + \mu$ in the cohomology class $u$. Using the coordinate $z = e^{\rho + \sqrt{-1}\theta}$ on $\mathbb{C}^* = \mathbb{C}P^1 \setminus \{i_0(0), \check{i}_0(0)\}$, we may write $\omega|_{\mathbb{C}P^1 \setminus \{i_0(0), \check{i}_0(0)\}} = f(\rho) d\rho \wedge d\theta = d(\int_{-\infty}^{\rho} f(t) dt. d\theta)$. On the other hand, as we have $d\mu_\eta = -i_\eta \omega = 2\pi f(\rho) d\rho$ with $\eta = 2\pi \frac{\partial}{\partial \theta}$, we can express the function $\mu_\eta$ as

$$\mu_\eta(z) = 2\pi \int_{-\infty}^{\log|z|} f(\rho) d\rho + \mu_\eta(i_0(0)).$$

Using Stokes theorem, we compute

$$\begin{aligned}\int_{\mathbb{C}P^1} \omega &= \lim_{\rho \to \infty} \left( \int_{|z| = e^\rho} \int_{-\infty}^{\rho} f(t) dt. d\theta - \int_{|z| = e^{-\rho}} \int_{-\infty}^{-\rho} f(t) dt. d\theta \right) \\ &= 2\pi \int_{-\infty}^{\infty} f(t) dt \\ &= \mu_\eta(\check{i}_0(0)) - \mu_\eta(i_0(0)).\end{aligned}$$

The sign can be checked with the above example: $\int_{\mathbb{C}P^1}(\omega_- + \mu) = -1$, $i_0^*\mu = 1$, $\check{i}^*\mu = 0$. See [GGK, Appendix C. 7] for a general localization formula.

The advantage of the Cartan model for our purpose is that when we consider an action by a product group $T \times K$, we have the following chain-level evaluation map

$$\begin{aligned} \mathrm{ev}_\xi : S^p(\mathfrak{t} \times \mathfrak{k})^\vee \otimes \Omega^{q-p}(X) &\to \bigoplus_{0 \le r \le p} S^r \mathfrak{k}^\vee \otimes \Omega^{q-p}(X) \\ \sum_{0 \le r \le p} \rho_{\mathfrak{t}}^{(p-r)} \cdot \rho_{\mathfrak{k}}^{(r)} \otimes \phi &\mapsto \sum_{0 \le r \le p} \rho_{\mathfrak{t}}^{(p-r)}(\xi) . \rho_{\mathfrak{k}}^{(r)} \otimes \phi \end{aligned} \tag{2.47}$$

for each vector $\xi \in \mathfrak{t}$, which we can treat on the fixed finite dimensional space $X$. This map is $T \times K$-equivariant and compatible with the chain-level proper push-forward map which we will define for equivariant currents. The map $\mathrm{ev}_\xi$ in general does not preserve the $d_K$-closedness of equivariant forms, however, it holds when $T$ acts on $X$ trivially. In this case, the evaluation map descends to the evaluation map on the equivariant cohomology $\mathrm{ev}_\xi : H^k_{T\times K}(X,\mathbb{R}) \to \bigoplus_{0\le l\le k} H^l_K(X,\mathbb{R})$ defined in (2.36). We will use this to show the convergence of a sequence in the $K$-equivariant cohomology of a base $B$ of a $T\times K$-equivariant map $\pi : X \to B$ with the trivial $T$-action on $B$. The sequence is constructed as the evaluation of the integration of a sequence of $T\times K$-equivariant cohomology classes on $X$.

---

Next we consider the dual construction, which corresponds to the equivariant locally finite homology. Firstly we review the current homology. Let $X$ be a connected $n$-dimensional smooth manifold. For a compact set $B\subset X$, let $\Omega^p_B$ denote the space of smooth $p$-forms supported on $B$ with the $C^\infty$-topology. We denote by $\mathcal{D}^p(X)$ the space of compactly supported smooth $p$-forms on $X$ endowed with the weakest topology which makes the natural inclusions $\Omega^p_B(X)\hookrightarrow \mathcal{D}^p(X)$ continuous for all compact sets $B\subset X$. Then $\mathcal{D}^p(X)$ is an LF-space. Let $\mathcal{D}'_p(X)$ denote the space of continuous linear functionals on $\mathcal{D}^p(X)$. We have a boundary map $\partial : \mathcal{D}'_p(X)\to\mathcal{D}'_{p-1}(X)$ adjoint to the differential map $d:\mathcal{D}_{p-1}(X)\to\mathcal{D}_p(X)$ (with an appropriate sign) and get the homology group $H^{\mathrm{dR}}_*(X,\mathbb{R})$ of this complex $(\mathcal{D}'_*(X),\partial)$.

A smooth $p$-chain $c:\Delta^p\to X$ defines an element of $\mathcal{D}'_p(X)$ by the integration $\phi\mapsto\int_{\Delta^p}c^*\phi$ and this gives a linear map $C^{\mathrm{lf}}_p\to\mathcal{D}'_p(X)$. It is known by [deR] that the homology $H^{\mathrm{dR}}_*(X,\mathbb{R})$ is isomorphic to the locally finite homology $H^{\mathrm{lf}}_*(X,\mathbb{R})$ via the map $C^{\mathrm{lf}}_p\to\mathcal{D}'_p(X)$ given as above.

For a proper smooth map $f:X\to Y$, we have a chain-level push-forward map $f_*:\mathcal{D}'_p(X)\to\mathcal{D}'_p(Y)$ adjoint to the proper pull-back $f^*:\mathcal{D}^p(Y)\to\mathcal{D}^p(X)$. This induces the push-forward map $f_*:H^{\mathrm{dR}}_p(X,\mathbb{R})\to H^{\mathrm{dR}}_p(Y,\mathbb{R})$. The cap product

$$\frown : H^{\mathrm{dR}}_p(X,\mathbb{R})\otimes H^q_{\mathrm{dR}}(X,\mathbb{R})\to H^{\mathrm{dR}}_{p-q}(X,\mathbb{R})$$

is induced from the chain-level map

$$\mathcal{D}'_p(X)\otimes\Omega^q(X)\to\mathcal{D}'_{p-q}(X):\sigma\otimes\phi\mapsto\sigma(\phi\wedge\cdot).$$

When $X$ is oriented and the action is orientation preserving, the closed current $\int_X:\mathcal{D}^n(X)\to\mathbb{R}$ gives the fundamental class $[X]\in H^{\mathrm{dR}}_n(X,\mathbb{R})$. All of these constructions are compatible with those counterpart of the locally finite homology $H_*^{\mathrm{lf}}(X,\mathbb{R})$.

In the equivariant setup, we consider the double complex

$$C_{p,q}^K := (S^{-p}\mathfrak{k}^\vee \otimes \mathcal{D}'_{q-p}(X))^K \tag{2.48}$$

with the differentials $\partial : C_{p,q}^K \to C_{p,q-1}^K$, $\delta : C_{p,q}^K \to C_{p-1,q}^K$ defined by $(\partial\sigma_\rho)(\phi) := (-1)^{n-(q-p)}\sigma_\rho(d\phi)$ and $(\delta\sigma_\rho)(\phi)(\xi) := (-1)^{n-(q-p)}\sigma_{\rho(\xi)}(i_\xi\phi)$, which are compatible with $(C_K^{p,q}, \delta, d)$ under the inclusion $C_K^{p,q} \hookrightarrow C_{-p,n-q}^K$ for oriented $X$. Here we put $S^{-p}\mathfrak{k}^\vee = 0$ for $p > 0$. We define the *Cartan model* $H_*^{\mathrm{dR},K}(X,\mathbb{R})$ *of equivariant current homology* to be the homology of the total complex $(\mathcal{D}')_*^K(X) := \bigoplus_{p+q=*} C_{p,q}^K = \bigoplus_{j-2i=*}(S^i\mathfrak{k}^\vee \otimes \mathcal{D}'_j(X))^K$. For a $K$-equivariant proper smooth map $f : X \to Y$, we have a chain-level push-forward map $f_* : (\mathcal{D}')_k^K(X) \to (\mathcal{D}')_k^K(Y)$ induced from $f_* : \mathcal{D}'_i(X) \to \mathcal{D}'_i(Y)$, which induces the push-forward map $f_* : H_i^{\mathrm{dR},K}(X,\mathbb{R}) \to H_i^{\mathrm{dR},K}(Y,\mathbb{R})$.

The equivariant cap product $\frown: H_k^{\mathrm{dR},K}(X,\mathbb{R}) \otimes H^l_{\mathrm{dR},K}(X,\mathbb{R}) \to H_{k-l}^{\mathrm{dR},K}(X,\mathbb{R})$ and the equivariant fundamental class $[X]^K \in H_n^{\mathrm{dR},K}(X,\mathbb{R})$ are given similarly as the non-equivariant case and are compatible with those of locally finite homology. We also have the evaluation map $\mathrm{ev}_\xi : S^p(\mathfrak{t}\times\mathfrak{k})^\vee \otimes \mathcal{D}'_{q+p}(X) \to \bigoplus_{0\le r\le p} S^r\mathfrak{k}^\vee \otimes \mathcal{D}'_{q+p}(X)$.

When $X$ is oriented, the inclusion $C_K^{p,q} \hookrightarrow C_{-p,n-q}^K$ gives the isomorphism $([X]_K \frown \cdot) : H^k_{\mathrm{dR},K}(X,\mathbb{R}) \to H_{n-k}^{\mathrm{dR},K}(X,\mathbb{R})$. We can check this using the spectral sequence associated to the double complexes $C_K^{p,q}$ and $\check{C}_K^{p,q} := C_{-p,n-q}^K$ (cf. [GS, Section 10.10 and 6.5]).

### 2.4.2 Equivariant proper push-forward

**Topology on $H_q^{\mathrm{dR},K}(B,\mathbb{R})$**

We consider the following topology on the space $(\mathcal{D}')_k^K(B) = (\bigoplus_{j-2i=k} S^i\mathfrak{k}^\vee \otimes \mathcal{D}'_j(B))^K$. We can naturally regard each element of $(\mathcal{D}')_k^K(B)$ as a sum of $K$-equivariant $i$-homogeneous polynomial maps $\rho^i : \mathfrak{k} \to \bigoplus \mathcal{D}'_{k+2i}(B)$ for $i = \lceil -k/2 \rceil, \dots, \lfloor (\dim B - k)/2 \rfloor$. We say a sequence $\{\sum_{i=\lceil -k/2\rceil}^{\lfloor(\dim B-k)/2\rfloor} \rho_m^i\}_{m\in\mathbb{N}}$ converges to $\sum_{i=\lceil -k/2\rceil}^{\lfloor(\dim B-k)/2\rfloor} \rho_\infty^i$ iff $\rho_m^i(\xi)(\phi) \to \rho_\infty^i(\xi)(\phi) \in \mathbb{R}$ for every $\xi \in \mathfrak{k}$, $\phi \in \mathcal{D}^{k+2i}(B)$ and each $i$.

We show that the quotient topology on the current homology $H_p^{\mathrm{dR},K}(B,\mathbb{R})$ induced from the topology on $(\mathcal{D}')_p^K(B)$ is Hausdorff, even for non-compact

$B$. We in particular show that the following *form-to-homology push-forward map* is continuous for every $K$-equivariant proper $C^\infty$-map $f : X \to B$:

$$f_* : \Omega_K^{n+p}(X) \cap (f_*)^{-1}\Big(\mathrm{Ker}[\partial_K : (\mathcal{D}')_{b-p}^K(B) \to (\mathcal{D}')_{b-p+1}^K(B)]\Big) \to H_{b-p}^K(B) \tag{2.49}$$

with respect to the unique Hausdorff topology on $H_K^p(X, \mathbb{R})$. Here we put $b := \dim B$ and $n := \dim X - \dim B$. Note the space $H_p^{\mathrm{dR},K}(X, \mathbb{R})$ has a unique Hausdorff topology defined by a norm since it is finite dimensional for each $p \in \mathbb{Z}$, however, the quotient topology is a priori unrelated to the norm topology. (Indeed, there is a non-Hausdorff cohomology theory such as $\bar{\partial}$-cohomology since Hodge decomposition does not work in the non-compact case. )

We apply the following de Rham's theorem and the spectral sequence of topological vector spaces associated with the double complex of the Cartan model.

**Proposition 2.4.7.** [deR, Chapter IV, Theorem 17'] A $p$-current $\sigma \in \mathcal{D}'_p(B)$ is exact if and only if $\sigma(\phi) = 0$ for every closed compactly supported $C^\infty$-form $\phi \in \mathcal{D}^p(B)$.

**Corollary 2.4.8.** The induced topology on the current homology $H_p^{\mathrm{dR}}(B, \mathbb{R})$ is Hausdorff.

*Proof.* The space of exact $p$-currents $\partial\mathcal{D}'_{p+1}(B) \subset \mathcal{D}_p(B)$ is a closed subset of $\mathcal{D}_p(B)$ since we have $\sigma_m(\phi) \to \sigma_\infty(\phi)$ for every convergent sequence $\sigma_m \to \sigma_\infty$. ☐

We use the following easy lemma in our spectral sequence argument.

**Lemma 2.4.9.** Let $V_1$ be a topological vector space and $V_2$ be a Hausdorff topological vector space. Suppose there is a continuous map $p : V_1 \to V_2$ such that the induced topology on the subspace $V_0 := p^{-1}(0)$ is Hausdorff, then $V_1$ is also Hausdorff.

*Proof.* The topological vector space $V_1$ is Hausdorff iff $\{0\} \in V_1$ is closed. The closure $W := \overline{\{0\}}$ in $V_1$ is a linear subspace of $V_1$ and $W \cap V_0 = \{0\}$ as $V_0$ is Hausdorff. The closure of $\{0\}$ in the quotient space $V_1/V_0$ is given by $(W + V_0)/V_0$. On the other hand, as $V_2$ is Hausdorff and $p$ is continuous, $V_0 = p^{-1}(0)$ is a closed subspace of $V_1$. It follows that the quotient $V_1/V_0$ is Hausdorff, so that we have $\{0\} = \overline{\{0\}} = (W + V_0)/V_0$. This proves $W = \{0\}$. ☐

**Proposition 2.4.10.** Let $(\{C^{p,q}\}_{p\geq 0,q\geq 0},\delta,d)$ be a first quadrant double complex of (Hausdorff) topological vector spaces with continuous derivatives $\delta,d$ whose $E_1$-page is a finite dimensional Hausdorff topological vector space with respect to the quotient topology induced from $C^{p,q}$. Then the cohomology $H^k(\prod_{p+q=\bullet}C^{p,q},\delta+d)$ of the total complex is a finite dimensional Hausdorff topological vector space with respect to the quotient topology induced from $\prod_{p+q=k}C^{p,q}$.

*Proof.* Let us recall the argument of spectral sequence. Put $C_l^k:=\prod_{p+q=k,p\geq l}C^{p,q}$. There is a decreasing filtration

$$H^k(\prod_{p+q=\bullet}C^{p,q},\delta+d)=H_0^k\supset\cdots\supset H_1^k\supset\cdots\supset H_k^k\supset 0$$

on the cohomology $H^k(\prod_{p+q=\bullet}C^{p,q},\delta+d)$ of the total complex derived from the decreasing filtration $\{C_l^k\cap\mathrm{Ker}(\delta+d)\}_{l=0}^k$ of $\mathrm{Ker}(\delta+d)$.

Now we consider the quotient topology on each $H_l^k$ induced from the subspace $C_l^k\cap\mathrm{Ker}(\delta+d)$ of the product $\prod_{p+q=k}C^{p,q}$ (endowed with the product topology). From the above lemma, it suffices to show the quotient topology on $E_\infty^{l,k-l}:=H_l^k/H_{l+1}^k$ induced from $H_l^k$ is Hausdorff. By the usual lemma of spectral sequence, we can (algebraically) compute the quotient vector space $E_\infty^{l,k-l}$ by computing the cohomologies of $E_r$-pages $E_r^{l,k-l}$ successively. We must see the successive computation of $E_r$-page also detects the Hausdorffness. It is a general lemma that if we have a topological vector space $V$ and its subspaces $W$ and $V',W'$ with $W'\subset V'$, then there is a natural linear bijective homeomorphism $(V/W)/(V'/W')\to V/(W+V')$ of topological vector spaces, where we take the usual algebraic quotient and sum. It follows that we have a linear bijective homeomorphism

$$E_\infty^{l,k-l}\to\frac{C_l^k\cap\mathrm{Ker}(\delta+d)}{\left(C_l^k\cap\mathrm{Im}(\delta+d)\right)+\left(C_{l+1}^k\cap\mathrm{Ker}(\delta+d)\right)},\tag{2.50}$$

so that it suffices to show that the right hand side is Hausdorff.

Recall the definition of the $E_r$-page:

$$E_r^{l,k-l}:=\frac{\left(C_l^k\cap(\delta+d)^{-1}C_{l+r}^{k+1}\right)}{\left(C_l^k\cap(\delta+d)C_{l+1-r}^{k-1}\right)+\left(C_{l+1}^k\cap(\delta+d)^{-1}C_{l+r}^{k+1}\right)}.$$

We consider the quotient topology on $E_r^{l,k-l}$ induced from $C_l^k \cap (\delta+d)^{-1}C_{l+r}^{k+1}$. Then since $E_r^{l,k-l}$ for $r > \max(l, k-l)$ coincides with the right hand side in (2.50) as topological vector spaces, the Hausdorffness of $E_\infty^{l,k-l}$ follows from that of $E_r^{l,k-l}$ for $r > \max(l, k-l)$.

Remember from the usual lemma on spectral sequence, we have a linear map $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$ and a linear bijection $E_{r+1}^{p,q} \to \mathrm{Ker}(d_r^{p,q})/\mathrm{Im}(d^{p-r,q+r-1})$. The linear map $d_r^{l,k-l} : E_r^{l,k-l} \to E_r^{l+r,(k-l)-r+1}$ is induced from the continuous map $\delta+d : C_l^k \cap (\delta+d)^{-1}C_{l+r}^{k+1} \to C_{l+r}^{k+1} \cap (\delta+d)^{-1}C_{l+2r}^{k+2}$ and the linear bijection $E_{r+1}^{p,q} \to \mathrm{Ker}(d_r^{p,q})/\mathrm{Im}(d^{p-r,q+r-1})$ is induced from the continuous inclusion $C_l^k \cap (\delta+d)^{-1}C_{l+r+1}^k \hookrightarrow C_l^k \cap (\delta + d)^{-1}C_{l+r}^k$, so that these maps are continuous linear bijection, while we do not state here the continuity of the inverse map as there is no open mapping theorem for general topological vector spaces. Thanks to the direction of the continuous bijection $E_{r+1}^{p,q} \to \mathrm{Ker}(d_r^{p,q})/\mathrm{Im}(d^{p-r,q+r-1})$, $E_{r+1}^{p,q}$ is Hausdorff when the quotient topology on $\mathrm{Ker}(d_r^{p,q})/\mathrm{Im}(d^{p-r,q-1+r})$ induced from $E_r^{p,q}$ is Hausdorff.

Now our assumption that $E_1^{p,q}$ are finite dimensional Hausdorff spaces implies every subspace of $E_1^{p,q}$ is closed, so that $E_2^{p,q}$ are again finite dimensional Hausdorff spaces by the above general argument. Running the induction, we conclude that $E_r^{p,q}$ are finite dimensional Hausdorff spaces for every $r \geq 1$, and so are the spaces $E_\infty^{p,q}$. □

**Proposition 2.4.11.** The quotient topology on $H_p^{\mathrm{dR},K}(B, \mathbb{R})$ induced from the weak topology on $(\mathcal{D}')_p^K(B)$ is Hausdorff for every $p \in \mathbb{Z}$.

*Proof.* This follows by applying the above proposition to the double complex of Cartan model with reversed index $E_0^{p,q} = (S^p\mathfrak{k}^\vee \otimes \mathcal{D}'_{n-(p+q)}(B))^K$, whose assumption is confirmed by Proposition 2.4.7 and the computation of $E_1$-term:

$$E_1^{p,q} = \left(S^p(\mathfrak{k}^\vee) \otimes H_{n-(p+q)}^{\mathrm{dR}}(B, \mathbb{R})\right)^K$$

as topological vector spaces. □

**Corollary 2.4.12.** The push-forward map (2.49) is continuous with respect to the Fréchet topology on $\Omega_K^{n+p}(X)$ and the Hausdorff topology on $H_{b-p}^K(B)$.

We apply this continuity result to the key construction in section 2.3.1, together with the following lemma.

**Lemma 2.4.13.** Let $V$ be a Fréchet space and $\{\|\cdot\|_l\}_{l\in\mathbb{Z}_{\geq 0}}$ be a collection of seminorms on $V$ defining its Fréchet structure. Let $W$ be a Banach space and $F : V \to W$ be a continuous linear map. Let $\{v_i\}_{i=0}^\infty \in V$ be a sequence such that $\sum_{i=0}^\infty \|v_i\|_l < \infty$ for every $l \geq 0$. Then the infinite series $\sum_{i=0}^\infty F(v_i)$ is absolutely convergent with respect to the norm of $W$.

*Proof.* Remember that a linear map $F : V \to W$ from Fréchet space to Banach space is continuous if and only if there exists a constant $C > 0$ and $N \in \mathbb{Z}_{\geq 0}$ such that

$$\|F(v)\|_W \leq C(\|v\|_0 + \cdots \|v\|_N)$$

for every $v \in V$. So the claim follows by

$$\sum_{i=0}^{\infty} \|F(v_i)\|_W \leq C \sum_{l=0}^{N} \sum_{i=0}^{\infty} \|v_i\|_l < \infty.$$

□

**Equivariant homology todd class and equivariant Grothendieck–Riemann–Roch theorem**

Here we recall the equivariant Grothendieck–Riemann–Roch theorem for algebraic schemes established by Edidin–Graham [EG2] as equivariant version of [Ful]. The equivariant Chow group $A_p^G(X)$ is studied in [EG1] which is defined in the same way as the equivariant locally finite homology. The statement is as follows.

**Theorem 2.4.14.** Let $G$ be an algebraic group. For each algebraic $G$-schemes $X$ over $\mathbb{C}$ (i.e. schemes locally of finite type over $\mathbb{C}$), we can assign a homomorphism

$$\tau_X^G : K(\mathrm{Coh}^G(X)) \to \hat{A}_{\mathbb{Q}}^G(X)$$

from the K-group $K(\mathrm{Coh}^G(X))$ of $G$-equivariant algebraic coherent sheaves on $X$ to the $G$-equivariant Chow group $\hat{A}_{\mathbb{Q}}^G(X) = \prod_{p\in\mathbb{Z}} A_p^G(X) \otimes \mathbb{Q}$ of $\mathbb{Q}$-coefficient enjoying the following properties.

1. (Grothendieck–Riemann–Roch) For any $G$-equivariant proper morphism $f : X \to Y$ of algebraic schemes, we have $f_*\tau_X^G(\alpha) = \tau_Y^G(f_!\alpha)$ for every $\alpha \in K(\mathrm{Coh}^G(X))$. Here $f_!\alpha$ for an element $\alpha = [\mathcal{F}]$ represented by a

$G$-equivariant coherent sheaf $\mathcal{F}$ denotes the element $\sum_i(-1)^i[R^if_*\mathcal{F}]$ in $K(\mathrm{Coh}^G(Y))$, where the higher direct image sheaves $R^if_*\mathcal{F}$ are $G$-linearized in a natural way.

2. For every $\alpha \in K(\mathrm{Coh}^G(X))$ and $\beta \in K(\mathrm{Vect}^G(X))$, we have $\tau_X^G(\alpha \otimes \beta) = \tau_X^G(\alpha) \frown \mathrm{ch}_G(\beta)$.

3. For closed subscheme $Z \subset X$ of pure dimension $p$, we have $\tau_X^G(\mathcal{O}_Z)_{\langle p\rangle} = [Z]_G \in A_n^G(X)$.

4. When $X$ is smooth, we have $\mathrm{PD}_G((\tau_X^G(\mathcal{O}_X))_{\langle p\rangle}) = \mathrm{Td}_G^{n-p}(X)$.

**Definition 2.4.15.** For a pure $n$-dimensional algebraic $G$-scheme $X$ over $\mathbb{C}$, we define the *equivariant canonical class* $\kappa_X^G \in H_{2n-2}^{\mathrm{lf},G}(X,\mathbb{Q})$ by

$$\kappa_X^G := -2cl^G(\tau_X^G(\mathcal{O}_X))_{\langle n-1\rangle} \tag{2.51}$$

under the equivariant cycle map $cl^G : A_p^G(X) \to H_{2p}^{\mathrm{lf},G}(X)$.

For a relatively pure dimensional $G$-equivariant proper flat morphism $\pi : \mathcal{X} \to B$ from an algebraic $G$-scheme $\mathcal{X}$ to a smooth $G$-variety $B$, we define the *relative equivariant canonical class* $\kappa_{\mathcal{X}/B}^G \in H_{2\dim\mathcal{X}-2}^{\mathrm{lf},G}(\mathcal{X},\mathbb{Q})$ by

$$\kappa_{\mathcal{X}/B}^G := -2cl^G(\tau_{\mathcal{X}}^G(\mathcal{O}_{\mathcal{X}}) \frown \pi^\star\tau_B^G(\mathcal{O}_B))_{\langle \dim\mathcal{X}-1\rangle} = \kappa_{\mathcal{X}}^G - [\mathcal{X}]^G \frown \pi^\star(\kappa_B^G), \tag{2.52}$$

where we put $\pi^\star := \pi^* \circ \mathrm{PD}_{G,B} : H_l^{\mathrm{lf},G}(B) \to H_G^{2\dim\mathcal{X}-l}(\mathcal{X})$.

Let $f : \tilde{X} \to X$ be a $G$-equivariant proper morphism of pure dimensional $G$-schemes which is isomorphic away from a codimension $k+1$ subscheme of the target $X$. (Namely, there is a subscheme $Z \subset X$ of codimension $k+1$ such that the restriction $f^{-1}(X \setminus Z) \to X \setminus Z$ gives an isomorphism. ) Then we have $f_*(\tau_{\tilde{X}}^G(\mathcal{O}_{\tilde{X}}))_{\langle \dim\tilde{X}-i\rangle} = \tau_X^G(\mathcal{O}_X)_{\langle \dim X-i\rangle}$ for $i \leq k$ as we have $f_!(\tau_{\tilde{X}}^G(\mathcal{O}_{\tilde{X}})) = \tau_X^G(f_*[\mathcal{O}_{\tilde{X}}])$ by the equivariant Grothendieck–Riemann–Roch and $f_![\mathcal{O}_{\tilde{X}}] - [\mathcal{O}_X] = [f_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X] + \sum_{i\geq 1}(-1)^i[R^if_*\mathcal{O}_{\tilde{X}}]$ is supported on $Z$. In particular, we have $\kappa_X^G = K_X^G$ for any normal variety $X$, where $K_X^G$ denote the locally finite homology class corresponding to the equivariant frist Chern class $c_1^G(\omega_{X^{\mathrm{reg}}}) = -c_1^G(X^{\mathrm{reg}}) \in H_G^2(X^{\mathrm{reg}},\mathbb{Z})$ via the isomorphism $H_G^2(X^{\mathrm{reg}}) \cong H_{2n-2}^{\mathrm{lf},G}(X^{\mathrm{reg}}) \cong H_{2n-2}^{\mathrm{lf},G}(X)$. If $X$ has only rational singularities, we have $f_!\mathcal{O}_{\tilde{X}} = \mathcal{O}_X$ for any equivariant resolution $f : \tilde{X} \to X$, so that $f_*\tau_{\tilde{X}}^G(\mathcal{O}_{\tilde{X}}) = \tau_X^G(\mathcal{O}_X)$.

**Corollary 2.4.16.** Let $\pi : \mathcal{X} \to B$ be a relatively pure $n$-dimensional $G$-equivariant proper flat morphism from an algebraic $G$-scheme $\mathcal{X}$ to a smooth $G$-variety $B$, $f : B' \to B$ be a $G$-equivariant morphism from another smooth $G$-variety $B'$. Put $\mathcal{X}' := \mathcal{X} \times_B B'$ and denote by $\hat{f} : \mathcal{X}' \to \mathcal{X}$ and $\pi' : \mathcal{X}' \to B'$ the projection to the first factor and the second factor, respectively.

For each $G$-equivariant line bundle $\mathcal{L}$ on $\mathcal{X}$, we have

$$f^*\Big(\pi_\star\big(\kappa^G_{\mathcal{X}/B} \frown c_1^G(\mathcal{L})^{\smile(n+p-1)}\big)\Big) = \pi'_\star\big(\kappa^G_{\mathcal{X}'/B'} \frown c_1^G(\hat{f}^*\mathcal{L})^{\smile(n+p-1)}\big)$$

in $H_G^{2p}(B, \mathbb{Q})$.

In other words, the assignment

$$\mathcal{L} \mapsto (\kappa^G_{\mathcal{X}/B}.e^{\mathcal{L}})_B \in \hat{H}_G^{\mathrm{even}}(B, \mathbb{Q}) \tag{2.53}$$

is base change stable.

*Proof.* By the equivariant Grothendieck–Riemann–Roch theorem, we have

$$\pi_\star\tau^G_{\mathcal{X}}(\mathcal{L}^{\otimes k}) = \mathrm{PD}_B^G\tau^G(\pi_*[\mathcal{L}^{\otimes k}]) = \tau^G_B(\mathcal{O}_B) \frown \mathrm{ch}_G(\pi_*[\mathcal{L}^{\otimes k}])$$

in $\hat{H}^{\mathrm{lf},G}(B, \mathbb{Q})$. Since $\tau^G_B(\mathcal{O}_B) = [B]_G \frown \mathrm{Td}_G(B)$ and $\mathrm{Td}_G(B)$ has the inverse element $\mathrm{Td}_G(-TB)$ with respect to the cup product, we obtain

$$\mathrm{ch}^G(\pi_*[\mathcal{L}^{\otimes k}]) = \mathrm{PD}_B^G(\pi_\star\tau^G_{\mathcal{X}}(\mathcal{L}^{\otimes k}) \frown \mathrm{Td}_G(-TB)).$$

We compute

$$\pi_\star\tau^G_{\mathcal{X}}(\mathcal{L}^{\otimes k}) = \pi_\star(\tau^G_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}) \frown \mathrm{ch}_G(\mathcal{L}^{\otimes k})) = \pi_\star(\tau^G_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}) \frown e^{kc_1^G(\mathcal{L})})$$

and obtain

$$\begin{aligned}
\big(\mathrm{ch}^G(\pi_*[\mathcal{L}^{\otimes k}])\big)^{\langle \dim B - q\rangle} &= \sum_{i=0}^{\infty}\frac{k^i}{i!}\Big(\pi_\star(\tau^G_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}) \frown \pi^*\mathrm{Td}_G(-TB)) \frown c_1^G(\mathcal{L})^{\smile i}\Big)^{\langle \dim B - q\rangle} \\
&= \sum_{i=0}^{\dim \mathcal{X} - p}\frac{k^i}{i!}\pi_\star((\tau^G_{\mathcal{X}}(\mathcal{O}_{\mathcal{X}}) \frown \pi^*\mathrm{Td}_G(-TB))_{\langle i+q\rangle} \frown c_1^G(\mathcal{L})^{\smile i}).
\end{aligned}$$

It follows that

$$\pi_\star(\kappa^G_{\mathcal{X}/B} \frown c_1^G(\mathcal{L})^{\smile(n+p-1)})$$

is the coefficient of degree $n+p-1$ of the polynomial map $(\mathrm{ch}^G(\pi_*[\mathcal{L}^{\otimes\bullet}]))^{\langle p\rangle} : \mathbb{Z} \to H_G^{2p}(B, \mathbb{Q})$. Now the claim follows from

$$f^*(\mathrm{ch}^G(\pi_*[\mathcal{L}^{\otimes k}])) = \mathrm{ch}^G(f^*\pi_*[\mathcal{L}^{\otimes k}]) = \mathrm{ch}^G(\pi'_*[(\hat{f}^*\mathcal{L})^{\otimes k}]).$$

$\square$

We denote by $NS_G(\mathcal{X},\mathbb{R}) \subset H^2_G(\mathcal{X},\mathbb{R})$ the subspace spanned by $\{c_1^G(\mathcal{L}) \mid \mathcal{L} \in \mathrm{Pic}^G(\mathcal{X})\}$. Since the map $H^2_G(\mathcal{X},\mathbb{R}) \to H^{2p}_G(B,\mathbb{R}) : c \mapsto \pi_\star(\kappa^G_{\mathcal{X}/B} \frown c^{\smile(n+p-1)})$ is continuous, the assignment $c \mapsto \pi_\star\big(\kappa^G_{\mathcal{X}/B} \frown e^c\big)$ is also base change stable for $c \in NS_G(\mathcal{X},\mathbb{R})$.

# Part II

# Moduli space of Fano manifolds with Kähler–Ricci solitons

# Introduction for Part II

In this part II, we give a theoretical framework and methods for construction of moduli space of Fano manifolds with Kähler–Ricci solitons. We begin with backgrounds and motivations for this new-type moduli space.

In the celebrated paper [FS2], Fujiki and Schumacher constructed the complex analytic moduli spaces of all compact smooth (polarized) Calabi–Yau ($K_X \equiv 0$) and canonically polarized manifolds ($K_X > 0$), as a higher dimensional analogue of the moduli spaces of Riemann surfaces of genus $g = 1$ and $g \geq 2$, respectively.

In contrast to these cases, it is known that 'the moduli space of all Fano manifolds' in a primitive sense behaves pathologically; it does not enjoy the $T_1$-separation axiom, in particular, it does not admit any nice geometric structure like a complex analytic structure. Indeed, the separation is obstructed by the existence of iso-trivial degenerations of Fano manifolds: there are (many) families $\mathcal{X} \to \Delta$ of Fano manifolds which is biholomorphically trivial over $\Delta^* = \Delta \setminus \{0\}$ and whose central fibre $\mathcal{X}_0$ is *not* biholomorphic to the general fibres; for example, small deformations of the Mukai–Umemura threefold (cf. [Tian-book, Chapter 7]) and the unique $G_2$-horospherical Fano manifold with Picard number one (cf. [PP] and Example 3.5.4) give such examples. In other words, the moduli space of all Fano manifolds does not even exist in the complex analytic framework.

Still, in view of the work of [FS2], one can imagine or hope that the existence of some 'canonical metrics' on Fano manifolds may play a role to ensure the separation property of the moduli space or stack. For Fano manifolds, Kähler–Einstein metric is a candidate for such 'canonical metrics'. However, it is known that Fano manifolds do not always admit Kähler–Einstein metrics, while Calabi–Yau and canonically polarized manifolds always admit Kähler–Einstein metrics, as observed in [Mat1, Fut]. We must exclude 'unstable' Fano manifolds in some sense.

Detecting a geometric condition of Fano manifolds equivalent to the existence of Kähler–Einstein metrics was a long standing problem and a conjecture on this concern was settled as the Yau–Tian–Donaldson conjecture for these two decades. Recently, Chen–Donaldson–Sun and Tian [CDS, Tian2] broke through this problem (for Fano manifolds): the existence of Kähler–Einstein metrics on a Fano manifold is equivalent to the K-stability of the Fano manifold, which is a pure algebro-geometric condition for polarized variety.

After this breakthrough, in the spirit that the existence of canonical metrics plays a role for the separation, the (algebro-geometric) moduli space of Fano manifolds *with Kähler–Einstein metrics* was constructed in [OSS, Oda2, Oda3, LWX1] as an algebraic space within a unified theoretical framework (not defeating one by one). Different from the case of [FS2], even the dimension of the automorphism groups of Fano manifolds may jump along deformation of complex structures. So they constructed the moduli space of Kähler-Einstein Fano manifolds by rather different new technologies from [FS2], while sharing the same spirit with [FS2] on the philosophical reason for the separation. After the construction of the moduli space, Li–Wang–Xu [LWX2] proves the quasi-projectivity of the moduli space. There are also more intensive studies as [SS, LiuXu] on the relation with a literal GIT construction, which a priori depends on some ad hoc data such as an embedding of varieties into a fixed projective space, for some special cases.

This additional stability assumption '*with Kähler–Einstein metrics*' is enjoyable for interest in particular examples: there are various important examples of Fano manifolds admitting Kähler–Einstein metrics. However, on the other hand, it is also known that there are many examples of Fano manifolds who do not admit any Kähler–Einstein metrics; even the one point blowing up of $\mathbb{C}P^n$.

Philosophically, constructing moduli spaces of varieties in the schematic or complex analytic category within a unified theoretical framework can be regard as giving a (schematic/complex analytic method for) classification of them. From a viewpoint of MMP, Odaka and Okada conjectured in [OO] that every *smooth* Fano manifold with Picard number one, which is one of the final outcome of the MMP, is *K-semistable*, so that they are members of the moduli space of Kähler–Einstein Fano manifolds and hence are classified. However, (infinitely) many counter-examples of this conjecture are discovered by Fujita [Fuj] and Delcroix [T. Del]. We face that the assumption *'with Kähler–Einstein metrics'* is restrictive for our interest on this classification concern.

In this part II, we extend the moduli space of Fano manifolds with Kähler–Einstein metrics to *the moduli space of Fano manifolds with Kähler–Ricci solitons*. Kähler–Ricci soliton, which consists of a Kähler metric and a holomorphic vector field, is a natural generalization of Kähler–Einstein metrics from the viewpoint of Kähler–Ricci flow. The uniqueness of Kähler–Ricci soliton modulo the identity component of the biholomorphism group is known by [TZ2] as in the case of Kähler–Einstein metrics [BM], so that we can regard Kähler–Ricci soliton as a kind of 'canonical metrics' on Fano manifolds. The equivalence with K-stability as developed in [DT, Tian1, Ber] and [CDS, Tian2] for Kähler-Einstein metrics are also covered for the case of Kähler–Ricci soliton in [Xio, BW] and [DaSz]. There are large amount of known examples of Fano manifolds admitting Kähler–Ricci solitons; Delcroix's infinite series of counter-examples of Odaka–Okada conjecture admit Kähler–Ricci solitons, while they do not admit Kähler–Einstein metrics.

We usually characterize moduli spaces in the schematic/complex analytic framework by a universal property. A category consisting of families of which we intend to construct the moduli space, which is usually called the *moduli stack*, is a convenient and essential tool for describing the universal property. In our moduli problem, we do not work with the usual moduli stack consisting of the usual families of Fano manifolds. In order to ensure the separation, and technically in order to apply GIT method, we instead consider another new moduli stack $\boldsymbol{\mathcal{K}}(n)$ consisting of some families of *pairs* $(X, \xi')$ of $n$-dimensional Fano manifolds and holomorphic vector fields, which is natural in view of the theory of Kähler–Ricci soliton. The moduli stack $\boldsymbol{\mathcal{K}}(n)$ is furthermore divided into clopen (closed and open, but not necessarily connected) sub-stacks $\boldsymbol{\mathcal{K}}_{T,\chi}$, where the associated holomorphic vector fields are deformed holomorphically. As we must review the theory of Kähler–Ricci soliton (in section 3.2) before explaining this unfamiliar moduli stack, here we do not explain the detail and postpone the precise description/definition until section 3.2 and Definition 3.4.1. The author hopes that Appendix in this chapter helps the readers unfamiliar to stacks to grasp some fundamental generalities on stacks over the category of complex spaces. Example 3.5.4 explains that this formulation of the moduli stack is essential and the reason why the usual stack does not serve our purpose. The readers will see in Remark 3.2.8 that the change of our moduli stacks does *not affect the sets* of what we intend to parametrize ($X$ or $(X, \xi')$) and these are naturally identified to each other (only) *as sets*.

# Chapter 3

# The moduli space of Fano manifolds with KRs

We construct a canonical Hausdorff complex analytic moduli space of Fano manifolds with Kähler–Ricci solitons. This enlarges the moduli space of Fano manifolds with Kähler–Einstein metrics. We discover a moment map picture for Kähler–Ricci solitons, and give complex analytic charts on the topological space consisting of Kähler–Ricci solitons, by studying differential geometric aspects of this moment map. Some stacky words and arguments on Gromov–Hausdorff convergence help to glue them together in the holomorphic manner.

The content corresponds to the paper [Ino1].

## 3.1 Introduction

Let $\mathcal{KR}_{GH}(n)$ be the set of biholomorphism classes of $n$-dimensional Fano manifolds admitting Kähler–Ricci solitons. We can endow $\mathcal{KR}_{GH}(n)$ with a natural topology induced by the 'complexified' Gromov–Hausdorff convergence (cf. [PSS]). Note that the set $\mathcal{K}_{0,GH}(n)$ of biholomorphism classes of $n$-dimensional Fano manifolds admitting Kähler–Einstein metrics forms a clopen subset of $\mathcal{KR}_{GH}(n)$. Our main theorem is the following.

**Theorem H** (Theorem 3.4.8 + Proposition 3.4.11)**.** The Hausdorff topological space $\mathcal{KR}_{GH}(n)$ admits a natural complex analytic structure which is uniquely characterized by the following universal property of a natural morphism $\boldsymbol{\mathcal{K}}(n) \to \mathcal{KR}_{GH}(n)$ from the moduli stack: any morphism

$\boldsymbol{\mathcal{K}}(n) \to B$ to any complex space $B$ holomorphically and uniquely factors through $\mathcal{KR}_{GH}(n)$.

In contrast to the current known construction ([Oda2, Oda3, LWX1]) of the moduli space of Fano manifolds with Kähler–Einstein metrics, our method for construction actually does not depend on the result in [DaSz], where they proved modified K-polystable Fano manifolds admit Kähler–Ricci solitons. Although, as some of the readers might prefer algebro-geometric formulation, we formulate things in terms of modified K-stability, which can be translated into the existence of Kähler–Ricci solitons via [DaSz].

Our main tool for the construction of complex analytic charts on $\mathcal{KR}_{GH}(n)$ is the following moment map.

**Key Observation** (Proposition 3.3.1 + Proposition 3.3.2)**.** Let $(M, \omega)$ be a $2n$-dimensional $C^\infty$-symplectic manifold underlying a Fano manifold with a Hamiltonian action of a closed real torus $T$. For any $\xi \in \mathfrak{t}$, there is a moment map

$$\mathcal{S}_\xi : \mathscr{J}_T(M, \omega) \to \mathrm{Lie}(\mathrm{Ham}_T(M, \omega))^\vee$$

on the space $\mathscr{J}_T(M, \omega)$ of $T$-invariant almost complex structures with respect to the modified symplectic structure $\Omega_\xi$ (see subsection 3.3.1) and the action of $\mathrm{Ham}_T(M, \omega)$. Moreover, integrable complex structures in $\mathcal{S}_\xi^{-1}(0)$ correspond to Kähler–Ricci solitons.

We firstly construct charts on the quotient space $(\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)/\mathrm{Ham}_T(M, \omega)$, where $\mathcal{S}_\xi^{\mathrm{int}}$ denotes the restriction of the moment map $\mathcal{S}_\xi$ to the subspace $\mathscr{J}_T^{\mathrm{int}}(M, \omega) \subset \mathscr{J}_T(M, \omega)$ consisting of integrable almost complex structures. The quotient space reveals to be identified with a clopen subspace of $\mathcal{KR}_{GH}(n)$.

To compare our constructions with [Oda2, Oda3, LWX1], we briefly review their methods here. They firstly prove the Zariski openness of the set of the K-(semi)stable points in any family of Fano manifolds. It follows that the usual moduli stack is Artin algebraic, so that they can apply the established theory of *good moduli spaces* of Artin algebraic stacks. Secondly they construct étale local charts on this stack of the form $[V/G]$, where each $V$ is an affine scheme and $G$ is a reductive algebraic group. Each quotient stack $[V/G]$ has the good moduli space $V /\!\!/ G$. We can glue them together, just applying the gluing theory of good moduli spaces developed in [Alp2]. Technically, the proofs of the Zariski openness and the existence of the étale local charts rely on the argument showing that the set of K-(semi/poly)stable points forms a constructible set of the parameter space in the Zariski topology. The CM line bundle, whose GIT weight equals to the Donaldson–Futaki invariant ([PT]), is used to prove the constructibility. (Compare [Don2] for another proof of the Zariski openness. )

However, in the case of Kähler–Ricci soliton, as there is no candidate for the CM-line bundle because of the irrationality of the modified Donaldson–Futaki invariant, we face a problem with the constructibility. So we will work with the real topology, in other words, with Artin *analytic* stacks. We can still construct local charts on this Artin analytic stack with good moduli spaces, however, the second nuisance appears when gluing the good moduli spaces together: there is no well-established theory of good moduli spaces for Artin analytic stacks so far. (At least to the author, it seems not so easy to show the uniqueness (universal) property of good moduli spaces of Artin analytic stacks, if it exists, which is obviously a key property for the good gluing theory (cf. [Alp1, Alp2]). The lack of nice counterpart of 'quasi-coherent sheaves' on complex analytic spaces seems critical. (cf. [EP-book, Section 4])

Alternatively, we glue our charts by a 'cooperation of virtual and real'. We construct analytic charts not only on the stack $\boldsymbol{\mathcal{K}}(n)$, but also on the topological spaces $(\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)/\mathrm{Ham}_T(M,\omega)$, which are related in a canonical way. The latter 'real side' is studied in section 3.3 and is used to show that the charts are actually homeomorphisms onto open subsets of $(\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)/\mathrm{Ham}_T(M,\omega)$. This is not treated in [Oda2, Oda3, LWX1] as they could apply Alper's gluing work of good moduli spaces, which works 'without reality'. The former 'virtual side' is studied in section 3.4 and is used to show that the coordinate changes are holomorphic. Finding holomorphic relations between the analytic charts are easier on the stack $\boldsymbol{\mathcal{K}}(n)$ than on the topological spaces $(\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)/\mathrm{Ham}_T(M,\omega)$. These holomorphic relations of stacks descend to the actual holomorphic relations between the analytic charts on $(\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)/\mathrm{Ham}_T(M,\omega)$ thanks to the universality of the local moduli spaces and the fundamental (2-categorical version of) Yoneda's lemma: the natural fully faithful embedding of the category $\mathbb{C}\mathbf{an}$ of complex analytic spaces to the 2-category of complex analytic stacks.

### Organization

The remainder of this chapter is organized as follows. In section 3.2, we review some known results on Kähler–Ricci soliton and rearrange K-stability notion modified to the soliton setting so that it fits into our moduli problem. It is explained that the pair $(X, \xi')$ can be converted into the action $X \curvearrowleft T$, where $T$ is the torus generated by the holomorphic vector field $\xi'$. We introduce *gentle Fano $T$-manifolds* as Fano $T$-manifolds inseparable from *smooth* Fano $T$-manifolds with Kähler–Ricci solitons, which are expected to be K-semistable. They form an adequate moduli stack in our moduli problem. Finally, we propose Proposition 3.2.18, which states the uniqueness of the central fiber of gentle degenerations. It will be proved after we complete Proposition 3.4.7, and play an essential role in the proof of Theorem 3.4.8 in subsection 3.4.2.

In section 3.3, we construct and study an infinite dimensional moment map $\mathcal{S}_\xi$ whose integrable zero points correspond to Kähler–Ricci solitons. We describe that local slices $\nu : B \to \mathfrak{k}$ of the moment map actually give charts $\nu^{-1}(0)/K \approx BK^c /\!\!/ K^c$ on the topological space consisting of Kähler–Ricci solitons. To achieve this, we need to study Banach completions of Fréchet manifolds, where we must pay attention to the treatment of the completions of $\mathrm{Ham}_T(M, \omega)$ as they are never Banach Lie groups. We also prove that, in any family of Fano $T$-manifolds, the set of gentle Fano $T$-manifolds forms an open subset in the parameter space of the family.

In section 3.4, the main theorem is proved. We introduce the stack $\boldsymbol{\mathcal{K}}_{T,\chi}$ of gentle Fano $T$-manifolds and show that it is an Artin analytic stack. We prove Proposition 3.2.18 in subsection 3.4.4, using the results in the former half of subsection 3.4.2. We use this proposition in the proof of the main theorem. In subsection 3.4.3, we show that our moduli space is related to the topological space $\mathcal{KR}_{GH}(n)$ endowed with the 'complexified' Gromov–Hausdorff topology, which is studied in [PSS].

In section 3.5, we review some examples of Fano manifolds with Kähler–Ricci solitons and propose some future studies. In particular, we find an iso-trivial degeneration of a Kähler–Einstein Fano manifold to another Fano manifold with non-Einstein Kähler–Ricci soliton, which implies that the usual moduli stack is not sufficiently separated and hence our new formulation of moduli stacks $\boldsymbol{\mathcal{K}}(n)$ and $\boldsymbol{\mathcal{K}}_{T,\chi}$ is essential.

## 3.2 Kähler–Ricci soliton and K-stability

### 3.2.1 Kähler–Ricci soliton

A Kähler metric $g$ on a Fano manifold $X$ is called a *Kähler–Ricci soliton* if it satisfies the following equation:

$$\mathrm{Ric}(g) - L_{\xi'} g = g$$

for some holomorphic vector field $\xi'$. The same term sometimes refers the pair $(g, \xi')$.

A fundamental feature of a Kähler–Ricci soliton $(g, \xi')$ is that it gives an eternal solution of the normalized Kähler–Ricci flow:

$$\partial_t g(t) = -\mathrm{Ric}(g(t)) + g(t).$$

Namely, for the 1-parameter smooth family $\phi_t : X \xrightarrow{\sim} X$ generated by $\mathrm{Re}(\xi')$, the following holds:

$$\partial_t(\phi_t^* g) = -\mathrm{Ric}(\phi_t^* g) + \phi_t^* g.$$

On a Fano manifold admitting Kähler–Ricci soliton, it is shown in [TZ3, TZZZ, DeSz] that the normalized Kähler–Ricci flow converges to a Kähler–Ricci soliton, starting from any Kähler metric in $2\pi c_1(M)$.

It is shown in [Zhu] that there is a solution $g$ of the equation

$$\mathrm{Ric}(g) - L_{\xi'} g = g_0$$

for any initial Kähler metric $g_0$. Let us consider the following smooth continuity path for Kähler–Ricci soliton:

$$\mathrm{Ric}(g_t) - L_{\xi'} g_t = t g_t + (1-t) g_0. \tag{3.1}$$

One can prove that

$$R_{\xi'}(X) := \sup\{t \in [0,1] \mid \text{a solution } g_t \text{ of (3.1) exists. }\}$$

is independent of the choice of the initial metrics $g_0$ and has the equality

$$R_{\xi'}(X) = \sup\{t \in [0,1] \mid \exists g \text{ s.t. } \mathrm{Ric}(g) - L_{\xi'} g > tg\}. \tag{3.2}$$

The proof of this equality is in [Szé3] for $\xi' = 0$ and in Kazuma Hashimoto's master thesis [Has] for the general case ($\xi' \neq 0$). A related invariant is also mentioned in [DGSW].

*Remark* 3.2.1. Kazuma Hashimoto was a master student of University of Tokyo supervised by Prof. Akito Futaki. He did not proceed to doctoral course and quit his research position. The proof of the equality (2) in his thesis is an analogy of [Szé3], using the functionals $\mathcal{M}_\xi := \mu_\omega$ originally defined in [TZ2] and the following $\mathcal{J}_{\alpha,\xi}$ instead of $\mathcal{M}, \mathcal{J}_\alpha$ in [Szé3]:

$$\mathcal{M}_\xi(\phi) := -\int_0^1 dt \int_X \dot{\phi}_t\Big(s(g_{\phi_t}) - n - \mathrm{tr}(\nabla_{g_{\phi_t}}\xi') + \xi'(h_{g_{\phi_t}} - \theta'_\xi(\phi_t))\Big)e^{\theta'_\xi(\phi_t)}\omega^n_{phi_t},$$

$$\mathcal{J}_{\alpha,\xi}(\phi) := \int_0^1 dt \int_X \dot{\phi}_t(\mathrm{tr}_{\omega_{\phi_t}}\alpha - n + \xi'\varphi_\alpha)e^{\theta'_\xi(\phi_t)}\omega^n_{\phi_t},$$

where $\varphi_\alpha$ is a function with $\alpha - \omega = \sqrt{-1}\partial\bar{\partial}\varphi_\alpha$.

The uniqueness and the existence results analogous to those of the Kähler–Einstein metrics [BM, CDS, Tian2] (and ) hold also for Kähler–Ricci solitons.

**Theorem 3.2.2** (Uniqueness, [TZ1, TZ2] (and [BW] for $\mathbb{Q}$-Fano variety with $t = 1$))**.** If $(g_1, \xi'_1)$ and $(g_2, \xi'_2)$ are two Kähler–Ricci solitons on a Fano manifold $X$, then there is an element $\phi \in \mathrm{Aut}^0(X)$ such that

$$g_2 = \phi^* g_1, \quad \xi'_2 = \phi_*^{-1}\xi'_1,$$

where $\mathrm{Aut}^0(X)$ is the identity component of the group $\mathrm{Aut}(X)$ of biholomorphisms of $X$. Moreover, a solution $g_t$ of the equation (3.1) is absolutely unique for any initial metric $g_0$ and $t \in [0, 1)$.

**Theorem 3.2.3** (Existence, [DaSz, CSW])**.** $R_{\xi'}(X) = 1$ for any K-semistable pair $(X, \xi')$. If in addition $(X, \xi')$ is K-polystable, there is a Kähler–Ricci soliton on $X$ with respect to $\xi'$.

We will see the definition of the K-stability of pairs $(X, \xi')$ in the next subsection. The above claim on K-semistability is also covered in [C. Li] for the Kähler–Einstein case, using [CDS, Tian2]. The opposite implication for K-polystablity is proved in [Ber, BW] including the $\mathbb{Q}$-Fano case as follows.

**Theorem 3.2.4** ([Ber, BW])**.** Let $X$ be a $\mathbb{Q}$-Fano variety. If $X$ admits a Kähler–Ricci soliton $(g, \xi')$, then $(X, \xi')$ is K-polystable.

In the Kähler–Einstein case (i.e. $\xi' = 0$), [Der, C. Li] shows that $X$ is K-semistable if $R(X) = 1$. So we can summarize as follows.

- $X$ is K-polystable $\iff$ $X$ admits a Kähler–Einstein metric.
- $X$ is K-semistable $\iff$ $R(X) = 1$.

Only the implication from the right to the left-hand side of the second item is still open for general $(X, \xi')$.

There is a version of Futaki invariant suitable for Kähler–Ricci soliton defined in [TZ2]. Let $H^0(X, \Theta_X)$ denote the space of holomorphic vector fields on $X$. Define a linear map $\mathrm{Fut}_{\xi'} : H^0(X, \Theta_X) \to \mathbb{C}$ by

$$\mathrm{Fut}_{\xi'}(v') := \int_X v'(h - \theta_{\xi'}) e^{\theta_{\xi'}} \omega^n,$$

where $\omega \in 2\pi c_1(M)$ is a Kähler form, $h$ is a real valued function satisfying $\sqrt{-1}\partial\bar{\partial}h = \mathrm{Ric}(\omega) - \omega$ and $\theta_{\xi'}$ is a complex-valued function characterized by

$$\begin{cases} L_{\xi'}\omega = \sqrt{-1}\partial\bar{\partial}\theta_{\xi'} \\ \int_X e^{\theta_{\xi'}} \omega^n = \int_X \omega^n. \end{cases}$$

The function $\theta_{\xi'}$ becomes real-valued when $\xi := \mathrm{Im}\xi'$ is a Killing vector. This linear function is independent of the choice of $\omega$, so it gives an invariant depending only on $X$ and $\xi'$, which is now called *the modified Futaki invariant.* This invariant obviously vanishes when $X$ admits a Kähler–Ricci soliton with respect to the vector field $\xi'$.

The following is a crucial fact in order to properly formulate our moduli problem.

**Proposition 3.2.5** ([TZ2])**.** Let $X$ be a Fano manifold, which does not necessarily have a Kähler–Ricci soliton, and $K \subset \mathrm{Aut}(X)$ be a compact subgroup. Then there is a unique holomorphic vector field $\xi'$ with $\mathrm{Im}(\xi') \in \mathrm{Lie}(K)$ such that

$$\mathrm{Fut}_{\xi'}(v') = 0, \ \forall v' \in \mathrm{Lie}(K^c),$$

where $K^c \subset \mathrm{Aut}(X)$ is the complexification of the group $K$.

*Remark* 3.2.6. In general, a reductive algebraic group $K^c$ does not uniquely determine its maximal compact subgroup $K$, but only up to conjugate. When $K^c$ is an algebraic torus, which is isomorphic to $(\mathbb{C}^*)^k$, its maximal compact subgroup $(U(1))^k$ is uniquely determined. This fact allows us to get away from a formulation relying on structures over the field $\mathbb{R}$ as we see in the next subsection and to formulate things over even a field of positive characteristic, which should be preferred by algebraic geometers.

The total biholomorphism group of a Fano manifold $X$ admitting a Kähler–Ricci soliton $(g, \xi')$ is not necessarily reductive. Instead, we have the following.

**Theorem 3.2.7** ([TZ2] ([BW] for the $\mathbb{Q}$-Fano case))**.** Suppose a $\mathbb{Q}$-Fano variety $X$ has a Kähler–Ricci soliton $(g, \xi')$, then the subgroup $\mathrm{Aut}^0(X, \xi') \subset \mathrm{Aut}^0(X)$ consisting of $\xi'$-preserving biholomorphisms is a maximal reductive subgroup of $\mathrm{Aut}^0(X)$. Moreover, the complexification of the identity component $\mathrm{Isom}^0(X, \xi')$ of the group of isometries preserving $\xi'$ coincides with the group $\mathrm{Aut}^0(X, \xi')$.

*Remark* 3.2.8. The reductivity of the automorphism groups of geometric structures of which we intend to construct a geometric moduli space, is crucial if one expect to apply local or global GIT to its construction and indeed indispensable in the doctrine of Alper's good moduli space (cf. [Alp1, Alp2]).

The uniqueness of Kähler–Ricci soliton implies that the set consisting of the isomorphism classes of the pairs $(X, \xi')$ with Kähler–Ricci solitons can be naturally identified with the set consisting of the biholomorphism classes of Fano manifolds $X$ with Kähler–Ricci solitons. So there is no change in the *support sets* of 'the moduli spaces' of the following two moduli stacks: one is the usual moduli stack associated with Fano manifolds $X$ admitting Kähler–Ricci solitons, and the other is the moduli stack associated with Fano pairs $(X, \xi')$ admitting Kähler–Ricci solitons.

However, there are nice geometric features in the latter stack compared to the former stack, such as the separation property and the reductivity of the stabilizer groups at K-polystable points, which is appropriate for the local GIT construction of the good moduli space.

So we will work with the latter stack, and precisely define it in subsection 3.4.1, replacing the pairs $(X, \xi')$ with the $T_{\xi'}$-action on $X$. This *may change the topology* of the moduli space, but it turns out that the latter stack is correct with regard to the 'complexified' Gromov–Hausdorff convergence considered in [PSS].

### 3.2.2 K-stability

Here we review the definition of K-stability and formulate it as the stability notion of a Fano manifold *with an algebraic torus action*. This enables us to introduce an adequate notion of 'deformations of Fano manifolds with Kähler–Ricci solitons' and leads us to the proper definition of the stack $\boldsymbol{\mathcal{K}}(n)$.

Recall that a $\mathbb{Q}$-*Fano variety* $X$ is a reduced irreducible normal complex space $X$ with the following property: there is a positive integer $\ell$ such that the sheaf $i_*((\det \Theta_{X^{\mathrm{reg}}})^{\otimes \ell})$, which is denoted by $\mathcal{O}(-\ell K_X)$, is isomorphic to the sheaf of sections of an ample line bundle on $X$, and $X$ has only log terminal singularities (see [EGZ]). The minimum $\ell$ satisfying this property is called the $\mathbb{Q}$-*Gorenstein index* of $X$. Obviously, $\mathbb{Q}$-Fano varieties can be embedded into some $\mathbb{C}P^N$, hence they are also considered as schemes, but we treat them in the category of complex spaces.

A $\mathbb{Q}$-*Fano $T$-variety* is a $\mathbb{Q}$-Fano variety $X$ with a holomorphic action $\alpha : X \times T \to X$, where we only consider an algebraic torus $T \cong (\mathbb{C}^*)^k$. When $X$ has no singularities, we call it *Fano $T$-manifold*. We denote by $\mathrm{Aut}_T(X)$ the centralizer of $T \subset \mathrm{Aut}(X)$:

$$\mathrm{Aut}_T(X) := \{g \in \mathrm{Aut}(X) \mid gt = tg \text{ for } \forall t \in T\}.$$

Let $T$ be an algebraic torus. We denote the character lattice of $T$ by $M := \mathrm{Hom}(T, \mathbb{C}^*)$ and its dual (1-psg) lattice by $N := \mathrm{Hom}(\mathbb{C}^*, T)$. Let $X$ be a $\mathbb{Q}$-Fano $T$-variety. Its $T$-action canonically lifts to the sheaf $\mathcal{O}(-m\ell K_X)$ and hence there is an action of $T$ on the cohomologies of $\mathcal{O}(-m\ell K_X)$. For a character $u \in M$, put

$$H^0_u(X, \mathcal{O}(-m\ell K_X)) := \{\sigma \in H^0(X, \mathcal{O}(-m\ell K_X)) \mid t.\sigma = u(t)\sigma \ \forall t \in T\}$$

and set

$$\begin{aligned} h^i_X(m) &:= \dim H^i(X, \mathcal{O}(-m\ell K_X)), \\ h^i_{X,u}(m) &:= \dim H^i_u(X, \mathcal{O}(-m\ell K_X)). \end{aligned}$$

We adopt the following pure algebraic definition of modified Futaki invariant exhibited in [BW], which is shown to coincide with $\mathrm{Fut}_{\xi'}$ in the previous section, up to a uniform positive factor. Note that the modified Futaki invariant for special degenerations is firstly introduced in [Xio] and reformulated in [WZZ]. (The author thanks the referees for telling the author these important references. )

**Definition 3.2.9** (modified algebraic Futaki invariant)**.** For a $\mathbb{Q}$-Fano $T$-variety $X$ and an element $\xi \in N_{\mathbb{R}}$, we define the *modified* (sometimes we omit this word in our $T$-equivariant setup) *algebraic Futaki invariant* $F_{X,\xi} : N \to \mathbb{R}$ by

$$F_{X,\xi}(\lambda) := -\lim_{m \to \infty} \frac{w_{X,\xi}(m; \lambda)}{m h^0_X(m)},$$

where

$$w_{X,\xi}(m;\lambda) := \sum_{u\in M} e^{\langle u,\xi\rangle/m} h^0_{X,u}(m)\langle u,\lambda\rangle.$$

We define the *Hilbert character* $\chi : \mathbb{Z} \to \mathbb{Z}[M]$ of a Fano $T$-manifold $X$ by

$$\chi_m := \sum_{i=0}^{\dim X} (-1)^i \sum_{u\in M} h^i_{X,u}(m)u \in \mathbb{Z}[M]. \tag{3.3}$$

We call a function $\chi : \mathbb{Z} \to \mathbb{Z}[M]$ a *Fano character* if there exists a Fano $T$-manifold whose Hilbert character given in (3.3) is the given $\chi$.

**Proposition 3.2.10** ([TZ2]). For every Fano $T$-manifold $X$, there exists a unique vector $\xi \in N_{\mathbb{R}}$ such that the modified algebraic Futaki invariant $F_{X,\xi}(\lambda)$ in the above sense vanishes on the 1-psg lattice $N$ of $T$ (equivalently, the modified Futaki invariant $\mathrm{Fut}_{\xi'}|_{\mathfrak{t}}$ restricted to the Lie algebra $\mathfrak{t}$ of $T$ vanishes). We call this vector $\xi$ the *K-optimal vector* of $(X, T)$.

Obviously from the definition of the modified algebraic Futaki invariant, the K-optimal vector $\xi$ of a Fano $T$-manifold $X$ depends only on the Hilbert character $(T, \chi)$. So it also makes sense to say that $\xi \in N_{\mathbb{R}}$ is the K-optimal vector of a Fano character $(T, \chi)$, which is a $T$-equivariant deformation invariant.

**Proposition 3.2.11** ([TZ2]). If a Fano manifold $X$ has a Kähler–Ricci soliton $(g, \xi')$, then the $\xi'$ is the K-optimal vector with respect to any algebraic torus containing the algebraic torus generated by $\xi'$. (Note that the closure of the exponential of the imaginary part of the holomorphic vector $\xi'$ associated to soliton gives a closed real torus, and the algebraic torus generated by $\xi'$ is just the complexification of this closed real torus. )

We define the K-optimality of Fano character (and in particular the K-optimality of a Fano $T$-manifold), not of vector, as follows.

**Definition 3.2.12** (K-optimal character). We call a Fano character $(T, \chi)$ *K-optimal* if there is no proper sub-lattice $\tilde{N} \subset N$ with $\xi \in \tilde{N}_{\mathbb{R}}$ for the K-optimal vector $\xi \in N_{\mathbb{R}}$ of $(T, \chi)$.

For a Fano manifold $X$, we call an algebraic action of an algebraic torus $T$ on $X$ is *K-optimal* if it is maximal (as actions on $X$) among all K-optimal characters $(\tilde{T}, \tilde{\chi})$ obtained from $\tilde{T}$-actions on $X$.

*Remark* 3.2.13. It is possible that both characters $(T_1, \chi_1) \subset (T_2, \chi_2)$ are K-optimal, where $\chi_i$ are related by the projection $\mathbb{Z}[M_2] \to \mathbb{Z}[M_1]$. Not only that, there is an example of a Fano manifold $X_1$ with a Kähler–Einstein metric $(g_1, \xi_1' = 0)$ admitting a deformation to a Fano manifold $X_2$ with a Kähler–Ricci soliton $(g_2, \xi_2')$ (Example 3.5.4), which shows that both actions $X_1 \curvearrowleft T_1 = 0$ and $X_2 \curvearrowleft T_2 = T(\xi_2') \neq 0$ are K-optimal with $(T_1, \chi_{X_1}) \subset (T_2, \chi_{X_2})$. This example illustrates that the torus equivariant formulation is essential for the separation of the moduli space of Fano manifolds with Kähler–Ricci solitons.

For a fixed Fano manifold $X$, K-optimal action $T \subset \mathrm{Aut}(X)$ is unique up to adjoint thanks to the uniqueness in Proposition 3.2.5.

Let $X$ be a $\mathbb{Q}$-Fano $T$-variety. A pair $(\pi : \mathfrak{X} \to \mathbb{C}, \theta)$ consisting of the following data is called a *special degeneration* of $X$.

1. $\mathfrak{X}$ is a normal complex space with an action of $T \times \mathbb{C}^*$ and $\pi : \mathfrak{X} \to \mathbb{C}$ is a $T \times \mathbb{C}^*$-equivariant proper flat $\mathbb{Q}$-Gorenstein surjective morphism whose central fiber $\mathfrak{X}_0$ is a $\mathbb{Q}$-Fano variety, where $T \times \mathbb{C}^*$ acts on $\mathbb{C}$ by $z.(t, s) = sz$.

2. $\theta$ is a $T \times \mathbb{C}^*$-equivariant isomorphism $\theta : X \times \mathbb{C}^* \xrightarrow{\sim} \pi^{-1}(\mathbb{C}^*)$.

We also assume that there is a holomorphic line bundle $\mathcal{L}$ on $\mathfrak{X}$ with an isomorphism $\theta^* \mathcal{L}|_{\pi^{-1}(\mathbb{C}^*)} \cong p_1^* \mathcal{O}(-\ell K_X)$ for some $\ell$. It is shown in [Ber, Lemma 2.2] that if such $\mathcal{L}$ exists, then $-\ell K_{\mathfrak{X}}$ becomes $\mathbb{Q}$-Cartier and the tensor bundle $\mathcal{L}^{\otimes m}$ is actually isomorphic to $\mathcal{O}(-m\ell K_{\mathfrak{X}/\mathbb{C}})$ for some $m$. So we exclude the datum $\mathcal{L}$ from the data of special degeneration.

**Definition 3.2.14** (K-stability)**.** Let $\xi \in N_{\mathbb{R}}$ be the K-optimal vector of a $\mathbb{Q}$-Fano $T$-variety $X$. Denote the vector $(\xi, 0) \in (N \times \mathbb{Z})_{\mathbb{R}}$ by the same symbol $\xi$. We call the $\mathbb{Q}$-Fano $T$-variety $X$

- *K-semistable* if for any special degeneration $(\pi : \mathfrak{X} \to \mathbb{C}, \theta)$ of $X$, the modified algebraic Futaki invariant $F_{X,\xi}(\pi, \theta) := F_{\mathfrak{X}_0,\xi}(\lambda)$ of the central fiber $\mathfrak{X}_0$ is nonnegative, where $\lambda$ is the one parameter subgroup defined by $\lambda : \mathbb{C}^* \to T \times \mathbb{C}^* : s \mapsto (1, s)$.

- *K-polystable* if $X$ is K-semistable and $F_{X,\xi}(\pi, \theta) = 0$ if and only if there exists a one parameter subgroup $\lambda : \mathbb{C}^* \to \mathrm{Aut}_T(X)$ such that $\theta(x\lambda(t)^{-1}, t)$ extends to an isomorphism of the total space $X \times \mathbb{C} \xrightarrow{\sim} \mathfrak{X}$.

- *K-stable* if the Fano $T$-variety $X$ is K-polystable and $\mathrm{Aut}^0_T(X) = T$.

*Remark* 3.2.15. A pair $(X, \xi')$ of a $\mathbb{Q}$-Fano variety $X$ and a holomorphic vector field $\xi'$ is called K-(semi/poly)stable if $\xi := \mathrm{Im}\xi'$ generates a closed real torus $T_{\mathbb{R}}$ and $(X, T)$ is K-(semi/poly)stable where $T$ denotes the complexification of the closed real torus $T_{\mathbb{R}}$. In this case, the vector $\xi \in \mathrm{Lie}(T_{\mathbb{R}}) = N_{\mathbb{R}}$ is of course K-optimal.

We call $X$ modified K-(semi/poly)stable if there exists a torus action $X \curvearrowleft T$ which makes $X$ K-(semi/poly)stable with respect to the action.

*Remark* 3.2.16. Note that a K-(semi/poly)stable Fano $T$-manifold is *not* necessarily a K-(semi/poly)stable Fano manifold (with respect to the trivial torus action), but only a *modified* K-(semi/poly)stable Fano manifold. However, suppose $X$ is a Fano $T$-manifold, $\tilde{T} \subset T$ is a sub-torus and the K-optimal vector $\tilde{\xi}$ with respect to the $\tilde{T}$-action coincides with the K-optimal vector $\xi$ with respect to the $T$-action (i.e. $\xi \in \mathrm{Lie}(\tilde{T})$), then the Fano $T$-manifold $X$ is K-(semi/poly)stable if and only if the Fano $\tilde{T}$-manifold $X$ is. This is proved in [DaSz] and recently proved by purely algebraic method in [**?**, LWX3] for the KE case ($\xi = 0, T = 0$).

We introduce a gentle Fano $T$-manifold as a Fano $T$-manifold inseparable from a smooth Fano $T$-manifold admitting Kähler–Ricci soliton.

**Definition 3.2.17** (gentle Fano)**.** A Fano $T$-manifold $X$ is called *gentle* if there is a $T$-equivariant deformation $\mathcal{X} \to \Delta$ with an isomorphism $\mathcal{X}|_{\Delta^*} \cong X \times \Delta^*$ such that its central fiber $\mathcal{X}_0$ is a smooth K-polystable Fano $T$-manifold. We call $\mathcal{X} \to \Delta$ a *gentle degeneration*.

From GIT viewpoint, it is naturally expected that any gentle Fano $T$-manifold is K-semistable. In this section, we do not pursue this expectation as it is not essential for the construction of our moduli space, while their K-semistability might be philosophically important. (This turns out to be true in the next section. ) Note that we always have $R_{\xi'}(X) = 1$ for a gentle Fano $T$-manifold $X$ with the K-optimal vector $\xi$, thanks to the equality (3.2). This fact helps us to prove the following proposition.

**Proposition 3.2.18.** Let $X$ be a gentle Fano $T$-manifold whose torus action is K-optimal. Then any two gentle degenerations of $X$ have the $T$-equivariant biholomorphic central fibers.

The proposition will be proved at the end of section 3.4, using Proposition 3.4.7 and a version of Donaldson-Sun's technology on Gromov–Hausdorff limit, and will be applied to the proof of Theorem 3.4.8. The logical order of our argument is "Proposition 3.4.7 $\Rightarrow$ Proposition 3.2.18 $\Rightarrow$ Theorem 3.4.8". It seems also possible to show this proposition without using a finiteness from Proposition 3.4.7 as in [LWX1]. However, the author thinks the finiteness simplifies our argument.

## 3.3 Local charts

We call a closed $C^\infty$-symplectic manifold $(M, \omega)$ *symplectic Fano* if its cohomology class $[\omega]$ is equal to $2\pi$ times the first Chern class $c_1(M, \omega)$ and there exists an $\omega$-compatible almost complex structure $J$ with positive Ricci curvature. Note that we have $b^1(M) = 0$ from familiar Bochner's theorem or Myers' theorem as we have a metric with $\mathrm{Ric} > 0$. Throughout this section, $T$ stands for a closed real torus and $(M, \omega)$ for a symplectic Fano manifold with a Hamiltonian effective action by $T$.

We denote by $\mathrm{Symp}(M, \omega)$ the group of symplectic diffeomorphisms and $\mathrm{Ham}^0(M, \omega)$ its subgroup generated by Hamiltonian diffeomorphisms. Thanks to Banyaga's theorem, in the case $b^1(M) = 0$, $\mathrm{Ham}^0(M, \omega)$ actually coincides with $\mathrm{Symp}^0(M, \omega)$, the identity connected component of $\mathrm{Symp}(M, \omega)$. (Even though it is easy to see that both groups have a natural Fréchet Lie group structures and their Lie algebras coincide, the coincidence at the level of Fréchet Lie group is *not* trivial because the Fréchet Lie group structures are not locally exponential. See [Neeb] for the generalities on Fréchet Lie groups. ) We must work with the group $\mathrm{Symp}(M, \omega)$ (resp. $\mathrm{Symp}_T(M, \omega)$) so that the complexification of the stabilizer group of cscK structure $J \in \mathcal{J}(M, \omega)$ (resp. Kähler–Ricci soliton structure $J \in \mathcal{J}_T(M, \omega)$) coincides with the biholomorphism group $\mathrm{Aut}(M, J)$ of $(M, J)$ (resp. $\mathrm{Aut}_T(M, J)$), not only it just includes the identity component $\mathrm{Aut}^0(M, J)$ (resp. $\mathrm{Aut}^0_T(M, J)$). Keeping Banyaga's theorem in our mind, we prefer using the notation $\mathrm{Ham}(M, \omega) := \mathrm{Symp}(M, \omega)$, which is *not necessarily connected*, as we always identify its Lie algebra with $C^\infty(M)/\mathbb{R}$.

We consider the space $\mathcal{J}_T(M, \omega)$ of $T$-invariant $\omega$-compatible almost complex structures and denote by $\mathcal{J}_T^{\mathrm{int}}(M, \omega)$ the subspace of integrable complex structures. It is well known that $\mathcal{J}_T(M, \omega)$ admits a natural Fréchet smooth manifold structure, which is identified with the space of $T$-equivariant sections of an associated $Sp(2n)/U(n)$-fibre bundle (see [Pal] for instance). The tangent space at $J \in \mathcal{J}_T(M, \omega)$ can be written as follows.

$$T_J\mathcal{J}_T(M, \omega) = \{A \in \Gamma_T^\infty(\mathrm{End}TM) \mid AJ + JA = 0, \omega(A\cdot, \cdot) + \omega(\cdot, A\cdot) = 0\}.$$

Similarly, the group $\mathrm{Ham}_T(M, \omega)$ of $T$-compatible symplectic diffeomorphisms can be endowed with a Fréchet smooth Lie group structure, whose Lie algebra can be identified with $C_T^\infty(M)/\mathbb{R}$. The left adjoint action is given by

$$\mathrm{Ham}_T(M, \omega) \times C_T^\infty(M)/\mathbb{R} \to C_T^\infty(M)/\mathbb{R} : (\phi, f) \mapsto f \circ \phi^{-1}.$$

The following right action

$$\mathcal{J}_T(M, \omega) \times \mathrm{Ham}_T(M, \omega) \to \mathcal{J}_T(M, \omega) : (J, \phi) \mapsto \phi^* J$$

is also smooth and its derivative is given by

$$C_T^\infty(M)/\mathbb{R} \to T_J\mathcal{J}_T(M, \omega) : f \mapsto L_{X_f} J,$$

where $X_f$ is the Hamiltonian vector field of $f$: $-df = i(X_f)\omega$.

### 3.3.1 The moment map

For a given $\xi \in \mathfrak{t} = \mathrm{Lie}(T)$, we let $\mu_\xi$ be a real valued function on $M$ given by

$$-d\mu_\xi = i_\xi \omega$$

with the prescribed normalization

$$\int_M \mu_\xi e^{-2\mu_\xi} \omega^n = 0.$$

This function is invariant under the action of $\mathrm{Ham}_T(M, \omega)$.

Set $\theta_\xi := -2\mu_\xi$. For each $J \in \mathcal{J}_T(M, \omega)$,

$$\xi'_J := J\xi + \sqrt{-1}\xi \in \mathcal{X}^{1,0}(M, J)$$

satisfies

$$\sqrt{-1}\bar{\partial}\theta_\xi = \sqrt{-1}(d(-2\mu_\xi) + \sqrt{-1}Jd(-2\mu_\xi))/2 = i_{\xi'_J}\omega.$$

We consider the following Riemannian metric on $\mathcal{J}_T(M,\omega)$, modified by $\xi$ from the usual one ([Don1]), defined as

$$(A,B)_\xi := \int_M g_J^{ij} g_{J,kl} A_i^k B_j^l \; e^{-2\mu_\xi}\omega^n$$

for tangent vectors $A, B \in T_J\mathcal{J}_T(M,\omega)$ and set

$$\Omega_\xi(A,B) := (JA,B)_\xi.$$

It is easy to see that $\Omega_\xi$ defines a non-degenerate closed 2-form on $\mathcal{J}_T(M,\omega)$. We also consider

$$(f,g)_\xi := \int_M fg \; e^{-2\mu_\xi}\omega^n$$

for $f, g \in C_T^\infty(M)$, which defines an inner product on the subspace

$$C_{T,\xi}^\infty(M,\omega) := \{f \in C_T^\infty(M) \mid \int_M f \; e^{-2\mu_\xi}\omega^n = 0\} \cong C_T^\infty(M)/\mathbb{R}.$$

Finally, we denote by $s(J)$ the Hermitian scalar curvature of $J$, defined by Donaldson [Don1]. We normalize $s(J)$ by a factor so that it is equal to the Kähler scalar curvature $-g_J^{i\bar{j}}\partial_{J,i}\partial_{J,\bar{j}}(\log\det g_J)$ for any integrable $J$, which is the half of the Riemannian scalar curvature. We denote by $\Delta_{g_J}$ the usual Riemannian Laplacian with positive eigenvalue, which is the twice of the $\bar{\partial}$-Laplacian $\bar{\square}_J = -g_J^{i\bar{j}}\partial_{J,i}\partial_{J,\bar{j}}$ when $J$ is integrable. Here is the moment map for our modified symplectic structure $\Omega_\xi$.

**Proposition 3.3.1.** Fix $\xi, \zeta \in \mathfrak{t}$. For each $J \in \mathcal{J}_T(M,\omega)$, we consider *the modified Hermitian scalar curvature* defined as

$$s_{\xi,\zeta}(J) := (s(J) - n) + \Delta_{g_J}\theta_\xi - \xi'_J\theta_\xi - \theta_\xi - \theta_\zeta,$$

where $\theta_\zeta$ is normalized as $\int_M \theta_\zeta e^{\theta_\xi}\omega^n = 0$. Then the map

$$\mathcal{S}_{\xi,\zeta} : \mathcal{J}_T(M,\omega) \to C_{T,\xi}^\infty(M,\omega)^\vee : J \mapsto (4s_{\xi,\zeta}, \cdot)_\xi$$

satisfies the property of the moment map with respect to the symplectic structure $\Omega_\xi$ and the action of $\mathrm{Ham}_T(M,\omega)$ on $\mathcal{J}_T(M,\omega)$. That is, $\mathcal{S}_{\xi,\zeta}$ is a $\mathrm{Ham}_T(M,\omega)$-equivariant smooth map satisfying

$$-\frac{d}{dt}\Big|_{t=0}\langle \mathcal{S}_{\xi,\zeta}(J_t), f\rangle = \Omega_\xi(L_{X_f}J_0, \dot{J}_0)$$

for any smooth curve $J_t \in \mathcal{J}_T(M,\omega)$ and $f \in C^\infty(M)$.

*Proof.* The equivariance of the map readily follows because the coadjoint right action is given by $(s,\cdot)_\xi \cdot \phi = (\phi^* s, \cdot)_\xi$ and $\mu'_\xi$ and $\mu'_\zeta$ are $\mathrm{Ham}_T(M,\omega)$-invariant.

The modified Hermitian scalar curvature can be divided in two parts as follows.

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}(4s_{\xi,\zeta}(J_t), f)_\xi &= \int_M \frac{d}{dt}\Big|_{t=0} 4s_{\xi,\zeta}(J_t) f\ e^{-2\mu_\xi}\omega^n \\ &= \int_M \frac{d}{dt}\Big|_{t=0} 4s(J_t) f e^{-2\mu_\xi}\ \omega^n - \frac{d}{dt}\Big|_{t=0}\int_M ((-4\Delta_t + 4\xi'_t)\theta_\xi) f\ e^{-2\mu_\xi}\omega^n \\ &= \frac{d}{dt}\Big|_{t=0}(4s(J_t), f e^{-2\mu_\xi}) - \frac{d}{dt}\Big|_{t=0}\int_M ((8\Delta_t - 8J_t\xi)\mu_\xi) f\ e^{-2\mu_\xi}\omega^n \end{aligned} \tag{3.4}$$

Now we use the following Donaldson's famous calculation [Don1] on the Hermitian scalar curvature with respect to the usual symplectic structure:

$$\frac{d}{dt}\Big|_{t=0}(4s(J_t), f) = (L_{X_f} J, JA)$$

for $A = \dot{J}_0$. The factor 4 comes from our convention of the metric $(\cdot,\cdot)_\xi$ (compare [Szé1, Proposition 2.2.1.]). Combined with the following basic identities: (a) $X_{fg} = fX_g + gX_f$, (b) $L_{fX}J = fL_X J - Jdf \otimes X + df \otimes JX$, (c) $L_\xi J = 0$, the first term of (3.4) can be arranged as follows.

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}(4s(J_t), fe^{-2\mu_\xi}) &= (L_{X_{f\exp(-2\mu_\xi)}} J, JA) \\ &= (L_{\exp(-2\mu_\xi)X_f} J, JA) + (-2)(L_{(f\exp(-2\mu_\xi))\xi} J, JA) \\ &= (e^{-2\mu_\xi} L_{X_f} J - Jd(e^{-2\mu_\xi}) \otimes X_f + d(e^{-2\mu_\xi}) \otimes JX_f, JA) \\ &\qquad + (-2)((fe^{-2\mu_\xi}) L_\xi J - Jd(fe^{-2\mu_\xi}) \otimes \xi + d(fe^{-2\mu_\xi}) \otimes J\xi, JA) \\ &= (L_{X_f} J, JA)_\xi \\ &\qquad - (-2)(Jd\mu_\xi \otimes X_f, JA)_\xi + (-2)(d\mu_\xi \otimes JX_f, JA)_\xi \\ &\qquad\quad - (-2)(Jdf \otimes \xi, JA)_\xi + (-2)(df \otimes J\xi, JA)_\xi \\ &\qquad\qquad - 4(fJd\mu_\xi \otimes \xi, JA)_\xi + 4(fd\mu_\xi \otimes J\xi, JA)_\xi. \end{aligned}$$

Now it suffices to show the following equalities.

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}((\Delta_t + (-2)J_t\xi)\mu_\xi, f)_\xi &= (Jd\mu_\xi \otimes X_f, JA)_\xi & (3.5)\\
&= -(d\mu_\xi \otimes JX_f, JA)_\xi & (3.6)\\
&= (Jdf \otimes \xi, JA)_\xi & (3.7)\\
&= -(df \otimes J\xi, JA)_\xi & (3.8)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}(-(J_t\xi)\mu_\xi, f)_\xi &= (fJd\mu_\xi \otimes \xi, JA)_\xi & (3.9)\\
&= -(fd\mu_\xi \otimes J\xi, JA)_\xi. & (3.10)
\end{aligned}$$

As for (3.5),

$$\begin{aligned}
(Jd\mu_\xi \otimes X_f, JA)_\xi &= \int_M g^{ij}g_{kl}(Jd\mu_\xi \otimes X_f)_i^k (JA)_j^l \ e^{-2\mu_\xi}\omega^n\\
&= \int_M (Jd\mu_\xi \otimes X_f)_i^k (JA)_k^i \ e^{-2\mu_\xi}\omega^n\\
&= \int_M -\mu_{\xi,p}(f_j\omega^{jk})A_k^p \ e^{-2\mu_\xi}\omega^n\\
&= \int_M -\mu_{\xi,p}f_j\omega^{pk}A_k^j \ e^{-2\mu_\xi}\omega^n\\
&= \frac{d}{dt}\Big|_{t=0}\int_M g_t^*(d\mu_\xi, df) \ e^{-2\mu_\xi}\omega^n\\
&= \frac{d}{dt}\Big|_{t=0}((\Delta_t + (-2)J_t\xi)\mu_\xi, f)_\xi.
\end{aligned}$$

We obtain (3.9) as follows.

$$\begin{aligned}
(fJd\mu_\xi \otimes \xi, JA)_\xi &= \int_M g^{ij}g_{kl}(Jd\mu_\xi \otimes \xi)_i^k (JA)_j^l \ fe^{-2\mu_\xi}\omega^n\\
&= \int_M (Jd\mu_\xi \otimes \xi)_i^k (JA)_k^i \ fe^{-2\mu_\xi}\omega^n\\
&= \int_M -\mu_{\xi,p}\xi^k A_k^p \ fe^{-2\mu_\xi}\omega^n\\
&= \frac{d}{dt}\Big|_{t=0}(-(J_t\xi)\mu_\xi, f)_\xi.
\end{aligned}$$

We can similarly compute the rest of them, using the following basic formulas: Let $A$ be an endomorphism of the tangent bundle $TM$ satisfying $JA + AJ = 0$ and $\omega(AX, Y) = \omega(AY, X)$. On a local coordinate, we denote by $\omega^{kj}$ the matrix valued function satisfying $\omega^{kj}\omega_{ij} = \delta_i^k$ and by $g^{kj}$ that satisfying $g^{kj}g_{ij} = \delta_i^k$. Then we have the following.

A. (a) $\omega_{ij} = -\omega_{ji}$, (b) $J_i^j J_j^k = -\delta_i^k$, (c) $g_{ij} = g_{ji}$.

B. (a) $\omega_{ij} = g_{pj} J_i^p = -g_{ip} J_j^p$, (b) $g_{ij} = \omega_{iq} J_j^q = -\omega_{qj} J_i^q$.

C. (a) $\omega^{kj} = -g^{qj} J_q^k = g^{kq} J_q^j = -\omega^{jk}$, (b) $g^{kj} = \omega^{pj} J_p^k = -\omega^{kp} J_p^j = g^{jk}$.

D. (a) $\omega^{kj}\omega_{ij} = \omega^{jk}\omega_{ji} = \delta_i^k$, (b) $g^{kj} g_{ij} = g^{jk} g_{ji} = \delta_i^k$.

E. $f_j = -X_f^i \omega_{ij} = X_f^i g_{pi} J_j^p = -X_f^i J_i^p g_{pj}$.

F. $X_f^k = -f_j \omega^{kj} = f_j g^{qj} J_q^k = -f_j J_q^j g^{qk}$.

G. (a) $(JA)_i^k = J_p^k A_i^p = -J_i^p A_p^k$, (b) $\omega_{kj} A_i^k = \omega_{ki} A_j^k$.

H. $g^{ij} g_{kl} (JA)_j^l = (JA)_k^i$.

$\square$

Now we observe that our moment map actually corresponds to Kähler–Ricci solitons.

**Proposition 3.3.2.** For simplicity, we let $s_\xi, \mathcal{S}_\xi$ stand for $s_{\xi,0}, \mathcal{S}_{\xi,0}$, respectively. The following (1)-(3) are equivalent for any integrable $J \in \mathcal{J}_T^{\mathrm{int}}(M, \omega)$.

1. $(g_J, \xi'_J)$ is a Kähler–Ricci soliton on $(M, J)$.
2. $s_\xi(J) = 0$.
3. $\mathcal{S}_\xi(J) = 0$.

*Proof.* Provided that $g_J$ satisfies the Kähler–Ricci soliton equation $\mathrm{Ric}(g_J) - L_{\xi'_J} g_J = g_J$. The trace of this formula gives

$$s(J) + \bar{\Box}\theta_\xi = n. \tag{3.11}$$

Since $\xi'_J$ is holomorphic, the Lie derivative by $\xi'_J$ can be arranged as follows.

$$\sqrt{-1}\partial\bar{\partial}(\bar{\Box}\theta_\xi - \xi'_J\theta_\xi) = \sqrt{-1}\partial\bar{\partial}\theta_\xi,$$

and hence $\bar{\Box}\theta_\xi - \xi'_J\theta_\xi - \theta_\xi$ is constant. Recall that the operator $(\bar{\Box} - \xi'_J)$ is a formally self-adjoint elliptic operator with respect to the inner product $(\cdot,\cdot)_{\theta_\xi}$ (see for example [Fut-book, Section 2.4], it is also shown in our Appendix B). It follows that the equation $(\bar{\Box} - \xi'_J)u = f$ has a solution $u$ if and only if $\int_M fe^{\theta_\xi}\omega^n = 0$. This shows

$$\bar{\Box}\theta_\xi - \xi'_J\theta_\xi = \theta_\xi \tag{3.12}$$

under the normalization condition $\int_M \theta_\xi e^{\theta_\xi}\omega^n = 0$. Substituting (3.12), the equation (3.11) can be reformulated as

$$(s(J) - n) + 2\bar{\Box}\theta_\xi - (\xi'_J\theta_\xi + \theta_\xi) = 0.$$

The left hand side of the equation is nothing but $s_\xi(J)$, and we obtain $s_\xi(J) = 0$.

Conversely, assume $\mathcal{S}_\xi(J) = 0$. Take a function $h$ so that $\sqrt{-1}\partial\bar{\partial}h = \mathrm{Ric}(\omega) - \omega$. Since $L_{\xi'_J}\omega = \sqrt{-1}\partial\bar{\partial}\theta_\xi$, it is enough to show that $h - \theta_\xi$ is actually constant. Similarly as before, the Lie derivative of $\sqrt{-1}\partial\bar{\partial}h = \mathrm{Ric}(\omega) - \omega$ gives

$$\sqrt{-1}\partial\bar{\partial}\xi'_J h = \sqrt{-1}\partial\bar{\partial}(\bar{\Box}\theta_\xi - \theta_\xi)$$

and hence

$$c_1 := \bar{\Box}\theta_\xi - \theta_\xi - \xi'_J h$$

is constant. We can rearrange the modified Hermitian scalar curvature as

$$\begin{aligned} s_\xi(J) &= -\bar{\Box}h + 2\bar{\Box}\theta_\xi - \xi'_J\theta_\xi - \theta_\xi \\ &= -\bar{\Box}h + 2\bar{\Box}\theta_\xi - \xi'_J\theta_\xi - (\bar{\Box}\theta_\xi - \xi'_J h - c_1) \\ &= -\bar{\Box}(h - \theta_\xi) + \xi'_J(h - \theta_\xi) + c_1. \end{aligned} \tag{3.13}$$

Now the assumption $\mathcal{S}_\xi(J) = 0$ implies that $c_2 := s_\xi(J)$ is a constant. Since $((\bar{\Box} - \xi'_J)(h - \theta_\xi), 1)_{\theta_\xi} = 0$, the constant $c_1 - c_2 = (\bar{\Box} - \xi'_J)(h - \theta_\xi)$ has to be zero and we have shown that $h - \theta_\xi$ is constant. □

*Remark* 3.3.3. As noted in [Don1] for the cscK (or Kähler–Einstein) case, our moment map picture enables us to interpret or even reproduce the following known results from more geometric viewpoint, which were originally proved in [TZ2]. (See also [Wang1]. )

- The invariance of the modified Futaki invariant.

- The reductiveness of $\mathrm{Aut}(X, \xi')$. (cf. [FO])
- The uniqueness of Kähler–Ricci soliton. (cf. [BB])

For instance, substituting (3.13), we obtain

$$\begin{aligned}
\langle \mathcal{S}_\xi(J), f\rangle &= \int_M s_\xi(J) f \; e^{\theta_\xi}\omega^n \\
&= \int_M 4(-(\bar{\Box} - \xi')(h - \theta_\xi) + c_1) f \; e^{\theta_\xi}\omega^n \\
&= -4 \int_M (\bar{\partial}(h - \theta_\xi), \bar{\partial} f) \; e^{\theta_\xi}\omega^n \\
&= -4 \int_M X'_f (h - \theta_\xi) \; e^{\theta_\xi}\omega^n \\
&= -4c \mathrm{Fut}_{\xi'}(X'_f)
\end{aligned}$$

for $X'_f \in \mathrm{Lie}(\mathrm{Stab}(J))$, where $c = \int e^{\theta_\xi}\omega^n / \int \omega^n$ is independent of $J$. Its invariance can be interpreted as coming from a general fact on moment maps: for any $x \in M$ and $v \in \mathrm{Lie}(K^c_x)$, $\langle \mu(xg), g^{-1}v\rangle$ is invariant for $g \in K^c$, where $\mu : M \to \mathfrak{k}^*$ is a moment map.

Proposition 3.3.1 in particular shows that $\mathrm{Fut}_{\xi'}|_{\mathfrak{t}}$ is invariant under $T$-equivariant complex deformation: it only depends on the $T$-equivariant symplectic structure.

We call a symplectic Fano $T$-manifold $(M, \omega, T)$ *K-optimal* if there is a $T$-invariant $\omega$-compatible integrable complex structure $J_0$ (in this case, $(M, J_0)$ is a Fano manifold) and whose Hilbert character $(T_{\mathbb{C}}, \chi(M, J_0))$ is K-optimal. Its Hilbert character $\chi_m$, seen as a (real analytic) function $\mathfrak{t} \to \mathbb{R}$ by $\xi \mapsto \sum_i \sum_{u \in M} h^i_{X,u}(m)\langle u, \xi\rangle$, can be computed by the equivariant Hirzebruch–Riemann–Roch formula ([Mei]):

$$\chi_m(\xi) = \int_M Ch_{\mathfrak{t}}(-K_{(M,\omega)}, \xi)\, Td_{\mathfrak{t}}(M, \omega, \xi)$$

near $\xi = 0$. Here the equivariant Chern character $Ch_{\mathfrak{t}}(-K_{(M,\omega)}, \cdot)$ and Todd character $Td_{\mathfrak{t}}(M, \omega, \cdot)$ is defined as the equivariant cohomology classes of the following $T$-equivariant forms, which is independent of the choice of $J \in$

$\mathscr{J}_T(M,\omega)$:

$$Ch_{\mathfrak{t}}(-K_{(M,\omega)},\cdot) := e^{\mathrm{tr}(\frac{\sqrt{-1}}{2\pi}F_{\mathfrak{t}}(g_J,\cdot))} = e^{\omega+\langle\mu,\cdot\rangle},$$
$$Td_{\mathfrak{t}}(M,\omega,\cdot) := \det\Big(\frac{\frac{\sqrt{-1}}{2\pi}F_{\mathfrak{t}}(g_J,\cdot)}{1-e^{-\frac{\sqrt{-1}}{2\pi}F_{\mathfrak{t}}(g_J,\cdot)}}\Big),$$

where $Td_{\mathfrak{t}}(M,\omega,\xi)$ is defined near $\xi=0$. Here the equivariant curvature $F_{\mathfrak{t}}(g_J,\xi)$ is given by

$$F_{\mathfrak{t}}(g_J,\xi) := F_{g_J} + 2\pi\sqrt{-1}(L_\xi - \nabla^{g_J}_\xi).$$

Although we firstly use the integrable complex structure $J_0$ to define the Hilbert character, its Hilbert character can be computed by the $T$-equivariant characteristic classes associated to the symplectic $T$-manifold $(M,\omega,T)$, which makes sense at least near $\xi=0$ even when there is no $T$-invariant integrable complex structures. In particular, the Fano character $(T_{\mathbb{C}},\chi(M,J))$ is independent of the choice of integrable $J\in\mathscr{J}_T(M,\omega)$ (in other words, it is well-defined for $(M,\omega,T)$) and is K-optimal for every $J$ if $(M,\omega,T)$ is K-optimal (i.e. if $(T_{\mathbb{C}},\chi(M,J_0))$ is K-optimal for some $J_0$). Beware that even when $(M,\omega,T)$ is K-optimal and the action $(M,J_0)\curvearrowleft T$ is K-optimal for some $J_0$, *the action* $(M,J)\curvearrowleft T$ might be not K-optimal for other integrable complex structure $J\in\mathscr{J}_T(M,\omega)$ as the action might be not maximal among actions with K-optimal characters.

We denote by $\mathcal{S}^{\mathrm{int}}_\xi$ the restriction of the moment map $\mathcal{S}_\xi:\mathscr{J}_T(M,\omega)\to C^\infty_{T,\xi}(M,\omega)^*$ to the subspace $\mathscr{J}^{\mathrm{int}}_T(M,\omega)$, which consists of integrable complex structures.

**Proposition 3.3.4.** Assume the action of $T$ on $(M,\omega)$ is K-optimal. Then the following two statements are equivalent for any integrable $J,J'\in(\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)$.

1. There is a $T$-equivariant $C^\infty$-diffeomorphism $\phi: M\xrightarrow{\sim} M$ such that $J=\phi^*J'$.

2. $[J]=[J']\in(\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)/\mathrm{Ham}_T(M,\omega)$.

*Proof.* It follows from the uniqueness of Kähler–Ricci soliton and $\mathrm{Aut}_T(X)=\mathrm{Aut}(X,\xi)$ from the K-optimal action. □

*Remark* 3.3.5. The above proposition would hold without K-optimal assumption. To see this, it suffices to prove the following uniqueness claim.

Claim: *If $g_1, g_2$ are two $T$-invariant Kähler–Ricci solitons on a Fano $T$-manifold $X$ (in this case, we have the same soliton vectors $\xi_1 = \xi_2 \in \mathfrak{t}$), then there is an element $\phi \in \mathrm{Aut}_T^0(X)$ such that $g_2 = \phi^* g_1$.*

In general, it seems not so easy to verify the K-optimality of a given torus action on a Fano manifold, especially when the dimension of the center of its maximal reductive subgroup is greater than one. From this point, it may be better to consider non K-optimal actions for studying explicit description of the moduli space of Fano manifolds with Kähler–Ricci solitons in some special cases. Indeed, for instance, the claim holds at least for a maximal torus, as the Weyl group $N_T/T$ can be represented by the elements of any maximal compact $K$ including the maximal compact torus $T_{\mathbb{R}} \subset T$.

It follows that the quotient $(\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)/\mathrm{Ham}_T(M,\omega)$ can be identified with the set of biholomorphism classes of Fano manifolds admitting Kähler–Ricci solitons with the fixed underlying symplectic structure $(M,\omega)$, as sets. Therefore, this quotient space must be the support set of our moduli space. The quotient topology on this set is Hausdorff (cf. [FS1]). We exhibit the proof for the readers' convenience.

**Proposition 3.3.6.** The action of $\mathrm{Ham}_T(M,\omega)$ on $\mathcal{J}_T(M,\omega)$ is proper. In particular, the quotient topological space $(\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)/\mathrm{Ham}_T(M,\omega)$ is Hausdorff.

*Proof.* We must show that the map

$$a : \mathcal{J}_T(M,\omega) \times \mathrm{Ham}_T(M,\omega) \to \mathcal{J}_T(M,\omega) \times \mathcal{J}_T(M,\omega) : (J,\phi) \mapsto (J, \phi^* J)$$

is proper. Take a sequence $(J_n, \phi_n)$ so that $J_n, \phi_n^* J_n$ converge to some $J_\infty, J'_\infty \in \mathcal{J}_T(M,\omega)$ in the given order. It suffices to show that a subsequence of $\phi_n$ converges to some $\phi_\infty \in \mathrm{Ham}_T(M,\omega)$ satisfying $\phi_\infty^* J_\infty = J'_\infty$. Let $g_\infty, g'_\infty$ denote the Riemannian metrics associated to $J_\infty, J'_\infty$, respectively.

Let us take a dense countable subset $S$ of $M$. The diagonal argument shows that we have a subsequence of $\phi_n$ so that $\phi_n(x)$ converges for any $x \in S$. We continue to write $\phi_n$ for this subsequence. We obtain a distance preserving map $\phi_{S,\infty} : (S, d_{g'_\infty}|_S) \to (M, d_{g_\infty})$ by putting $\phi_{S,\infty}(x) := \lim_{n\to\infty} \phi_n(x)$. Then this map can be uniquely extended to a distance preserving map $\phi_\infty : (M, d_{g'_\infty}) \to (M, d_{g_\infty})$. Similarly we obtain a distance preserving map $\psi_\infty : (M, d_{g_\infty}) \to (M, d_{g'_\infty})$ as a limit of $\phi_n^{-1}$. It follows from [BBI-book, Theorem 1.6.14] that the distance preserving endomorphism $\phi_\infty \circ \psi_\infty$ is surjective, and we conclude $\phi_\infty$ is a continuous bijective map.

Thanks to Myers-Steenrod theorem, we see that $\phi_\infty$ is a $C^\infty$-diffeomorphism with $\phi_\infty^* g_\infty = g'_\infty$. Moreover, since $g_n = g_{n,ij} dx^i dx^j$, $\phi_n^* g_n = g_{n,pq}(d\phi_n)^p_i (d\phi_n)^q_j dx^i dx^j$ respectively converge to $g_\infty, \phi_\infty^* g_\infty$ in $C^\infty$-topology, we see the $C^\infty$-convergence of the coefficients $(d\phi_n)^p_i (d\phi_n)^q_j$ to $(d\phi_\infty)^p_i (d\phi_\infty)^q_j$ with respect to a fixed $C^\infty$-coordinate. In particular, we have

$$(\partial_k (d\phi_n)^p_i)(d\phi_n)^p_i \xrightarrow{C^\infty} (\partial_k (d\phi_\infty)^p_i)(d\phi_\infty)^p_i \tag{3.14}$$

and $(d\phi_n)^p_i = \sqrt{((d\phi_n)^p_i)^2}$ convereges to $(d\phi_\infty)^p_i$ in $C^0$-topology. It follows from (3.14) that the $C^k$-convergence of $d\phi_n$ induces the $C^{k+1}$-convergence of them. This shows that $\phi_n$ converges to $\phi_\infty \in \mathrm{Ham}_T(M, \omega)$ in the $C^\infty$-topology and $\phi_\infty^* J_\infty = J'_\infty$. $\square$

### 3.3.2 Local slice

The materials in this subsection are parallel to [Szé2], where the cscK case is treated.

Let $X$ be a Fano $T$-manifold with a Kähler–Ricci soliton $(g, \xi')$, $\phi : (M, J_0) \xrightarrow{\sim} X$ be a biholomorphism, where $M$ is a $C^\infty$-manifold and $J_0$ is a complex structure on $M$. Put $\omega := (\phi^* g)(J_0 \cdot, \cdot)$, $K := \{h \in \mathrm{Ham}(M, \omega) \mid h^* J_0 = J_0\}$, which is a compact Lie group, and $\mathfrak{k} := \{f \in C^\infty_T(M) \mid L_{X_f} J_0 = 0, \int_M f e^{\theta_\xi} \omega^n = 0\}$, which can be identified with the Lie algebra of $K$. Consider the following $L^2_k$-completion of the moment map in the last subsection

$$\mathcal{S}_\xi : \mathcal{J}_T(M, \omega)^2_k \to L^2_{k-2,T}(M, \omega)^\vee.$$

We denote by $\Theta$ the holomorphic tangent sheaf of $X$ and by $H^i_T(X, \Theta)$ the $T$-invariant subspace of the $i$-th cohomology $H^i(X, \Theta)$. Note that we have $H^i(X, \Theta) = 0$ for every $i \geq 2$ and a smooth Fano manifold $X$, thanks to Serre duality and Kodaira vanishing.

**Proposition 3.3.7.** There are an open ball $B \subset H^1_T(X, \Theta)$ centered at the origin, a $K$-equivariant holomorphic deformation $\varpi : \mathcal{X} \to B$ of $X$ with a holomorphic morphism $\iota : X \hookrightarrow \mathcal{X}_0$ inducing a biholomorphism to the central fiber, a $K$-equivariant $C^\infty$-smooth map $\mathfrak{J} : B \to \mathcal{J}_T(M, \omega)^2_k$ and a $T_\mathbb{R}$-equivariant $L^2_k$-regular diffeomorphism $\Phi : B \times M \xrightarrow{\sim} \mathcal{X}$ with the following properties.

1. The holomorphic family $X \xhookrightarrow{\iota} \mathcal{X} \xrightarrow{\varpi} (B, 0)$ is a semi-universal family of $X$.

2. For each $b \in B$, $\mathfrak{J}(b)$ is an $L^2_k$-regular integrable complex structure satisfying $s_\xi(\mathfrak{J}(b)) \in \mathfrak{k}$ and $\mathfrak{J}(0) = J_0$.

3. The diffeomorphism $\Phi$ satisfies $\varpi \circ \Phi^{-1} = p_B$ and $\Phi(0, \cdot) = \phi$, where $p_B : B \times M \to B$ is the projection. The restricted map $\Phi(b, \cdot) : (M, \mathfrak{J}(b)) \to \mathcal{X}_b$ is a biholomorphism for each $b \in B$.

*Proof.* Let $\varpi : \mathcal{X} \to B$ be the Kuranishi family of $T$-equivariant deformation of $X$ (see [Kur1, Kur2, Dou1] for its construction). From its construction, we have a holomorphic $K$-action on $\mathcal{X}$ and $B$ so that $\varpi$ is $K$-equivariant and a holomorphic map $\mu : B \to \mathcal{J}_T(M)^2_k$ whose image $\mu(b)$ is a real analytic (with respect to the real analytic structure on $X$) integrable complex structure for each $b \in B$ with a biholomorphism $\mathcal{X}_b \cong (M, \mu(b))$. As $-\ell K_{\mathcal{X}/B}$ is relatively very ample for large $\ell \in \mathbb{N}$, all the higher direct images of $\mathcal{O}(-\ell K_{\mathcal{X}/B})$ vanishes and thus $\varpi_* \mathcal{O}(-\ell K_{\mathcal{X}/B})$ is a $K$-equivariant vector bundle on $B$. Taking smaller $B$ and using a $K$-equivariant isomorphism $\varpi_* \mathcal{O}(-\ell K_{\mathcal{X}/B}) \cong \underline{H^0(X, \mathcal{O}(-\ell K_{\mathcal{X}/B}))}_{B,K} = B \times H^0(X, \mathcal{O}(-\ell K_{\mathcal{X}/B}))$ of vector bundles on $B$ (see Lemma 3.4.6 for the $K$-action), we can embed these Fano manifolds into a uniform projective space $\mathbb{C}P^N = \mathbb{P}(H^0(X, \mathcal{O}(-\ell K_{\mathcal{X}/B}))^\vee)$ so that $\mathcal{X}_{s.g} = \mathcal{X}_s.g$, where in the latter we consider the $K$-action on $\mathbb{C}P^N$ induced from the action on $H^0(X, \mathcal{O}(-\ell K_{\mathcal{X}/B}))$. Pulling back the Fubini-Study metric, we obtain a $K$-equivariant smooth family of Kähler metrics $\{\omega_b\}_{b \in B}$, where each $\omega_b$ can be identified with a Kähler metric on $(M, \mu(b))$. Taking smaller $B$ again, we can assume that closed forms $\omega_{b,t} := \omega_0 + t(\omega_b - \omega_0)$ are non-degenerate for each $b \in B$ and $t \in [0, 1]$. Then we can find a $K$-equivariant family of diffeomorphisms $\{f_b\}_{b \in B}$ so that $f_b^* \omega_b = \omega_0$. Putting $\mathfrak{J}'(b) := f_b^* \mu(b)$, we obtain a $K$-equivariant smooth map $\mathfrak{J}' : B \to \mathcal{J}_T(M, \omega)^2_k$, whose image $\mathfrak{J}'(b)$ is a smooth complex structure for each $b \in B$.

It suffices to show that we can find an equivariant perturbation $\mathfrak{J}$ of $\mathfrak{J}'$ so that $\mathfrak{J}(b) = g_b^* \mathfrak{J}'(b)$ for each $b$ and $s_\xi(\mathfrak{J}(b)) \in \mathfrak{k}$. Let $U^2_{k+2} \subset L^2_{T,k+2}(M, \omega)$ be a small ball of the origin. For each $\phi \in U^2_{k+2}$ and an almost complex structure $J \in \mathcal{J}_T(M, \omega)$, we can find an $L^2_k$-regular vector field $X^{\phi,J}_t$ on $M$ so that $i(X^{\phi,J}_t)(\omega_0 - t dJd\phi) = -Jd\phi$. This vector field is actually $L^2_{k+1}$-regular. In fact, it is sufficient to show that $[X_f, X^{\phi,J}_t]$ is $L^2_k$-regular for any smooth function $f$. For any smooth vector field $Z$, we have

$$\begin{aligned} i([X_f, X^{\phi,J}_t])\omega_t(Z) &= -(L_{X_f}\omega)(X_{\phi,J,t}, Z) + X_f(\omega(X^{\phi,J}_t, Z)) - \omega(X^{\phi,J}_t, [X_f, Z]) \\ &= X_f(-Jd\phi(Z)) + Jd\phi([X_f, Z]) \in L^2_k. \end{aligned}$$

Thus $[X_f, X_t^{\phi,J}]$ is $L^2_k$, as we expected. The flow $f_t^{\phi,J}$ of this time-dependent vector fields is $L^2_{k+1}$-regular and satisfy $(f_t^{\phi,J})^*(\omega_0 - tdJd\phi) = \omega_0$. To see the regularity, it is sufficient to show that $(f_t^{\phi,J})_*Y$ is a $L^2_k$-regular vector field for each smooth vector field $Y$ on $M$. Note that $(d/dt)(f_t^{\phi,J})_*Y = [X_t^{\phi,J}, Y]$ is $L^2_k$-regular and $(f_t^{\phi,J})_*Y$ can be written as $\int_0^t [X_s^{\phi,J}, Y]ds$. Then for each $l \le k$, we obtain the following estimate, so $f_t^{\phi,J}$ is $L^2_{k+1}$-regular.

$$\begin{aligned}
\int_M |\nabla^l (f_t^{\phi,J})_*Y|^2 \omega^n &= \int_M \left| \int_0^t \nabla^l [X_s^{\phi,J}, Y] ds \right|^2 \omega^n \\
&\le \int_M t \int_0^t |\nabla^l [X_s^{\phi,J}, Y]|^2 ds\ \omega^n \\
&\le t \int_0^t \|[X_s^{\phi,J}, Y]\|^2_{L^2_k} ds < \infty.
\end{aligned}$$

It follows that $(f_1^{\phi,J})^* J \in \mathcal{J}_T(M,\omega)^2_k$. Consider the orthogonal decomposition $L^2 = \mathfrak{k} \oplus \mathfrak{k}_\perp$ with respect to $L^2$-norm $(\cdot,\cdot)_\xi$. Put $\mathfrak{k}^2_{k,\perp} := L^2_k \cap \mathfrak{k}_\perp$ and let $\Pi_\perp : L^2_{k-2} \to \mathfrak{k}^2_{k-2,\perp}$ be the $L^2$-projection. Note that

$$(D(\phi \mapsto (f_1^{\phi,J_0})^* J_0))_0(\psi) = \frac{d}{dt} f_t^* J_0 = J_0 P(\psi),$$

where $P$ denotes the linear differential operator $P : L^2_{T,k+2} \to T_J \mathcal{J}_T(M,\omega)^2_k : \psi \mapsto L_{X_\psi} J_0$, and $(Ds_\xi)_J(A) = P^* JA$, where $P^*$ is the formal adjoint of $P$ with respect to the norm $(-,-)_\xi$. It follows that

$$G : B \times U \to \mathfrak{k}^2_{k-2,\perp} : (b,\phi) \mapsto \Pi_\perp s_\xi((f_1^{\phi,\mathfrak{J}'(b)})^* \mathfrak{J}'(b))$$

is a $K$-equivariant smooth map with the derivative

$$DG_{(0,0)}(0,\psi) = -P^*P(\psi).$$

Since $P^*P$ is a self-adjoint fourth order elliptic differential operator, it gives the isomorphism $P^*P : \mathfrak{k}^2_{k+2,\perp} \to \mathfrak{k}^2_{k-2,\perp}$. Applying the implicit function theorem, we can find a new $K$-equivariant smooth map $\mathfrak{J} : B \to \mathcal{J}_T(M,\omega)^2_k$ so that $\Pi_\perp s_\xi(\mathfrak{J}(b)) = 0$, hence $s_\xi(\mathfrak{J}(b)) \in \mathfrak{k}$, taking smaller $B$ if necessary. $\square$

Pulling back the symplectic structure $\Omega_\xi$ on $\mathcal{J}_T(M,\omega)^2_k$ by the $K$-equivariant smooth map $\mathfrak{J} : B \to \mathcal{J}_T(M,\omega)^2_k$, we obtain a $K$-equivariant smooth symplectic structure (by taking smaller $B$ if necessary), which we denote by the same notation. Then $\nu : B \to \mathfrak{k}^* : b \mapsto \mathcal{S}_\xi(\mathfrak{J}(b))$ is a moment map with respect to this symplectic structure.

**Proposition 3.3.8** (See Postscript Remark below.). If $B \subset H^1_T(X,\Theta)$ in the Proposition 3.3.7 is sufficiently small, then the following two statements are equivalent for any $b \in B$.

1. The fiber $\mathcal{X}_b$ of the family $\varpi : \mathcal{X} \to B$ has a Kähler–Ricci soliton.
2. The orbit $b \cdot \mathrm{Aut}_T(X) \subset H^1_T(X,\Theta)$ is closed. That is, $b$ is polystable with respect to the $\mathrm{Aut}_T(X)$-action.

*Remark* 3.3.9. While discussing with R. Dervan and P. Naumann, the author realized that there was a gap in the following proof with regards to the implication "the existence of Kähler–Ricci soliton $\Rightarrow$ GIT-polystability". To be precise, what we prove here is that the following are equivalent for $b \in B$:

1. $\nu^{-1}(0) \cap B \cap b \cdot G \neq \emptyset$.
2. $\nu_0^{-1}(0) \cap B \cap b \cdot G \neq \emptyset$.
3. The point $b \in H^1_T(X,\Theta)$ is polystable with respect the $\mathrm{Aut}_T(X)$-action.

Of course, this (1) implies the existence of Kähler–Ricci soliton on $\mathcal{X}_b$. On the other hand, however, the existence of Kähler–Ricci soliton on $\mathcal{X}_b$ only implies that there is a unique orbit $B \cap b_0 \cdot G$ in the closure $B \cap \overline{b \cdot G}$ such that $\nu^{-1}(0) \cap B \cap b_0 \cdot G \neq \emptyset$ and $\mathcal{X}_{b'}$ is isomorphic to $\mathcal{X}_b$ for any $b' \in B \cap b_0 \cdot G$ (thanks to K-polystability). This in particular implies that $\nu^{-1}(0)/K \approx BK^c /\!\!/ K^c$ (Corollary 3.3.15 below) can be naturally identified with the isomorphism classes of Fano manifolds admitting Kähler–Ricci solitons who appear in the family $\varpi : \mathcal{X} \to B$ (thanks to Corollary 3.3.14 below).

The author emphasizes that we do not use the original statement of Proposition 3.3.8 to prove all the statements in the rest of this chapter and we only use the equivalence stated in this remark. The original proof is still fine to show this equivalence.

*Remark* 3.3.10. After the publication of [Ino1], Yue Fan kindly informed me that the following proof has a trouble. We propose an alternative proof in chapter 4, which makes use of Theorem G. See the explanation between Theorem 4.1.5 and Proposition 4.1.6.

*Proof of Proposition 3.3.8.* Let $\Omega_0$ be the linearization of $\Omega_\xi$ at $0 \in B$, i.e., $\Omega_0 = (d_0\mathfrak{J}\cdot, J_0 d_0\mathfrak{J}\cdot)_\xi$ under the identification $T_bB = \mathbb{H}^1_T = T_0B$. Consider the map $\nu_0 : \mathbb{H}^1_T \to \mathfrak{k}^*$ defined by

$$\langle f, \nu_0(b)\rangle = \Omega_0(L_{X_f} b, b) = (L_{X_f} d\mathfrak{J}_0 b, J_0 d\mathfrak{J}_0 b)_\xi.$$

Then $\nu_0$ is a moment map with respect to the symplectic structure $\Omega_0$. The Kempf-Ness theorem says that $b \in \mathbb{H}^1_T$ is polystable with respect to $K^c = \mathrm{Aut}_T(X)$ if and only if $bK^c \cap \nu_0^{-1}(0) \neq \emptyset$.

Since

$$\begin{aligned}\frac{d^2}{dt^2}\Big|_{t=0}\langle f, \nu(tb)\rangle &= \frac{d^2}{dt^2}\Big|_{t=0}(f, s_\xi(\mathfrak{J}(tb)))_\xi \\ &= \frac{d}{dt}\Big|_{t=0}(L_{X_f}\mathfrak{J}(tb), \mathfrak{J}(tb)\dot{\mathfrak{J}}(tb))_\xi \\ &= \langle f, \nu_0(b)\rangle,\end{aligned}$$

the moment map $\nu : B \to \mathfrak{k}^*$ can be expanded as

$$\nu(tb) = \nu(0) + td_0\nu(b) + t^2\nu_0(b)/2 + O(t^3).$$

Since $0 \in B$ corresponds to Fano manifolds with Kähler–Ricci soliton $(M, J_0, \omega)$, $\nu(0) = \mathcal{S}_\xi(J_0) = 0$ from Proposition 3.3.2. Moreover, since 0 is a fixed point of the $K$-action, we have $d_0\nu = 0$. Therefore we get

$$\nu(tb) = t^2\nu_0(b)/2 + O(t^3).$$

Since the action of $K$ on $\mathbb{H}^1_T$ is linear, the stabilizer group $K_b \subset K$ of $b$ satisfies $K_{tb} = K_b$. So we have

$$\frac{d}{dt}\langle f, \nu(tb)\rangle = \Omega_{tb}(b, \sigma_{tb}(f)) = 0$$

for any $f \in \mathfrak{k}_b$, where $\sigma_b : \mathfrak{k} \to T_xB$ is the differential of the action. Then it follows that $\nu(b) \in \mathfrak{k}_b^\perp$ and $\nu_0(b) \in \mathfrak{k}_b^\perp$.

Now we cite the following general lemma from [Szé2, Proposition 9] and [Don3, Proposition 17. ].

**Lemma 3.3.11.** Let $(B, \Omega)$ be a symplectic manifolds with a $K$-action, $\nu : B \to \mathfrak{k}$ be a moment map with respect to the $K$-aciton ($\mathfrak{k}$ is endowed with a inner product). Suppose $b \in B$ satisfies $\nu(b) \in \mathfrak{k}_b^\perp$ and $\lambda, \delta > 0$ with $\lambda\|\nu(b)\| < \delta$ satisfies $\|(\sigma^*_{e^{iv}b}\sigma_{e^{iv}b})^{-1}\| \leq \lambda$ for any $v \in \mathfrak{k}$ with $\|v\| < \delta$. Then there is $v_b \in \mathfrak{k}$ such that $\nu(e^{iv_b}b) = 0$ and $\|v_b\| \leq \lambda\|\nu(b)\|$.

Fix a small $\delta > 0$ so that there is $C > 0$ such that for any $v \in \mathfrak{k}$ with $\|v\| < \delta$ and any $f \in \mathfrak{k}^\perp_{e^{iv}b}$

$$\|\sigma_{e^{iv}b}(f)\|^2_{\Omega_0} \geq C\|f\|^2$$

holds. Take smaller $B$ so that $\Omega_\xi \geq \frac{1}{2}\Omega_0$. Since $\sigma_{tx}(f) = t\sigma_x(f)$ and

$$(\sigma_{tx}^*\sigma_{tx}(f), f)_\xi = \|\sigma_{tx}(f)\|_\Omega^2 \geq \frac{1}{2}Ct^2\|f\|^2,$$

we obtain $\|(\sigma_{tx}^*\sigma_{tx})^{-1}\| \leq C't^{-2}$. Replacing $\Omega$ with $\Omega_0$, we obtain the similar estimate for the adjoint of $\sigma$ with respect to $\Omega_0$.

Suppose $b \in B$ is polystable. Then there exists a point $b' \in bK^c \cap \nu_0^{-1}(0)$. In regards of the linear symplectic form, $b'$ is given by minimizing the norm of $b'$ in the $K^c$-orbit of $b$, so $b'$ is also in $B$. Since the points in the same $K^c$-orbit give the isomorphic complex structures, we can assume $\nu_0(b) = 0$. It follows that $\nu(tb) = O(t^3)$. Then we can take $t$ small so that $C't^{-2}\|\nu(tb)\| < \delta$. Applying the above lemma, we find a point $tb' \in B$ in the $K^c$-orbit of $tb$ satisfying $\nu(tb') = 0$. It follows that $(M, \mathfrak{J}(tb)) \cong (M, \mathfrak{J}(tb'))$ admits Kähler–Ricci soliton. Note the polystability of $b$ and $tb$ is equivalent as we consider a linear action.

Conversely, suppose $(M, \mathfrak{J}(b))$ admits Kähler–Ricci soliton. Then similarly we can show that there is a point $b' \in bK^c$ satisfying $\nu_0(b') = 0$. This shows $b$ is polystable. $\square$

The following corollary exhibits one of good features of our $T$-equivariant formulation. We use this to show the Artinianity of our moduli stack in the next section.

**Corollary 3.3.12.** Any $T$-equivariant small deformation of Fano $T$-manifold with Kähler–Ricci soliton is gentle. In particular, for any $T$-equivariant family $\mathcal{M} \to S$ of complex manifolds, the following subset

$$S^\circ := \{ s \in S \mid \mathcal{M}_s \text{ is a gentle Fano manifold } \}$$

is an open subset of $S$ (with respect to the real topology).

*Proof.* Suppose the Fano manifold $(M, \mathfrak{J}(b))$ does not admit Kähler–Ricci soliton for the point $b \in B$. From the above proposition, $b \in B$ is not polystable. Then we can find a polystable point $b_0 \in B$ in the closure of the orbit $bK^c$ by minimizing the norm $\Omega_0(-, J_0-)$. Since $K^c$ is reductive, we can find a regular morphism $\lambda : \mathbb{C}^* \to \mathbb{H}^1_T$ so that $\lambda(t) \to b_0$. We can extend this to a regular morphism $\tilde{\lambda} : \mathbb{C} \to \mathbb{H}^1_T$. Pulling back the family $\varpi : \mathcal{X} \to B$, we obtain a $T$-equivariant holomorphic family $\mathcal{M} \to \Delta$ whose central fiber $(M, \mathfrak{J}(b_0))$ has Kähler–Ricci soliton because there is some $b_0' \in b_0K^c$ such that

$\nu(b_0) = \mathcal{S}_\xi(\mathfrak{J}(b_0')) = 0$. So $\mathcal{M} \to \Delta$ gives a gentle degeneration of $\mathcal{X}_b$, hence $\mathcal{X}_b$ is gentle. Since the family $\varpi : \mathcal{X} \to B$ parametrizes all isomorphism classes of complex structures near $\mathcal{X}_b$ for any $b \in B$, we have shown the assertion. $\square$

### 3.3.3 Completion

The topological space $\mathrm{Ham}_T(M,\omega)^2_{k+1}$ of $L^2_{k+1}$-regular symplectic diffeomorphisms admits a natural Banach smooth manifold structure (cf. [IKT, KM]). The compositions and the inverses of morphisms in $\mathrm{Ham}_T(M,\omega)^2_{k+1}$ are again in $\mathrm{Ham}_T(M,\omega)^2_{k+1}$. However, the following maps

$$\mathrm{Ham}_T(M,\omega)^2_{k+1} \times \mathrm{Ham}_T(M,\omega)^2_{k+1} \to \mathrm{Ham}_T(M,\omega)^2_{k+1} : (\phi,\psi) \mapsto \phi \circ \psi$$
$$\mathrm{Ham}_T(M,\omega)^2_{k+1} \to \mathrm{Ham}_T(M,\omega)^2_{k+1} : \phi \mapsto \phi^{-1}$$

are *not* differentiable with respect to the Banach smooth manifold structure, but are just continuous (see [IKT]). Therefore we can not treat $\mathrm{Ham}_T(M,\omega)^2_{k+1}$ as a Banach Lie group.

Nevertheless, we can consider the following $C^1$-smooth map

$$\mathcal{H} : B \times^K \mathrm{Ham}_T(M,\omega)^2_{k+1} \to \mathcal{J}_T(M,\omega)^2_k : [b,\phi] \mapsto \phi^*\mathfrak{J}(b)$$

by working with a slightly regular target of $\mathfrak{J}$ in Proposition 3.3.7, say, by working with $\mathfrak{J} : B \to \mathcal{J}_T(M,\omega)^2_{k+2}$. Note, first of all, the quotient $B \times^K \mathrm{Ham}_T(M,\omega)^2_{k+1} := B \times \mathrm{Ham}_T(M,\omega)^2_{k+1}/K$ is endowed with a unique Banach smooth manifold structure whose quotient map is a submersion, as the finite dimensional compact Lie group $K$ acts freely on $B \times \mathrm{Ham}_T(M,\omega)^2_{k+1}$. The $C^1$-smoothness of $\mathcal{H}$ follows from the $C^\infty$-smoothness of $\mathfrak{J} : B \to \mathcal{J}_T(M,\omega)^2_{k+2}$ and the $C^1$-smoothness of

$$\begin{array}{ccc} \mathcal{J}_T(M,\omega)^2_{k+2} \times \mathrm{Ham}_T(M,\omega)^2_{k+1} & \to & \mathcal{J}_T(M,\omega)^2_k \\ (J \quad , \quad \phi) & \mapsto & \phi^* J, \end{array}$$

which follows from the main theorem of [IKT].

As Proposition 3.3.6, the map

$$\begin{array}{cccc} a^2_k : & \mathcal{J}_T(M,\omega)^2_k \times \mathrm{Ham}_T(M,\omega)^2_{k+1} & \to & \mathcal{J}_T(M,\omega)^2_k \times \mathcal{J}_T(M,\omega)^2_k \\ & (J \quad , \quad \phi) & \longmapsto & (J \quad , \quad \phi^* J), \end{array}$$

is proper for any large $k$ ($L^2_k \subset C^2$ is sufficient). To see this, take a sequence $(J_n, \phi_n) \in \mathcal{J}_T(M,\omega)^2_k \times \mathrm{Ham}_T(M,\omega)^2_{k+1}$ so that $g_n, \phi_n^* g_n$ converge to $g_\infty, g'_\infty$ in $L^2_k$-topology. Construct $\phi_\infty$ as in the proof of Proposition 3.3.6. Again, thanks to Myers-Steenrod theorem, $\phi_\infty$ is $C^2$-smooth and satisfies $\phi_\infty^* g_\infty = g'_\infty$. Then $\phi_\infty$ is a harmonic map between $(M, g_\infty)$ and $(M, g'_\infty)$. Hence it satisfies the elliptic equation

$$\Delta_{g'_\infty} \phi^\alpha_\infty - \Gamma^\alpha_{\beta\gamma} \frac{\partial \phi^\beta_\infty}{\partial x^i} \frac{\partial \phi^\gamma_\infty}{\partial x^j} {g'}^{ij}_\infty = 0,$$

where the coefficients of the Laplacian $\Delta_{g'_\infty}$ and the Levi-Civita connection $\Gamma^\alpha_{\beta\gamma}$ are $L^2_{k-1}$-regular. It follows that $\phi_\infty$ is $L^2_{k+1}$-regular.

Let us see that $\phi_n$ converges to $\phi_\infty$ in $L^2_{k+1}$-topology. Since $g_n \to g_\infty$ and $g'_n := \phi_n^* g_n \to g'_\infty$ in $L^2_k$-topology, we have $\Gamma^\alpha_{\beta\gamma,n} \to \Gamma^\alpha_{\beta\gamma}$ and $\Delta_{g'_n} \to \Delta_{g'_\infty}$ in $L^2_{k-1}$-topology. Now we use the following uniform elliptic estimates for the elliptic operators $\Delta_{g'_n}$ ($n = 1, 2, \ldots, \infty$) with $L^2_{k-1}$-bounded coefficients and $0 \le \ell \le k-1$.

$$\|u\|_{L^2_{\ell+2}(g_0)} \le C_{k-1}(\|\Delta_{g'_n} u\|_{L^2_\ell(g_0)} + \|u\|_{L^2_\ell(g_0)}),$$

where $C_{k-1}$ is independent of $n = 1, 2, \ldots, \infty$ and $g_0$ is a fixed reference smooth metric. (Note $L^2_{k-1} \subset C^1$. We used this to the above uniform elliptic estimates. See for instance the proof of the elliptic estimates in the Appendix of [Kod-book]. Note also Sobolev multiplication works. ) First, the $C^1$-convergence of $\phi_n \to \phi_\infty$ follows by the same argument as before. Then we know that $\Delta_{g'_n} \phi^\alpha_\infty = \Gamma^\alpha_{\beta\gamma,n} \partial_i \phi^\beta_n \partial_j \phi^\gamma_n g^{ij}_n$ converges to $\Delta_{g'_\infty} \phi^\alpha_\infty = \Gamma^\alpha_{\beta\gamma} \partial_i \phi^\beta_\infty \partial_j \phi^\gamma_\infty g^{ij}_\infty$ in $L^2$-topology (actually in $C^0$-topology). Combined with the $L^2_{k-1}$-convergence of $\Delta_{g'_n} \phi^\alpha_\infty \to \Delta_{g'_\infty} \phi^\alpha_\infty$, we obtain $\|\Delta_{g'_n}(\phi^\alpha_n - \phi^\alpha_\infty)\|_{L^2(g_0)} \to 0$. It follows from the above uniform elliptic estimate that

$$\|\phi^\alpha_n - \phi^\alpha_\infty\|_{L^2_2(g_0)} \le C_{k-1}(\|\Delta_{g'_n}(\phi^\alpha_n - \phi^\alpha_\infty)\|_{L^2(g_0)} + \|\phi^\alpha_n - \phi^\alpha_\infty\|_{L^2(g_0)}) \to 0,$$

and we obtain $\phi_n \to \phi_\infty$ in $L^2_2$-topology. We can repeat this process until we conclude the $L^2_{k+1}$-convergence of $\phi_n \to \phi_\infty$.

Now we can prove the following.

**Proposition 3.3.13.** The $C^1$-smooth map

$$\mathcal{H} : B \times^K \mathrm{Ham}_T(M,\omega)^2_{k+1} \to \mathcal{J}_T(M,\omega)^2_k$$

is injective for any sufficiently small neighbourhood $B \subset H^1_T(X, \Theta)$ of the origin.

*Proof.* The derivative of $\mathcal{H}$ at $[0, \mathrm{id}]$ is given by

$$\mathbb{H}^1_T \times L^2_{T,k+2}(M)_0/\mathfrak{k} \to \Omega^{0,1}_T(T^{1,0})^2_k : (\rho, f) \mapsto d\mathfrak{J}_0(\rho) + \bar{\partial} X'_f.$$

It is easy to see that this map is injective and has a closed split range. Then the implicit function theorem shows that $\mathcal{H}$ gives an immersion in a neighbourhood of $[0, \mathrm{id}]$. In particular, $\mathcal{H}$ is locally injective at $[0, \mathrm{id}]$.

Suppose $\mathcal{H}$ is not (globally) injective for any sufficiently small $B$. Then we can take sequences $b_n, b'_n \to 0 \in B$ and $\phi_n, \phi'_n \in \mathrm{Ham}_T(M, \omega)^2_{k+1}$ satisfying

$$[b_n, \phi_n] \neq [b'_n, \phi'_n] \quad \text{and} \quad \mathcal{H}([b_n, \phi_n]) = \mathcal{H}([b'_n, \phi'_n]).$$

In particular, we have $\mathfrak{J}(b_n) = (\phi'_n \circ \phi_n^{-1})^* \mathfrak{J}(b'_n)$ and both $\mathfrak{J}(b_n), \mathfrak{J}(b'_n)$ converge to $\mathfrak{J}(0)$ in $\mathscr{J}_T(M, \omega)^2_k$. From the properness of $a^2_k$, we have a subsequence of $\phi'_n \circ \phi_n^{-1}$ which converges to some $\phi_\infty$ in the stabilizer $K$ of $\mathfrak{J}(0)$. Hence, after taking a subsequence, both $[b_n, \mathrm{id}]$ and $[b'_n, \phi'_n \circ \phi_n^{-1}]$ converge to the same $[0, \mathrm{id}] = [0, \phi_\infty]$ with the same images $\mathcal{H}([b_n, \mathrm{id}]) = \mathcal{H}([b'_n, \phi'_n \circ \phi_n^{-1}])$. Since $\mathcal{H}$ is injective near $[0, \mathrm{id}]$, we conclude $[b_n, \mathrm{id}] = [b'_n, \phi'_n \circ \phi_n^{-1}]$ for sufficiently large $n$. This contradicts to the choice of the sequences $[b_n, \phi_n] \neq [b'_n, \phi'_n]$ and we have shown that $\mathcal{H}$ is injective for some (hence any) sufficiently small $B$. $\square$

The restriction of the map $\mathfrak{J} : B \to \mathscr{J}_T(M, \omega)^2_k$ gives a continuous map $\nu^{-1}(0) \to (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k$ and induces another continuous map

$$\nu^{-1}(0)/K \to (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k / \mathrm{Ham}_T(M, \omega)^2_{k+1}.$$

The following corollaries are essential in the proof of the main theorem.

**Corollary 3.3.14.** The induced map $\nu^{-1}(0)/K \to (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k / \mathrm{Ham}_T(M, \omega)^2_{k+1}$ is a homeomorphism onto an open subset.

*Proof.* The injectivity follows from the above Proposition. From the Proposition in section 2 of [Kur2], there is a point $b \in B$ such that $(M, \mathfrak{J}(b)) \cong (M, J)$ for any integrable $L^2_k$-regular $J$ sufficiently close to $J_0$ in $L^2_k$-topology. (Here we can work with $L^2_{l=k}$ rather than $L^2_{l=k+2}$ by taking smaller $B$ if necessary, thanks to the uniqueness of Kuranishi family independent of its construction. ) Furthermore, if $J \in (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k$ we have $(M, \mathfrak{J}(b')) \cong (M, \mathfrak{J}(b))$ for any $b' \in bK^c$ and $bK^c \cap \nu^{-1}(0) \neq \emptyset$, so we can take such $b$ from $\nu^{-1}(0) \subset B$ for any $J \in (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k$. Therefore the image of $\nu^{-1}(0)/K$

covers an open neighbourhood of $J_0 \in (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k/\mathrm{Ham}_T(M,\omega)^2_{k+1}$. Since $(\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k/\mathrm{Ham}_T(M,\omega)^2_{k+1}$ is a Hausdorff space, it follows that the map $\nu^{-1}(0)/K \to (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k/\mathrm{Ham}_T(M,\omega)^2_{k+1}$ becomes a homeomorphism onto an open subset, by taking smaller $B$ if necessary. $\square$

**Corollary 3.3.15.** Suppose the torus action on $(M,\omega)$ is K-optimal. The inclusion map $\nu^{-1}(0) \hookrightarrow BK^c$ induces a homeomorphism $\nu^{-1}(0)/K \to BK^c /\!\!/ K^c$.

*Proof.* The analytic GIT quotient $BK^c /\!\!/ K^c$ is identified as a topological space with the quotient space of $BK^c$ by the equivalence relation $b \sim b' \iff \overline{bK^c} \cap \overline{b'K^c} \neq \emptyset$. Take elements $b, b' \in \nu^{-1}(0)$ so that $b \sim b'$. Since both $b, b'$ are polystable with respect to the $K^c$-action, their $K^c$-orbits are closed and hence it follows that $bK^c = b'K^c$. As mentioned before, we obtain $(M, \mathfrak{J}(b)) \cong (M, \mathfrak{J}(b'))$. Then it follows from Proposition 3.3.4 that we get $[\mathfrak{J}(b)] = [\mathfrak{J}(b')] \in (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k/\mathrm{Ham}_T(M,\omega)^2_{k+1}$. From the above corollary, we obtain $[b] = [b'] \in \nu^{-1}(0)/K$ and we have shown the map $\nu^{-1}(0)/K \to BK^c /\!\!/ K^c$ is injective.

We know that $BK^c /\!\!/ K^c$ consists of closed $K^c$-orbits and closed $K^c$-orbit has non-empty intersection with $\nu^{-1}(0)$. This shows the surjectivity of $\nu^{-1}(0)/K \to BK^c /\!\!/ K^c$. Since both the spaces are locally compact Hausdorff, by taking smaller $B$ if necessary, the map becomes a homeomorphism. $\square$

**Corollary 3.3.16.** For any $b \in \nu^{-1}(0)$, $\mathrm{Aut}_T(\mathcal{X}_b)$ can be identified with the complexification of the stabilizer group $K_b$ of the action of $K$.

*Proof.* From the proof of Theorem 3.2.7, we know that $\mathrm{Aut}_T(\mathcal{X}_b) \cong \mathrm{Aut}_T(M, \mathfrak{J}(b))$ is the complexification of the compact group $\mathrm{Stab}(\mathfrak{J}(b)) \subset \mathrm{Ham}_T(M,\omega)^2_{k+1}$. Since $\mathfrak{J} : B \to \mathcal{J}_T(M,\omega)^2_{k+2}$ is $K \subset \mathrm{Ham}_T(M,\omega)^2_{k+1}$-equivariant, there is an inclusion $K_b \subset \mathrm{Stab}(\mathfrak{J}(b))$. For $\phi \in \mathrm{Stab}(\mathfrak{J}(b))$, we have $\mathcal{H}([b,\phi]) = \phi^*\mathfrak{J}(b) = \mathfrak{J}(b) = \mathcal{H}([b, \mathrm{id}])$, then the injectivity of $\mathcal{H}$ shows that $[b,\phi] = [b, \mathrm{id}]$, hence $\phi \in K_b$. $\square$

*Remark* 3.3.17. Corollary 3.3.16 enables us to prove Corollary 3.5.12 under the uniqueness statement of (2) in Conjecture 3.5.11, which we do not follow in this chapter. So the last corollary will be never used in any proofs of this paper. Recently, R. Dervan and P. Naumann find an another pure analytic approach to construct the moduli space of cscK manifolds that makes use of this corollary.

In the next section, we will construct complex structures on the following Hausdorff topological spaces.

**Definition 3.3.18.** We set

$$L^2_k K(M,\omega,T) := (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k/\mathrm{Ham}_T(M,\omega)^2_{k+1},$$
$$K(M,\omega,T) := (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)/\mathrm{Ham}_T(M,\omega)$$

and

$$K_{T,\chi} := \coprod_{\chi(M,\omega,T)=\chi} K(M,\omega,T)$$

for a Fano character $\chi = \{\chi_m \in \mathbb{Z}[M]\}_{m\in\mathbb{Z}}$ where $(M,\omega,T)$ runs K-optimal symplectic Fano $T$-manifolds whose Hilbert character $\chi(M,\omega,T)$ (see the description before Proposition 3.3.4) is equal to the given Fano character $\chi$.

Note that the homeomorphism $f^* : K(M',\omega',T) \xrightarrow{\sim} K(M,\omega,T)$ induced by a $T$-equivariant symplectic diffeomorphism $f : (M,\omega) \xrightarrow{\sim} (M',\omega')$ is independent of the choice of $f$, so the space $K_{T,\chi}$ is free from the choice of the representatives $(M,\omega,T)$ in the symplectic diffeomorphism class $[M,\omega,T]$.

## 3.4 Gluing of local charts

### 3.4.1 The moduli stack $\boldsymbol{\mathcal{K}}_{T,\chi}$

In this subsection, we prepare some standard terminologies around stack and introduce the moduli stack $\boldsymbol{\mathcal{K}}_{T,\chi}$ and see its Artinness (Definition 3.4.2). Though it is a simple task to check the Artinness under Corollary 3.3.12, the author believes that knowing its proof must help the readers to properly handle the moduli stack $\boldsymbol{\mathcal{K}}_{T,\chi}$ in the next subsection. See Appendix A for generalities on stacks over the category of complex spaces, which we call $\mathbb{C}$**an**-stacks.

We denote by $\mathbb{C}$**an** the category of complex analytic spaces which are not assumed to be reduced nor irreducible. The set of holomorphic morphisms between complex spaces $U$ and $V$ will be denoted by $\mathrm{Holo}(U,V)$. We also denote by $\mathbb{C}\mathbf{an}_S$ the category of complex spaces over $S$ and by $\mathrm{Holo}_S(U,V)$ its set of morphisms.

Let $S$ be a complex space. A morphism of complex spaces $\pi : \mathcal{M} \to S$ is called a *family of complex manifolds over* $S$ if it is surjective, proper, smooth (equivalent to submersive when $\mathcal{M}$ and $S$ are complex manifolds) and has connected fibers. Recall that a smooth morphism between singular complex spaces is by definition a morphism of complex spaces $f : X \to Y$ with the following property: There are open coverings $\{V_\alpha \subset Y\}_\alpha$, $\{U_\alpha \subset X\}_\alpha$ of $Y$ and $X$, respectively, an indexed set of smooth complex manifolds $\{W_\alpha\}_\alpha$ and an indexed set of biholomorphisms $\{\phi_\alpha : V_\alpha \times W_\alpha \xrightarrow{\sim} U_\alpha\}_\alpha$ satisfying $f \circ \phi_\alpha = p_1$, where the morphism $p_1$ denotes the projection $V_\alpha \times W_\alpha \to V_\alpha$.

Let $T \cong (\mathbb{C}^*)^k$ be an algebraic torus. A *fibrewise $T$-action on a family* $\pi : \mathcal{M} \to S$ is a holomorphic morphism $\alpha : \mathcal{M} \times T \to \mathcal{M}$ which satisfies the following conditions.

1. (Fibrewise) The morphism $\alpha$ is an $S$-morphism. Namely, $\pi \circ \alpha = \pi \circ p_1 : \mathcal{M} \times T \to S$.

2. (Group action) $\alpha \circ (\alpha \times \mathrm{id}_T) = \alpha \circ (\mathrm{id}_{\mathcal{M}} \times \mu) : \mathcal{M} \times T \times T \to \mathcal{M}$, where $\mu : T \times T \to T$ is the multiplication.

A fibrewise $T$-action on a family $\pi : \mathcal{M} \to S$ is called *effective* if for every $s \in S$ the induced group morphism $T \to \mathrm{Aut}(\mathcal{M}_s)$ is injective. Finally, an *$S$-family of complex $T$-manifolds* is defined to be a family of complex manifolds over $S$ together with an effective fibrewise $T$-action in the above sense.

Now we introduce the stack $\boldsymbol{\mathcal{K}}_{T,\chi}$. A $\mathbb{C}$**an**-stack (Definition 3.6.9) is a category $\boldsymbol{\mathcal{F}}$ together with a functor $\boldsymbol{\mathcal{F}} \to \mathbb{C}\mathbf{an}$ satisfying some natural geometric axioms. Here we give the category of our interest.

**Definition 3.4.1** (category/stack $\boldsymbol{\mathcal{K}}_{T,\chi}$)**.** Let $T$ be an algebraic torus and $\chi$ be a Fano character. Object in $\boldsymbol{\mathcal{K}}_{T,\chi}$ is a family of complex $T$-manifolds $\pi : \mathcal{M} \to S$ whose fibers are *gentle* (see Definition 3.2.17) Fano $T$-manifolds whose Hilbert characters are $\chi$.

A morphism from $\xi := (\pi : \mathcal{M} \to S, \alpha : \mathcal{M} \times T \to \mathcal{M})$ to $\xi' := (\pi' : \mathcal{M}' \to S', \alpha' : \mathcal{M}' \times T \to \mathcal{M}')$ is a pair $(f, \phi)$ where $f : S \to S'$ is a morphism of complex spaces and $\phi : \mathcal{M} \to \mathcal{M}'$ is a $T$-equivariant morphism which is compatible with $\pi, \pi', f$ and induces a biholomorphism $\tilde{\phi} : \mathcal{M} \to f^*\mathcal{M}'$, where $f^*\mathcal{M}' := S \times_{f,S',\pi'} \mathcal{M}' \subset S \times \mathcal{M}'$. Here, the morphism $\phi$ is said to be $T$-equivariant if $\alpha' \circ (\phi \times \mathrm{id}_T) = \phi \circ \alpha : \mathcal{M} \times T \to \mathcal{M}'$.

Note that there may be no gentle Fano manifolds whose K-optimal Hilbert characters coincide with a given K-optimal Fano character $(T, \chi)$. In other words, there might be no object in the category $\boldsymbol{\mathcal{K}}_{T,\chi}$ for a K-optimal Fano character $(T, \chi)$.

We denote by $\mathcal{K}^s_{T,\chi}$ the subcategory of $\mathcal{K}_{T,\chi}$ consisting of families of K-stable Fano $T$-manifolds and by $\mathcal{K}(n)$ the disjoint union of the categories $\mathcal{K}_{T,\chi}$ where $(T,\chi)$ run all the K-optimal Fano characters of $n$-dimensional Fano manifolds. Both categories $\mathcal{K}_{T,\chi}$ and $\mathcal{K}^s_{T,\chi}$ are $\mathbb{C}\mathbf{an}$-stacks with the forgetful functors $\mathcal{K}_{T,\chi}, \mathcal{K}^s_{T,\chi} \to \mathbb{C}\mathbf{an}$ given by $(\pi : \mathcal{M} \to S) \mapsto S$. (See Lemma 3.6.5 and 3.6.13 in Appendix A. )

We denote by $\mathcal{K}_{T,\chi}(S)$ the subcategory consisting of objects $(\pi : \mathcal{M} \to S, \alpha)$ with fixed base $S$ and whose morphisms are pairs $(\mathrm{id}_S, \phi)$. For any two objects $\xi = (\mathcal{M} \to S, \alpha), \eta = (\mathcal{M}' \to S, \alpha') \in \mathcal{K}_{T,\chi}(S)$, we define the contravariant functor $\mathit{Isom}_S(\xi, \eta)$ from $\mathbb{C}\mathbf{an}_S$ to $\mathbf{Sets}$ by mapping an object $f : U \to S$ to the set $\mathrm{Hom}_{\mathcal{K}_{T,\chi}}(f^*\xi, f^*\eta)$ and a morphism $h : (U, f) \to (V, g)$ to the map given by the canonical identifications $f^*\xi \cong h^*g^*\xi$, $f^*\eta \cong h^*g^*\eta$.

The following definition is just an analogy of Artin algebraic stack.

**Definition 3.4.2** (Artin $\mathbb{C}\mathbf{an}$-stack)**.** A $\mathbb{C}\mathbf{an}$-stack $\mathcal{F}$ is called an *Artin stack* if it satisfies the following two conditions.

1. The diagonal morphism $\Delta : \mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable by complex spaces.

2. There exists a smooth surjective morphism $U \to \mathcal{F}$ from a complex space $U$.

Or equivalently,

1. For every complex space $S$ and any $\xi, \eta \in \mathrm{Obj}(\mathcal{F})$, there exists a complex space $S_{\xi,\eta}$ and an isomorphism $\mathrm{Holo}_S(-, S_{\xi,\eta}) \cong \mathit{Isom}_S(\xi, \eta)$ of contravariant functors from $\mathbb{C}\mathbf{an}_S$ to $\mathbf{Sets}$.

2. There exists a morphism $U \to \mathcal{F}$ of fibred categories from a complex space $U$ such that the 2-fibre product $U \times_{\mathcal{F}} V$ is isomorphic as stacks to a complex space $f : V_U \to V$ smooth over $V$ with surjective $f$, for any morphism $V \to \mathcal{F}$ from any complex space $V$.

The 2-fibre product $U \times_{\mathcal{F}} V$ of stacks is always isomorphic to some complex space $W$ over $V$ and $U$ from the first condition (cf. [SPA, Tag 045G]).

We frequently use the following representability result in our analytic category in this thesis. The representability of the following moduli functor (and analogical functor in the schematic category) in both analytic/schematic category is well-known and the projectivity is also well-known in the schematic category. While the compactness of the Douady space is proved by Fujiki for class $\mathcal{C}$ space, it seems we must see the equivalence of the Douady space and the Hibert scheme, or must imitate the construction of Hilbert scheme in the analytic category, to show the projectivity of the Douady space when $X$ is projective. Our concerns here are just whether the analytification of the Hilbert scheme represents the moduli functor of the Douady space, and if we can make things $T$-equivariant. Though these concerns might be exhibited somewhere in literatures, we place a proof here since the author could not find an appropriate reference and the author believes it is better for the readers (including the author).

**Proposition 3.4.3** ($T$-Hilbert scheme)**.** Suppose $\pi : \mathcal{X} \to B$ is a holomorphic morphism of complex spaces and $\alpha : \mathcal{X} \times T \to \mathcal{X}$ is a holomorphic action with $f \circ \alpha = f \circ p_1$. Consider the functor $\mathit{Hilb}_{T,\pi} : \mathbb{C}\mathbf{an}_B \to \mathbf{Sets}$ given by

$$\mathit{Hilb}_{T,\pi}(S) := \left\{ \mathcal{Z} \subset S \times_B \mathcal{X} \mid \begin{array}{c} \mathcal{Z} \text{ is a } T\text{-invariant closed analytic subspace} \\ \text{and } \mathcal{Z} \to S \text{ is a flat family} \end{array} \right\}.$$

Then there exists a Hausdorff complex space $\mathrm{Hilb}_{T,\pi}$ representing the functor $\mathit{Hilb}_{T,\pi}$. Moreover, suppose $B = \mathrm{pt}$ and $X = \mathcal{X}$ is projective with an ample line bundle $L$, then the subfunctor $\mathit{Hilb}_{T,\chi,X} \subset \mathit{Hilb}_{T,X}$ consisting of families $\mathcal{Z} \to S$ with a fixed Hilbert polynomial $\chi$ is representable by a projective complex space $\mathrm{Hilb}_{T,\chi,X} \subset \mathbb{C}P^N$.

*Proof.* When $T$ is trivial, the existence of the Hausdorff complex space $\mathrm{Hilb}_\pi = \mathrm{Hilb}_{T,\pi}$ follows from [Dou2] for $B = \mathrm{pt}$ case and from [Pou] for general base $B$. The projectivity follows from Grothendieck's existence theorem of the Hilbert scheme, which represents an analogical functor defined on the category of schemes $\mathbf{Sch}_\mathbb{C}$, and the coincidence of the functors when they restricted to the subcategory $\mathbf{Def}_\mathbb{C}$ of the spectrum of finitely generated Artin algebras over $\mathbb{C}$, which is naturally embedded into both $\mathbb{C}\mathbf{an}$ and $\mathbf{Sch}_\mathbb{C}$. Actually, a morphism $f : X \to Y$ between complex spaces is an isomorphism if and only if it induces an isomorphism of functors $h_X|_{\mathbf{Def}_\mathbb{C}} \to h_Y|_{\mathbf{Def}_\mathbb{C}}$.

When $T$ is non-trivial, we can consider the action of $T$ on the set $\mathit{Hilb}_\pi(S)$ for each $S \in \mathbb{C}\mathbf{an}_B$, whose fixed point subset is nothing but the subset $\mathit{Hilb}_{T,\pi}(S) \subset \mathit{Hilb}_\pi(S)$. Then the existence in the category of complex spaces follows from the following two general statements.

1. Suppose $H$ is a (not necessarily reduced) complex space with $T$-action and $x \in H$ is a $T$-fixed point. Then there is a $T_\mathbb{R}$-invariant open

neighbourhood $U \subset H$ of $x$ and a $T_{\mathbb{R}}$-equivariant closed embedding $\varphi : U \to V$ into an open neighbourhood $V \subset T_x H$ of the origin, where $T_{\mathbb{R}}$ denotes the maximal compact subgroup of $T$ and $T_x H$ denotes the Zariski tangent space ([Akh-book, Subsection 2.2]).

2. Let $W \subset T_x H$ be the set of $T$-invariant points, which forms a $T$-invariant linear subspace. Then the complex space $U_T := U \times_V (W \cap V) \subset U$, considered as a closed subspace of $U$, enjoys the following universal property: for any holomorphic morphism $f : S \to H$ invariant under the $T$-action on $H$, the restricted holomorphic morphism $f|_{f^{-1}(U)} : f^{-1}(U) \to H$ holomorphically and uniquely factors through $U_T$.

On the other hand, the existence in the category of schemes follows from [Fog]. The rest of the proof is parallel to the first paragraph. □

The proof of the next proposition is a routine for the readers familiar with Artin stack. We exhibit the proof for the others.

**Proposition 3.4.4.** The $\mathbb{C}$**an**-stacks $\boldsymbol{\mathcal{K}}_{T,\chi}$ is an Artin $\mathbb{C}$**an**-stack. If $(T, \chi)$ is K-optimal, then $\boldsymbol{\mathcal{K}}^s_{T,\chi}$ is also Artin and is an open sub-stack of $\boldsymbol{\mathcal{K}}_{T,\chi}$.

*Proof.* By considering the graphs $\mathcal{M} \times_{\phi,\mathcal{N},\mathrm{id}} \mathcal{N} \subset \mathcal{M} \times \mathcal{N}$ of morphisms $\phi : \mathcal{M} \to \mathcal{N}$, the functor $\mathit{Isom}_S(\xi, \eta)$ is identified with a subfunctor of $\mathit{Hilb}_{T,\mathcal{M}\times\mathcal{N}/S}$. Then it is easy to see that $\mathit{Isom}_S(\xi, \eta)$ is representable by an open subspace of $\mathrm{Hilb}_{T,\mathcal{M}\times\mathcal{N}/S}$ (cf. [FGA-book, 5.6.2.]).

Next we construct a smooth surjective morphism $U \to \boldsymbol{\mathcal{K}}_{T,\chi}$. Let us consider a uniform $T$-equivariant embedding of Fano manifolds in $\boldsymbol{\mathcal{K}}_{T,\chi}$ into some fixed $\mathbb{C}P^N$. Then all Fano manifolds in $\boldsymbol{\mathcal{K}}_{T,\chi}$ emerge in $\mathrm{Hilb}_{T,\mathbb{C}P^N}$. From Corollary 3.3.12, there is an open subset $U$ of $\mathrm{Hilb}_{T,\mathbb{C}P^N}$, in the real topology, such that the restricted universal family $\mathcal{U}|_U \to U$ exactly consists of gentle Fano manifolds in $\boldsymbol{\mathcal{K}}_{T,\chi}$. Note the family $\mathcal{U}|_U \to U$ naturally carries a $T$-action and hence is considered as an object in $\boldsymbol{\mathcal{K}}_{T,\chi}$. So we have the induced morphism $U \to \boldsymbol{\mathcal{K}}_{T,\chi} : (f : S \to U) \mapsto (f^*\mathcal{U}|_U \to S)$, which is readily surjective from its construction. Consider a morphism $X \to \boldsymbol{\mathcal{K}}_{T,\chi}$; it is equivalent to give a family of gentle Fano $T$-manifolds $\mathcal{M} \to X$. For a sufficiently large $m$, the direct image sheaf $\pi_*(\mathcal{O}(-mK_{\mathcal{M}}))|_{U_\alpha}$ becomes locally free. Take a covering $\mathcal{U} = \{U_\alpha\}_\alpha$ of $X$ that trivializes the vector bundle $\pi_*(\mathcal{O}(-mK_{\mathcal{M}}))|_{U_\alpha}$ so that we can consider morphisms $U_\alpha \to U$ corresponding to trivializations of $\pi_*(\mathcal{O}(-mK_{\mathcal{M}}))|_{U_\alpha}$. There is a unique $PGL_T$-equivariant extension $U_\alpha \times PGL_T \to U$ of these morphisms. Then from the universality of the 2-fibre product $U \times_{\mathcal{K}_{T,\chi}} U_\alpha$ , we get morphisms $U_\alpha \times PGL_T \to U \times_{\mathcal{K}_{T,\chi}} U_\alpha$. We have the inverse morphisms of these morphisms given as follows. Take an object $(S, \xi : S \to U, \eta : S \to U_\alpha, \phi : \xi^* \mathcal{U} \xrightarrow{\sim} \eta^* \mathcal{M}|_{U_\alpha})$ of $U \times_{\mathcal{K}_{T,\chi}} U_\alpha$. Since $\eta^* \mathcal{M}|_{U_\alpha}$ can be considered as being embedded in $S \times \mathbb{C}P^N$, the isomorphism $\phi$ corresponds to a morphism $\tilde{\phi} : S \to PGL_T$. Then we have a morphism $\eta \times \tilde{\phi} : S \to U_\alpha \times PGL_T$, which gives an object in $\mathbb{C}\mathbf{an}_{U_\alpha \times PGL_T}$. Therefore $U \times_{\mathcal{K}_{T,\chi}} X \to X$ is locally written as $U_\alpha \times PGL_T \to U_\alpha$. So it is a smooth morphism.

It is shown in [H. Li, Theorem 3.4] that K-stable Fano manifolds form an open subset in the parameter space of any family of complex manifolds, without introducing the K-stability of Fano $T$-manifolds. From the exactly same argument as above, we conclude that $\mathcal{K}^s_{T,\chi}$ is Artin and is an open sub-stack of $\mathcal{K}_{T,\chi}$. □

In the above proof, the only non-routine part is Corollary 3.3.12, i.e. the openness of the subset consisting of gentle Fano manifolds in the parameter space of a family. Our method in Corollary 3.3.12 cannot be applied to prove the Zariski openness. This is the reason why we must work in the category of complex spaces rather than the category of algebraic spaces, so that we cannot apply Alper's theory on good moduli spaces over the category of algebraic spaces, at least so far.

### 3.4.2 Main construction

In this subsection, we prove our main theorem. First we prepare two general lemmas.

**Lemma 3.4.5.** Let $K$ be a compact Lie group and $K^c$ be its complexification, $V$ be a representation of $K^c$ and $B \subset V$ be a $K$-invariant Stein open neighbourhood of the origin. Let $s \times t : R \to B \times B$ be the holomorphic groupoid obtained by pulling back the holomorphic action groupoid $a : V \times K^c \to V \times V : (v, k) \mapsto (v, vk)$ along the inclusion $B \times B \subset V \times V$. Then the following holds.

1. The stack $[B/R]$ associated to the holomorphic groupoid $(B, R, s, t, c)$ as in Appendix A is isomorphic, induced by the inclusion $B \hookrightarrow BK^c$, to the quotient stack $[BK^c/K^c]$ as $\mathbb{C}\mathbf{an}$-stacks, where $BK^c$ denotes the $K^c$-orbit $\{bg \in V \mid b \in B, g \in K^c\}$ of $B$, which is $K^c$-invariant open.

2. There is a natural morphism $[BK^c/K^c] \to BK^c /\!\!/ K^c$ of $\mathbb{C}\mathbf{an}$-stacks to the analytic GIT quotient $BK^c /\!\!/ K^c$ which enjoys the following universal property: any morphism from the quotient stack $[BK^c/K^c]$ to any complex space $X$ uniquely factors through $BK^c /\!\!/ K^c$.

*Proof.* We identify $[BK^c/K^c]$ with the $\mathbb{C}\mathbf{an}$-stack $[\![BK^c/K^c]\!]$ in Example 3.6.3. Consider a morphism $\sigma$ from the fibred category $[B/R]_p$ to $[BK^c/K^c]$ sending an object $\xi : S \to X$ in $[B/R]_p$ to the object $(S, S \times K^c, a \circ (\xi \times \mathrm{id}))$ in $[BK^c/K^c]$ (cf. the description right after Example 3.6.3). Let $S$ be a complex space and $(S, P, \xi')$ be an object in $[BK^c/K^c](S)$. Take a local trivialization $\{P \cong U_\alpha \times K^c\}_\alpha$ of the principal $K^c$-bundle $P$ and consider the associated $K^c$-equivariant morphisms $\xi'_\alpha : U_\alpha \times K^c \to BK^c$. After taking smaller $U_\alpha$, we can find a holomorphic morphism $\xi_\alpha : U_\alpha \to B$ and a holomorphic morphism $g : U_\alpha \to K^c$ so that $\xi_\alpha(x)g(x) = \xi'_\alpha(x, e)$. It follows that the object $(U_\alpha, U_\alpha \times K^c, \xi'_\alpha)$ in $[BK^c/K^c]$ is isomorphic to $\sigma(\xi_\alpha) = (U_\alpha, U_\alpha \times K^c, a \circ ((\xi_\alpha g) \times \mathrm{id}_{K^c}))$. Moreover, it is easily seen that $\mathit{Isom}_{[B/R]_p,S}(\xi, \eta) \to \mathit{Isom}_{[BK^c/K^c],S}(\sigma(\xi), \sigma(\eta))$ is a sheafification of the functor $\mathit{Isom}_{[B/R]_p,S}(\xi, \eta) : \mathbb{C}\mathbf{an}_S \to \mathbf{Sets}$. It follows that $[BK^c/K^c]$ is a stackification of the fibred category $[B/R]_p$. Therefore, it is isomorphic to the stackification $[B/R]$ of $[B/R]_p$.

Since $B$ is a reduced Stein space, $BK^c$ is also a reduced Stein space and there exists a categorical quotient $BK^c /\!\!/ K^c$, which is also a reduced Stein space (see [Hei], [Snow]). Take an object $(S, P, \xi)$ in $[BK^c/K^c]$ and a local trivialization $\{P \cong U_\alpha \times K^c\}_\alpha$ of $P$. Then we have holomorphic morphisms $\tilde{\xi}_\alpha : U_\alpha \to U_\alpha \times K^c \to BK^c \to BK^c /\!\!/ K^c$. Since $\xi_\alpha : U_\alpha \times K^c \to BK^c$ agree on the overlaps $U_\alpha \cap U_\beta$ up to the action of $K^c$, and $BK^c \to BK^c /\!\!/ K^c$ is $K^c$-invariant, holomorphic morphisms $\tilde{\xi}_\alpha$ coincide on the overlaps $U_\alpha \cap U_\beta$ and define a holomorphic morphism $S \to BK^c /\!\!/ K^c$, glued together. This construction gives the morphism $[BK^c/K^c] \to BK^c /\!\!/ K^c$. The universal property follows from the fact that any $K^c$-invariant holomorphic morphism $BK^c \to X$ uniquely factors through $BK^c /\!\!/ K^c$. □

The complex space $BK^c /\!\!/ K^c$ is moreover normal as it is an open subspace of the algebraic GIT quotient $V /\!\!/ K^c$, which is normal whenever $V$ is normal (cf. [MFK-book, Section 0.2]).

**Lemma 3.4.6.** Let $K$ be a compact Lie group, $B$ be a complex manifold with holomorphic $K$-action and $E \to B$ be a $K$-equivariant holomorphic vector bundle. Suppose $0 \in B$ is a fixed point of $K$-action. Since the fiber $E_0$ can be considered as $K$-representation, we can construct a $K$-equivariant holomorphic vector bundle $\underline{E_0}_{B,K} := B \times E_0$ whose action is given by $(b,v)k := (bk, vk)$. Then $E$ is $K$-equivariantly isomorphic to $\underline{E_0}_{B,K}$ on some neighbourhood of $0 \in B$.

*Proof.* Consider the frame bundle $\pi : P \to B$ of the holomorphic vector bundle $E$ and fix a point $p_0 \in \pi^{-1}(0) \subset P$. We have a right holomorphic action of $K$ on $P$ defined by

$$pk : \mathbb{C}^r \xrightarrow{p_0} E_0 \xrightarrow{k^{-1}} E_0 \xrightarrow{p_0^{-1}} \mathbb{C}^r \xrightarrow{p} E_b \xrightarrow{k} E_{bk}$$

for $p : \mathbb{C}^r \xrightarrow{\sim} E_b \in P$ and $k \in K$. The point $p_0 \in P$ is a fixed point of this action and $\pi : P \to B$ is a $K$-equivariant submersion. So we have a $K$-equivariant holomorphic section $\sigma : B \to P$ with $\sigma(0) = p_0$ by taking smaller $B$ if necessary. Now the map $B \times GL(r) \to P : (b,g) \mapsto \sigma(b)g$ gives a $K$-equivariant isomorphism of principal $GL(r)$-bundles and hence induces a $K$-equivariant isomorphism of the adjoint bundles $\underline{\mathbb{C}^r}_{B,K} \xrightarrow{\tilde{p}_0} \underline{E_0}_{B,K}$. $\square$

Let $X$ be a Fano manifold with a Kähler–Ricci soliton $(g, \xi')$, $T \cong (\mathbb{C}^*)^{\times k}$ be the algebraic torus generated by $\xi'$, $K := \mathrm{Isom}_{\xi'}(X, g)$ be the (possibly non-connected) isometry group preserving $\xi'$ and $H^1_T(X, \Theta) \subset H^1(X, \Theta)$ denote the $T$-invariant subspace of the first cohomology of the tangent sheaf. Recall in Proposition 3.3.7 and in Corollary 3.3.12 we obtained a family $\varpi : \mathcal{X} \to B$ of gentle Fano $T$-manifolds over a small ball $B \subset H^1_T(X, \Theta)$. Moreover, $\mathcal{X}$ admits a holomorphic $K$-action so that $\varpi$ is $K$-equivariant with respect to the linear action on $B$, and a $T$-equivariant biholomorphism $\mathcal{X}_0 \cong X$. This in particular defines a morphism $B \to \boldsymbol{\mathcal{K}}_{T,\chi}$ of Artin $\mathbb{C}$**an**-stacks. Now we prove the following.

**Proposition 3.4.7.** Let $X$ be a Fano $T$-manifold with Kähler–Ricci soliton and the Hilbert character $(T, \chi)$. Then by taking smaller $B$ if necessary, the morphism $B \to \boldsymbol{\mathcal{K}}_{T,\chi}$ factors through an étale morphism $[B/R] \to \boldsymbol{\mathcal{K}}_{T,\chi}$ with finite fibres, where $R$ is defined as in Lemma 3.4.5. In other words, for any morphism $S \to \boldsymbol{\mathcal{K}}_{T,\chi}$ of $\mathbb{C}$**an**-stacks, there is an étale morphism $S' \to S$ of complex spaces with finite fibres and an $S$-isomorphism of $\mathbb{C}$**an**-stacks from $S'$ to the 2-base change $S \times_{\boldsymbol{\mathcal{K}}_{T,\chi}} [B/R] \to S$.

*Proof.* The family $\varpi : \mathcal{X} \to B$ in Proposition 3.3.7 defines a morphism $B \to \boldsymbol{\mathcal{K}}_{T,\chi}$. Now we will show that this morphism factors through the quotient morphism $B \to [B/R]$. It is equivalent to the existence of a natural $T$-equivariant $R$-biholomorphism $s^*\mathcal{X} \xrightarrow{\sim} t^*\mathcal{X}$. We prove this by relating our analytic family to an algebraic family *as groupoids*. As a consequence, the induced morphism $[B/R] \to \boldsymbol{\mathcal{K}}_{T,\chi}$ is shown to be étale with finite fibres.

Since $\varpi : \mathcal{X} \to B$ is a $K$-equivariant family and $\mathcal{O}(-K_{\mathcal{X}/B})$ is relatively ample, we can find a large $\ell \in \mathbb{N}$ so that the direct image sheaf $\varpi_*\mathcal{O}(-\ell K_{\mathcal{X}/B})$ is $K$-equivariantly isomorphic to the sheaf of sections of a $K$-equivariant holomorphic vector bundle $E$. Lemma 3.4.6 shows that there is a $K$-equivariant isomorphism $\underline{H^0(X, \mathcal{O}(-\ell K_X))}_{B,K} \cong E$, so we have a $K$-equivariant $B$-embedding $\mathcal{X} \hookrightarrow B \times \mathbb{P}^N$, where we identify $\mathbb{P}^N$ with $\mathbb{P}(H^0(X, \mathcal{O}(-\ell K_X))^*)$. From the universality of $\mathrm{Hilb}_{T,\mathbb{P}^N}$, we obtain a $K$-equivariant holomorphic morphism $h : B \to \mathrm{Hilb}_{T,\mathbb{P}^N}$ together with an isomorphism $h^*\mathcal{U} \cong \mathcal{X}$.

From the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}(1)^{\oplus N+1} \to \Theta_{\mathbb{P}^N} \to 0,$$

we obtain $H^1_T(X, i^*\Theta_{\mathbb{P}^N}) = 0$ and $H^0_T(X, i^*\Theta_{\mathbb{P}^N}) \cong H^0_T(\mathbb{P}^N, \Theta_{\mathbb{P}^N})$. Combining with this with the following exact sequence

$$\begin{aligned} 0 &\to H^0_T(X, \Theta_X) \to H^0_T(X, i^*\Theta_{\mathbb{P}^N}) \to H^0_T(X, N_{X/\mathbb{P}^N}) \\ &\to H^1_T(X, \Theta_X) \to H^1_T(X, i^*\Theta_{\mathbb{P}^N}) \to H^1_T(X, N_{X/\mathbb{P}^N}) \to 0 \end{aligned}$$

shows that the sequence

$$0 \to H^0_T(X, \Theta_X) \to H^0_T(\mathbb{P}^N, \Theta_{\mathbb{P}^N}) \to H^0_T(X, N_{X/\mathbb{P}^N}) \to H^1_T(X, \Theta_X) \to 0$$

is exact and $H^1_T(X, N_{X/\mathbb{P}^N})$ vanishes. So we conclude that $\mathrm{Hilb}_{T,\mathbb{P}^N}$ is smooth at $[X] = h(0) \in \mathrm{Hilb}_{T,\mathbb{P}^N}$, whose tangent space is given by $H^0_T(X, N_{X/\mathbb{P}^N})$ (cf. [FGA-book, subsection 6.4.]).

Now we work in the category of algebraic spaces in the blink of an eye. Since $\mathrm{Aut}_T(X)$ is reductive, we can apply the étale slice theorem [AHR, Theorem 2.1], which generalizes the Luna's étale slice theorem to non-affine cases, and then obtain the following: a smooth affine $\mathrm{Aut}_T(X)$-variety $(W, w)$, an $\mathrm{Aut}_T(X)$-equivariant morphism $\phi : (W, w) \to (\mathrm{Hilb}_{T,\mathbb{P}^N}, h(0))$ which induces a $PGL_T(N+1)$-equivariant étale morphism $W \times^{\mathrm{Aut}_T(X)} PGL_T \to \mathrm{Hilb}_T$, and a $\mathrm{Aut}_T(X)$-equivariant étale morphism $(W, w) \to (H^1_T(X, \Theta), 0)$.

$$\begin{tikzcd}
W \times^{\mathrm{Aut}_T(X)} PGL_T \arrow[d, "\text{étale}"'] & (W, w) \arrow[l, hook'] \arrow[dl, "\phi"'] \arrow[dr, "\text{étale}"] & \\
(\mathrm{Hilb}_T, h(0)) & & (H^1_T(X, \Theta_X), 0)
\end{tikzcd}$$

Note that the quotient morphism $W \times PGL_T \to W \times^{K^c} PGL_T$ is a $K^c$-equivariant submersion, under the right action of $K^c = \mathrm{Aut}_T(X)$ on $W \times PGL_T$ defined by $(x, g_0)g_1 = (xg_1, g_1^{-1}g_0g_1)$ and on $W \times^{K^c} PGL_T$ defined by $[x, g_0]g_1 = [x, g_0g_1]$. Since the point $(w, e) \in W \times PGL_T$ is fixed by this $K^c$-action, we obtain a $K$-equivariant holomorphic section $\sigma$ from a neighbourhood of $[w, e] \in W \times^{K^c} PGL_T$ with $\sigma([w, e]) = (w, e)$. Therefore, taking smaller $B$ if necessary, we can assume that $h : B \to \mathrm{Hilb}_T$ factors through $W \times PGL_T \to \mathrm{Hilb}_T$. We can moreover assume that the composed morphism $(B, 0) \to (W, w)$ of a lifting $B \to W \times PGL_T$ passing through $(w, e)$ with the projection to the first factor is $K$-equivariant holomorphic open embedding. Note that we do not know whether this morphism $(B, 0) \to (W, w)$ is a section of the étale morphism $(W, w) \to (H^1_T(X, \Theta_X), 0)$.

Set $\mathrm{Hilb}^\circ_T := h(B) \cdot PGL_T \subset \mathrm{Hilb}_T$. Since $\varpi : \mathcal{X} \to B$ is a complete family at any point $b \in B$, $\mathrm{Hilb}^\circ_T$ is an open subset. The restricted $PGL_T$-equivariant universal family $\mathcal{U}^\circ \to \mathrm{Hilb}^\circ_T$ parametrizes only gentle Fano $T$-manifolds and hence induces an open embedding $[\mathrm{Hilb}^\circ_T/PGL_T] \to \boldsymbol{\mathcal{K}}_{T,\chi}$. We fix this subset $\mathrm{Hilb}^\circ$ while we later take smaller $B$.

It follows from [Snow, Proposition 5.1] that we have a $K^c$-invariant open neighbourhood $W^\circ \subset \phi^{-1}(\mathrm{Hilb}^\circ_T)$ of $w$ so that the restriction $W^\circ \to H^1_T(X, \Theta)$ is an $K^c$-invariant open embedding. Taking smaller $B$, we have the restricted morphism $B \to W^\circ \times PGL_T$. Let $g : B \to PGL_T$ be the composition of this morphism with the projection to the second factor. Denote by $h_\circ : B \to \mathrm{Hilb}^\circ$ the composition $\alpha_{\mathrm{Hilb}_T} \circ (h \times g^{-1}) : B \to \mathrm{Hilb}^\circ_T \times PGL_T \to \mathrm{Hilb}^\circ_T$. Then the holomorphic morphism $h_\circ$ is $K$-equivariant and factors through the $K^c$-equivariant holomorphic morphism $W^\circ \to \mathrm{Hilb}^\circ_T$. Moreover, we have an induced isomorphism $h_\circ^* \mathcal{U} \cong \mathcal{X}$.

Since the differential of the induced morphism $B \to W^\circ$ at $0 \in B$ is a $K$-equivariant isomorphism, we can assume that $B \to W^\circ$ is a $K$-equivariant open embedding. Let us denote by $\beta : B \to H^1_T(X, \Theta)$ the composition of this morphism $B \to W^\circ$ with $W^\circ \to H^1_T(X, \Theta)$. Then $\beta$ is also a $K$-equivariant open embedding.

Note that both $BK^c \subset H^1_T(X, \Theta)$ and $\beta(B)K^c \subset H^1_T(X, \Theta)$ are the complexification, in the sense of [Hei], of $B$ with respect to the action of $K$ . From the uniqueness of the complexification, there is a $K^c$-equivariant biholomorphism $\gamma : BK^c \to \beta(B)K^c$ which is compatible with the $K$-equivariant morphisms $B \subset BK^c$ and $\beta : B \to \beta(B)K^c$.

Now we have the following cartesian diagrams

$$\begin{tikzcd}
R \arrow[r] \arrow[d, "s\times t"'] & BK^c \times K^c \arrow[r, "\hookrightarrow\boxtimes\mathrm{id}"] \arrow[d, "\alpha_{BK^c}"] & H^1_T \times K^c \arrow[d, "\alpha_{H^1_T}"] \\
B \times B \arrow[r, "\hookrightarrow^{\boxtimes 2}"] & BK^c \times BK^c \arrow[r, "\hookrightarrow^{\boxtimes 2}"] & H^1_T \times H^1_T
\end{tikzcd}$$

$$\begin{tikzcd}
R_W \arrow[r] \arrow[d, "s_W\times t_W"'] & \beta(B)K^c \times K^c \arrow[r, "\hookrightarrow\boxtimes\mathrm{id}"] \arrow[d, "\alpha_{\beta(B)K^c}"] & H^1_T \times K^c \arrow[d, "\alpha_{H^1_T}"] \\
B \times B \arrow[r, "\beta^{\boxtimes 2}"] & \beta(B)K^c \times \beta(B)K^c \arrow[r, "\hookrightarrow^{\boxtimes 2}"] & H^1_T \times H^1_T
\end{tikzcd}$$

Since $\gamma : BK^c \xrightarrow{\sim} \beta(B)K^c$ is $K^c$-equivariant, it satisfies $\gamma^{\boxtimes 2}\circ\alpha_{BK^c} = \alpha_{\beta(B)K^c}\circ(\gamma \boxtimes \mathrm{id})$ and hence gives an isomorphism of the groupoids $(p_1 \circ \alpha_{BK^c}, p_2 \circ \alpha_{BK^c}) : BK^c \times K^c \to BK^c$ and $(p_1 \circ \alpha_{\beta(B)K^c}, p_2 \circ \alpha_{\beta(B)K^c}) : \beta(B)K^c \times K^c \to \beta(B)K^c$. It follows that there is an isomorphism $(\rho, \mathrm{id}_B) : (R, B) \xrightarrow{\sim} (R_W, B)$ of the groupoids $s \times t : R \to B \times B$ and $s_W \times t_W : R_W \to B \times B$. Hence there is an isomorphism $[B/R] \cong [B/R_W]$ of the quotient $\mathbb{C}$**an**-stacks.

On the other hand, since $\beta : B \to H^1_T$ factors through the $K^c$-equivariant open embedding $W^\circ \to H^1_T$, the groupoids $s_W \times t_W : R_W \to B \times B$ also appears in the following cartesian diagram.

$$\begin{tikzcd}
R_W \arrow[r] \arrow[d, "s_W\times t_W"'] & W^\circ \times K^c \arrow[d, "\alpha_{W^\circ}"] \\
B \times B \arrow[r, "\beta^{\boxtimes 2}"] & W^\circ \times W^\circ
\end{tikzcd}$$

Therefore we obtain an open embedding of the quotient $\mathbb{C}$**an**-stacks $[B/R_W] \hookrightarrow [W^\circ/K^c]$.

Moreover, the étale finite morphism $W^\circ \times^{K^c} PGL_T \to \mathrm{Hilb}^\circ$ induces an étale finite morphism of the quotient $\mathbb{C}$**an**-stacks $[W^\circ/K^c] \cong [W^\circ \times^{K^c} PGL_T/PGL_T] \to [\mathrm{Hilb}^\circ_T/PGL_T]$.

Now combining all, we obtain an étale morphism $[B/R] \to \boldsymbol{\mathcal{K}}_{T,\chi}$ with finite fibers, which obviously commutes with $B \to \boldsymbol{\mathcal{K}}_{T,\chi}$ and $B \to [B/R]$ from its construction. □

Here is our main theorem.

**Theorem 3.4.8.** There exists a Hausdorff complex analytic space $\mathcal{K}_{T,\chi}$, which we call *the moduli space of Fano manifolds with Kähler–Ricci solitons*, and a morphism $\boldsymbol{\mathcal{K}}_{T,\chi} \to \mathcal{K}_{T,\chi}$ from the Artin $\mathbb{C}$**an**-stack $\boldsymbol{\mathcal{K}}_{T,\chi}$ such that any morphism from $\boldsymbol{\mathcal{K}}_{T,\chi}$ to any complex space $B$ holomorphically and uniquely factors through $\mathcal{K}_{T,\chi}$. Moreover, this moduli space enjoys the following property.

1. The complex space $\mathcal{K}_{T,\chi}$ is normal and homeomorphic to the space $K_{T,\chi}$ in Definition 3.3.18. We will prove this in Proposition 3.4.11 after constructing a moduli space with the following property.

2. The morphism $\boldsymbol{\mathcal{K}}_{T,\chi} \to \mathcal{K}_{T,\chi}$ induces a bijection $|\boldsymbol{\mathcal{K}}_{T,\chi}|/\sim \quad \to \mathcal{K}_{T,\chi}$ where $|\boldsymbol{\mathcal{K}}_{T,\chi}|$ denotes the set of points of the stack $\boldsymbol{\mathcal{K}}_{T,\chi}$, which is canonically identified with the set of the isomorphism classes of gentle Fano manifolds, and $[X] \sim [X']$ if the central fibers of the gentle degenerations of gentle Fano manifolds $X$ and $X'$ coincide.

As we have already noted, the logical order of our argument is "Proposition 3.4.7 $\Rightarrow$ Proposition 3.2.18 $\Rightarrow$ Theorem 3.4.8". Here we apply Proposition 3.2.18 before we prove it. The author recommend the readers who prefer following the proof in the logical order to read section 3.4.4 firstly.

*Proof.* The image of the étale morphism $[B/R] \to \boldsymbol{\mathcal{K}}_{T,\chi}$ defines an open substack $\mathrm{Im}[B/R] \subset \boldsymbol{\mathcal{K}}_{T,\chi}$. Object in $\mathrm{Im}[B/R]$ is an object $(\pi : \mathcal{M} \to S, \alpha)$ in $\boldsymbol{\mathcal{K}}_{T,\chi}$ whose fibers are gentle Fano $T$-manifolds appearing in the Kuranishi family $\varpi : \mathcal{X} \to B$. Firstly, we prove that the morphism $[B/R] \to BK^c /\!\!/ K^c$ in Lemma 3.4.5 factors through $\mathrm{Im}[B/R]$. We construct a morphism $\mathrm{Im}[B/R] \to BK^c /\!\!/ K^c$. Take an object $(\pi : \mathcal{M} \to S, \alpha)$ in $\mathrm{Im}[B/R]$ and consider the following cartesian diagram.

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & [B/R] \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathrm{Im}[B/R] \end{array}$$

Since $[B/R] \to \mathrm{Im}[B/R]$ is étale, $\tilde{S} \to S$ is also étale. Then we can take local slices $s_\alpha : U_\alpha \to \tilde{S}$ of $\tilde{S} \to S$ so that $\{U_\alpha\}_\alpha$ covers $S$ and obtain morphisms $U_\alpha \to [B/R]$, hence also obtain holomorphic morphisms $\phi_\alpha : U_\alpha \to BK^c /\!\!/ K^c$. From its construction, we know that $x \in U_\alpha$ maps to the point $\phi_\alpha(x) \in BK^c /\!\!/$

$K^c = \nu^{-1}(0)/K$ representing the central fiber of some gentle degeneration of $\mathcal{M}_x$, which is unique due to Proposition 3.2.18. So if $S$ is reduced, these morphisms $\phi_\alpha : U_\alpha \to BK^c /\!\!/ K^c$ coincide on the overlaps, hence they give a holomorphic morphism $\phi : S \to BK^c /\!\!/ K^c$, glued together. When $S$ is not reduced, since any Fano $T$-manifold has reduced semi-universal family, we can locally extend the morphism $S \to \mathrm{Im}[B/R]$ to some $T \to \mathrm{Im}[B/R]$ with reduced $T$. Take a point $x \in U_\alpha \cap U_\beta$ and a small neighbourhood $U$ of $x$ so that $U \to \mathrm{Im}[B/R]$ extends to some $T \to \mathrm{Im}[B/R]$ with reduced domain $T \supset U$ containing $U$ as a closed subspace. Taking smaller $T$, sections $s_\alpha|_U$, $s_\beta|_U$ extend to some sections $t_\alpha, t_\beta : T \to \tilde{T}$, where $\tilde{T}$ is given similar as $\tilde{S}$. Therefore the morphisms $\phi_\alpha|_U, \phi_\beta|_U : U \to BK^c /\!\!/ K^c$ extend to some morphisms $\psi_\alpha, \psi_\beta : T \to BK^c /\!\!/ K^c$. As we have already observed that $\psi_\alpha$ and $\psi_\beta$ coincide, the restrictions $\phi_\alpha|_U = \psi_\alpha \circ i_U, \phi_\beta|_U = \psi_\beta \circ i_U : U \to BK^c /\!\!/ K^c$ also coincide as holomorphic morphisms, where $i_U : U \to T$ is the closed immersion. Therefore we obtain a morphism $\phi : S \to BK^c /\!\!/ K^c$ by gluing the morphisms $\phi_\alpha : U_\alpha \to BK^c /\!\!/ K^c$. It is easy to see that this construction is functorial, so we obtain the expected morphism $\mathrm{Im}[B/R] \to BK^c /\!\!/ K^c$, which inherits the universal property from the morphism $[B/R] \to BK^c /\!\!/ K^c$.

Now consider two morphisms $[B/R] \to \boldsymbol{\mathcal{K}}_{T,\chi}$ and $[B'/R'] \to \boldsymbol{\mathcal{K}}_{T,\chi}$ with different domains. We also consider two maps $i : BK^c /\!\!/ K^c \to L^2_k K(M, \omega, T)$ and $i' : B'K'^c /\!\!/ K'^c \to L^2_k K(M, \omega, T)$. For any point $x \in L^2_k K(M, \omega, T)$ in the overlaps $\mathrm{Im} i \cap \mathrm{Im} i'$, we can find another étale morphism $[B''/R''] \to \boldsymbol{\mathcal{K}}_{T,\chi}$ and a map $i'' : B''K''^c /\!\!/ K''^c \to L^2_k K(M, \omega, T)$ with $i''([0]) = x$ so that $\mathrm{Im}[B''/R''] \subset \mathrm{Im}[B/R] \cap \mathrm{Im}[B'/R'] \subset \boldsymbol{\mathcal{K}}_{T,\chi}$ and $\mathrm{Im} i'' \subset \mathrm{Im} i \cap \mathrm{Im} i'$. Especially we have a natural inclusion morphism $\mathrm{Im}[B''/R''] \to \mathrm{Im}[B/R]$ and hence obtain a morphism $\mathrm{Im}[B''/R''] \to BK^c /\!\!/ K^c$. From the universality of the morphism $\mathrm{Im}[B''/R''] \to B''K''^c /\!\!/ K''^c$, we obtain a holomorphic morphism $B''K''^c /\!\!/ K''^c \to BK^c /\!\!/ K^c$. This holomorphic morphism is clearly compatible with $i''$ and $i$ as maps, so especially it is a homeomorphism onto its open image, after taking smaller $B''$ if necessary. Since the analytic GIT quotient spaces $BK^c /\!\!/ K^c$ are normal, this holomorphic homeomorphism is actually a biholomorphism. This argument shows that the coordinate change $i'^{-1} \circ i$ is biholomorphic. Thus we obtain a complex space $\mathcal{K}(M, \omega, T)$ by giving a complex structure on the topological space $L^2_k K(M, \omega, T)$ defined from the above holomorphic charts. Set $\mathcal{K}_{T,\chi} := \coprod_{\chi(M,\omega,T)=\chi} \mathcal{K}(M, \omega, T)$. Clearly from its construction, there is a morphism $\boldsymbol{\mathcal{K}}_{T,\chi} \to \mathcal{K}_{T,\chi}$ enjoying the universal property. It follows from section 3.3 and Proposition 3.2.18 that this morphism induces a bijection $|\boldsymbol{\mathcal{K}}_{T,\chi}|/\sim\ \to \mathcal{K}_{T,\chi}$. We prove in the next subsection that the space $\mathcal{K}(M,\omega,T)$, which is homeomorphic to $L_k^2K(M,\omega,T)$, is actually homeomorphic to $K(M,\omega,T)$. $\square$

**Corollary 3.4.9.** The $\mathbb{C}$**an**-stack $\boldsymbol{\mathcal{K}}^s_{T,\chi}$ admits a tame moduli space $\boldsymbol{\mathcal{K}}^s_{T,\chi} \to \mathcal{K}^s_{T,\chi}$ (see [Alp1, Definition 7.1]), with the same universal property as the moduli space $\boldsymbol{\mathcal{K}}_{T,\chi} \to \mathcal{K}_{T,\chi}$. Moreover, the complex space $\mathcal{K}^s_{T,\chi}$ is a Hausdorff complex orbifold.

This corollary follows from the construction in the proof of the main theorem, the openness property of the K-stable Fano $T$-manifolds and the fact that $[B/R] \to \boldsymbol{\mathcal{K}}^s_{T,\chi}$ is an open embedding in this case, which is an easy consequence of the injectivity of the map $|[B/R]| \to BK^c /\!\!/ K^c \approx \nu^{-1}(0)/K \to K_{T,\chi}$ and the bijection $|\boldsymbol{\mathcal{K}}^s_{T,\chi}| \to K_{T,\chi}$. The orbifold coordinates are given by open neighbourhoods of the origin in the spaces $H^1_T(X,\Theta) /\!\!/ (\mathrm{Aut}_T(X)/T)$. We can also consider a separated smooth Deligne-Mumford $\mathbb{C}$**an**-stack $\boldsymbol{\mathcal{K}}^*_{T,\chi}$ associated to the $\mathbb{C}$**an**-stack $\boldsymbol{\mathcal{K}}^s_{T,\chi}$.

### 3.4.3 Consistency

In the previous section, we constructed a complex analytic space structure on the spaces $L_k^2K(M,\omega,T) = (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k/\mathrm{Ham}_T(M,\omega)^2_{k+1}$ and proved that it has a certain universality independent of $k$, which is described in terms of the stack $\boldsymbol{\mathcal{K}}_{T,\chi}$. Since the universality determines a complex space uniquely up to biholomorphisms, the complex spaces $(\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k/\mathrm{Ham}_T(M,\omega)^2_{k+1}$ are all canonically biholomorphic to each other. In particular, we deduce that they are all homeomorphic through the following natural maps

$$\mathcal{I}_{l,k} : (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_l/\mathrm{Ham}_T(M,\omega)^2_{l+1} \to (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k/\mathrm{Ham}_T(M,\omega)^2_{k+1} : [J] \mapsto [J].$$

Now we show the continuous map

$$\mathcal{I}_k : (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)/\mathrm{Ham}_T(M,\omega) \to (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_k/\mathrm{Ham}_T(M,\omega)^2_{k+1} : [J] \mapsto [J]$$

is also homeomorphic, using that $\mathcal{I}_{l,k}$ is homeomorphic.

In the proof of the following proposition, we hope to apply an $L^2_k$-version of Newlander–Nirenberg theorem. Unfortunately, the author could not find in literatures a precise $L^2_k$-version of Newlander–Nirenberg theorem for our purpose, but only find the reference [NW]. As the Newlander–Nirenberg type theorem in [NW] losses some regularity, we make use of the above fact obtained from the universality of our moduli space.

**Proposition 3.4.10.** The continuous map $\mathcal{I}_k$ is a homeomorphism.

*Proof.* Take two elements $J, J' \in (\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)$ and suppose there is a $L^2_{k+1}$-regular map $\phi \in \mathrm{Ham}_T(M,\omega)^2_{k+1}$ such that $\phi^* J = J'$. Then $\phi$ is $C^\infty$-smooth by Myers-Steenrod theorem. This shows that $\mathcal{I}_k$ is injective.

Next we show the surjectivity. It is sufficient to show that for any $J \in (\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)^2_k$, there is a $L^2_{k+1}$-regular map $\phi \in \mathrm{Ham}_T(M,\omega)^2_{k+1}$ such that $\phi^* J \in (\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)$. Take a large integer $m \geq 2$ and $l$ so that $L^2_l \subset C^{m-1,\alpha} \subset C^{m-1} \subset L^2_{k+1}$. Since $\mathcal{I}_{l,k}$ is a homeomorphism, we can find a $L^2_{k+1}$-regular map $\phi_0 \in \mathrm{Ham}_T(M,\omega)^2_{k+1}$ so that $\phi_0^* J \in (\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)^2_l$. Then it follows from [NW] that there is a $C^{m,\alpha/n}$-smooth diffeomorphism $\phi_1 : M' \to M$ such that $\phi_1^* \phi_0^* J$ is a $C^\infty$-smooth integrable complex structure, where on the other hand $\phi_1^*\phi_0^*\omega$ and $\phi_1^*\phi_0^* g_J$ is only $C^{m-1,\alpha/n}$-regular, in particular, $L^2_{k+1}$-regular. We can choose a $C^\infty$-smooth diffeomorphism $\phi_2 : M \to M'$, which we can additionally suppose that it is sufficiently close to $\phi_1^{-1}$ in $C^m$-topology. Note that $\phi_0 \circ \phi_1 \circ \phi_2$ is sufficiently close to $\phi_0$ in $L^2_{k+2}$-topology. The pull-back metric $\phi_2^*(\phi_0 \circ \phi_1)^* g_J$ is a $L^2_{k+1}$-regular metric which is a Kähler–Ricci soliton with respect to $C^\infty$-smooth integrable complex structure $\phi_2^*(\phi_0 \circ \phi_1)^* J$. The elliptic regularity argument shows that $(\phi_0 \circ \phi_1 \circ \phi_2)^* g_J$ is in fact $C^\infty$-smooth. Hence $(\phi_0 \circ \phi_1 \circ \phi_2)^*\omega$ is also $C^\infty$-smooth. Since we further assume that $(\phi_0 \circ \phi_1 \circ \phi_2)^*\omega$ is close to $\phi_0^*\omega = \omega$ in $L^2_{k+1}$-topology, both $C^\infty$-smooth symplectic form $\omega, (\phi_0 \circ \phi_1 \circ \phi_2)^*\omega$ have the same cohomology classes and $\omega_t := t\omega + (1-t)(\phi_0 \circ \phi_1 \circ \phi_2)^*\omega$ is non-degenerate for any $t \in [0,1]$. From Moser's theorem, we obtain a $C^\infty$-smooth diffeomorphism $\phi_3$ satisfying $\phi_3^*(\phi_0 \circ \phi_1 \circ \phi_2)^*\omega = \omega$, which is close to $\mathrm{id}_M$ in $L^2_{k+1}$-topology as in the proof of Proposition 3.3.7. Now we have obtained the expected $L^2_{k+1}$-regular map $\phi := \phi_0 \circ \phi_1 \circ \phi_2 \circ \phi_3$. From the construction, we know $\phi$ can be taken sufficiently close to $\phi_0$ in $L^2_{k+1}$-topology.

Finally we prove that $\mathcal{I}_k$ is actually a homeomorphism. Take a convergent sequence $J_n \to J_\infty \in (\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)^2_k$. It suffices to show that there are elements $\phi_n, \phi_\infty \in \mathrm{Ham}_T(M,\omega)^2_{k+1}$ such that $\phi_n^* J_n, \phi_\infty^* J_\infty$ belong to $(\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)$ and the sequence $\phi_n^* J_n$ converges to $\phi_\infty^* J_\infty$ in the $C^\infty$-topology, by taking a subsequence if necessary (thanks to the injectivity of $\mathcal{I}_k$). Since $\mathcal{I}_k$ is surjective, we can find an element $\phi_\infty \in \mathrm{Ham}_T(M,\omega)^2_{k+1}$ so that $\phi_\infty^* J_\infty$ is $C^\infty$-smooth and hence there is no loss of generality in supposing $C^\infty$-smoothness of $J_\infty$ from the beginning. Since $\mathcal{I}_{l,k}$ is a homeomorphism, we can find a sequence $\phi_{n,l} \in \mathrm{Ham}_T(M,\omega)^2_{k+1}$ so that $\phi_{n,l}^* J_n \in (\mathcal{S}_\xi^{\mathrm{int}})^{-1}(0)^2_l$ and $\phi_{n,l}^* J_n$ converges to

$J_\infty$ in the $L^2_l$-topology. We define a set

$$\Sigma_l(J) := \{\phi \in \mathrm{Ham}_T(M,\omega)^2_{l+1} \mid \phi^* J \in (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)\}$$

for $J \in (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_l$. Since $\mathcal{I}_l$ is surjective, $\Sigma_l(J)$ is a non-empty set. Moreover, the density of $\mathrm{Ham}_T(M,\omega) \subset \mathrm{Ham}_T(M,\omega)^2_{l+1}$, which we can deduce from Weinstein's tubular neighbourhood theorem, shows that $\Sigma_l(J) \subset \mathrm{Ham}_T(M,\omega)^2_{l+1}$ is also dense. Therefore we can perturb $\phi_{n,l}$, as small as $n$ goes to the infinity, so that $\phi^*_{n,l} J_n$ are $C^\infty$-smooth and preserve the $L^2_l$-convergence $\phi^*_{n,l} J_n \to J_\infty \in (\mathcal{S}^{\mathrm{int}}_\xi)^{-1}(0)^2_l$. Now we can proceed to the diagonal argument with respect to $(n,l)$ and conclude that a subsequence $\phi^*_{n_l,l} J_{n_l}$ converges to $J_\infty$ in $C^\infty$-topology. $\square$

There is another topological space consisting of biholomorphism classes of Fano manifolds with Kähler–Ricci solitons, which is considered in [PSS].

$$\mathcal{KR}_{GH}(n,F) := \left\{ [M,J,g,\xi'] \,\middle|\, \begin{array}{c} (M,J,g,\xi') \text{ is a Fano manifold } (M,J) \text{ with} \\ \text{a Kähler–Ricci soliton } (g,\xi') \text{ and } \int_M |\xi'|^2_g \omega^n \leq F. \end{array} \right\}$$

In [PSS], a topological compactification $\overline{\mathcal{KR}_{GH}(n,F)}$ of this space is considered in regard to the 'complexified' Gromov–Hausdorff convergence. It was open whether the space $\mathcal{KR}_{GH}(n,F)$ is stable for large $F$, which was expected in [PSS]. As we have the equality $\mathrm{Fut}(\xi') = 2\int_M |\xi'|^2_g \omega^n$ for Kähler–Ricci soliton $(g,\xi')$, this is equivalent to say that the invariants $\mathrm{Fut}(\xi')$ are uniformly bounded from above for $n$-dimensional Fano manifolds with Kähler–Ricci solitons.

We show that this invariant is actually bounded, not only for Fano manifolds with Kähler–Ricci solitons, but also for all $n$-dimensional Fano $T$-manifolds with the maximal $K$-optimal action. We also compare $\mathcal{KR}_{GH}(n,F)$ with our $K(M,\omega,T)$ and $K_{T,\chi}$, which a priori have different topologies.

**Proposition 3.4.11.** Set

$$\mathcal{KR}_{GH}(n) := \left\{ [M,J,g,\xi'] \,\middle|\, \begin{array}{c} (M,J,g,\xi') \text{ is a Fano manifold } (M,J) \\ \text{with a Kähler–Ricci soliton } (g,\xi'). \end{array} \right\}.$$

Then $\mathcal{KR}_{GH}(n) = \mathcal{KR}_{GH}(n,F)$ for large $F$ and the map

$$K(M,\omega,T) \to \mathcal{KR}_{GH}(n) : [J] \mapsto [M,J,g_J]$$

gives a homeomorphism onto a clopen (closed and open) subset of $\mathcal{KR}_{GH}(n)$, for any $2n$-dimensional symplectic Fano manifold $(M,\omega)$ with K-optimal $T$-action.

*Proof.* Since Fano manifolds are bounded [KMM], we have a sufficiently large Hilbert scheme Hilb of $\mathbb{C}P^N$ with bounded Hilbert polynomials so that for any Fano manifold $X$ we can find a point $[X] \in \mathrm{Hilb}$, uniquely up to $\mathrm{Aut}(\mathbb{P}^N) = PGL(N+1)$-action, representing an anti-canonically embedded $X \subset \mathbb{C}P^N$. We denote by $\mathrm{Hilb}_{\mathrm{Fano}}$ the Zariski open locus parametrizing the anti-canonically embedded Fano manifolds. Obviously, $PGL(N+1)$ preserves the subset $\mathrm{Hilb}_{\mathrm{Fano}}$.

Fix a maximal algebraic torus $T$ of $PGL(N+1)$ and consider its action on $\mathrm{Hilb}_{\mathrm{Fano}}$. Note as we have $\mathrm{Stab}([X].g) = g^{-1}\mathrm{Stab}([X])g \subset PGL(N+1)$ for $g \in PGL(N+1)$, we can find a point $[X] \in \mathrm{Hilb}_{\mathrm{Fano}}$ so that $\mathrm{Stab}([X]) \cap T \subset \mathrm{Stab}([X]) \cong \mathrm{Aut}(X)$ is a maximal torus. Indeed, for a maximal torus $T_X \subset \mathrm{Stab}([X])$, pick a maximal torus $T' = gTg^{-1} \subset PGL(N+1)$ so that $T' \cap \mathrm{Stab}([X]) = T_X$, then we have a maximal torus $\mathrm{Stab}([X].g) \cap T = g^{-1}T_X g \subset \mathrm{Stab}([X].g)$.

Next, consider the normalization $\widetilde{\mathrm{Hilb}} \to \mathrm{Hilb}$, where $\widetilde{\mathrm{Hilb}}$ is a normal projective variety and the morphism is a finite surjective morphism. Then we have a $T$-equivariant embedding of $\widetilde{\mathrm{Hilb}}$ into some $\mathbb{P}(V)$, where $V$ is a $T$-representation ([MFK-book, Corollary 1.6]). Since $V$ decomposes into 1-dimensional representations as $V \cong \mathbb{C}_{\chi_1} \oplus \cdots \oplus \mathbb{C}_{\chi_{\dim V}}$, the stabilizer $T_x \subset T$ of any point $x \in \mathbb{P}(V)$ can be written as $\chi_{i_1}^{-1}(1) \cap \cdots \cap \chi_{i_j}^{-1}(1)$, hence the possibilities are finite. It also follows that every fiber $S_T^{-1}(T')$ of the following map

$$S_T : \mathbb{P}(V) \to \{ \text{ sub torus of } T \ \} : x \mapsto T_x$$

is a (possibly non irreducible) subvariety in $\mathbb{P}(V)$. Therefore, we obtain a finite stratification $\{S_T^{-1}(T_i) \subset \mathbb{P}(V)\}$ and $\{H_i \subset \widetilde{\mathrm{Hilb}}_{\mathrm{Fano}}\}$ by its restriction. We refine this stratification by taking connected components of each $H_i$ and continue to write $\{H_i \subset \widetilde{\mathrm{Hilb}}_{\mathrm{Fano}}\}$. Since the restricted family $\mathcal{U}|_{H_i} \to H_i$ gives a family of Fano $T_i$-manifolds, we can consider the K-optimal vector $\xi_i \in (N_i)_{\mathbb{R}}$ with respect to the $T_i$-action on the Fano manifolds $X_s$ ($s \in H_i$), which is independent of $s \in H_i$. Let $T_i' \subset T_i$ be the sub-torus generated by $\xi_i$.

Now from the construction, every Fano manifold $X$ with a maximal K-optimal $T'$-action finds some $H_i$ satisfying $[X] \in H_i$ and $T' = T_i'$. Since the Futaki invariant of $\xi_i'$ on $X_s$ is independent of the choice of $s \in H_i$, we conclude that there are only finitely many possibilities of the values of $\mathrm{Fut}_X(\xi')$ for the pairs $(X, \xi')$ of Fano manifolds with vanishing modified Futaki invariant $\mathrm{Fut}_{X,\xi'}$. In particular, $\mathrm{Fut}_X(\xi')$ is bounded for $(X, g, \xi') \in \mathcal{KR}_{GH}(n)$ and hence $\mathcal{KR}_{GH}(n, F) = \mathcal{KR}(n)$ for large $F$.

Next we see that the given map $K(M, \omega, T) \to \mathcal{KR}_{GH}(n)$ is a homeomorphism by a standard argument as follows. The continuity of the map is obvious. For every $[M, J, g] \in \mathcal{KR}_{GH}(n)$ and any two representatives $(M_1, J_1, g_1), (M_2, J_2, g_2) \in [M, J, g]$, we have a diffeomorphism $\phi : M_1 \to M_2$ satisfying $\phi^* J_2 = J_1$, $\phi^* g_2 = g_1$ and $(\phi^{-1})_* \xi'_2 = \xi'_1$, where $\xi'_i$ is the unique holomorphic vector field satisfying $\mathrm{Ric}(g_i) - L_{\xi'_i} g_i = g_i$. It follows that the map $K(M, \omega, T) \to \mathcal{KR}_{GH}(n)$ is injective for K-optimal $(M, \omega, T)$, and the images of $K(M_1, \omega_1, T_1), K(M_2, \omega_2, T_2) \to \mathcal{KR}_{GH}(n)$ intersect iff there is an isomorphism $\theta : T_1 \xrightarrow{\sim} T_2$ and a $(T_1, T_2)$-equivariant symplectic diffeomorphism $(M_1, \omega_1) \xrightarrow{\sim} (M_2, \omega_2)$.

Since the images of the maps for distinct pairs $(M_1, \omega_1, T_1)$, $(M_2, \omega_2, T_2)$ are disjoint, it suffices to prove that the maps are closed. Actually, if the maps are closed, then the maps are homeomorphisms onto their images and the images are open from the above finiteness of the possibilities of the K-optimal pairs $(M, \omega, T)$. To see the closedness, take a sequence $[J_n] \in K(M, \omega, T)$ which has the convergent images $[M, J_n, g_{J_n}] \to [M_\infty, J_\infty, g_\infty]$ in $\mathcal{KR}_{GH}(n)$. As remarked before Proposition 6.1 in [PSS], we have a sequence $[M, J_n, g_{J_n}] \in \mathrm{Hilb}_T^\circ$ which converges to $[M_\infty, J_\infty, g_\infty] \in \mathrm{Hilb}_T^\circ$, where $\mathrm{Hilb}_T^\circ$ denotes the open subset of $\mathrm{Hilb}_T$ parametrizing gentle Fano $T$-manifolds with bounded Hilbert polynomial. Now we have a canonical continuous (holomorphic) map $\mathrm{Hilb}_T^\circ \to K(M, \omega, T) \subset \mathcal{KR}_{T,\chi}$ induced by the universality of $\mathcal{K}_{T,\chi}$. The image of the sequence $[M, J_n, g_{J_n}]$ is nothing but the original sequence $[J_n]$, so we obtain the convergence of $[J_n]$ to the image of $[M_\infty, J_\infty, g_\infty]$ in $K(M, \omega, T)$. $\square$

*Remark* 3.4.12. The compactification $\overline{\mathcal{KR}_{GH}}(n)$ of $\mathcal{KR}_{GH}(n)$ constructed in [PSS] is a compact Hausdorff space with a countable basis (cf. [DoSu2]) and the boundary $\overline{\mathcal{KR}_{GH}}(n) \setminus \mathcal{KR}_{GH}(n)$ is closed. The closedness of the boundary is easily confirmed as follows. Suppose $[X_n, g_n]$ is a sequence in $\overline{\mathcal{KR}_{GH}}(n) \setminus \mathcal{KR}_{GH}(n)$ converging to $[X_\infty, g_\infty]$ in $\overline{\mathcal{KR}_{GH}}(n)$. Take a sequence $[X_{n,i}, g_{n,i}] \in \mathcal{KR}_{GH}(n)$ for each $n$ converging to $[X_n, g_n]$ in $\overline{\mathcal{KR}_{GH}}(n)$. We can suppose that $\mathrm{Hilb}(X_{n,i}, g_{n,i}) \to \mathrm{Hilb}(X_n, g_n)$ in $\mathrm{Hilb}_T$. Then we can find a subsequence of $[X_n, g_n]$ so that $\mathrm{Hilb}(X_n, g_n) \to \mathrm{Hilb}(X_\infty, g_\infty)$ in $\mathrm{Hilb}_T$ by the diagonal argument. Since the subset of $\mathrm{Hilb}_T$ parametrizing singular subspaces of $\mathbb{C}P^N$ forms a closed subset, the limit $[X_\infty, g_\infty]$ must be also singular, hence $[X_\infty, g_\infty] \in \overline{\mathcal{KR}_{GH}}(n) \setminus \mathcal{KR}_{GH}(n)$

### 3.4.4 The promised proof of Proposition 3.2.18

If $X$ is a gentle Fano $T$-manifold, then $R_{\xi'}(X) = 1$ for the K-optimal vector $\xi'$. So there exists a unique solution $\omega_t = \omega_t(\alpha)$ of the following equation

$$\mathrm{Ric}(\omega_t) - L_\xi \omega_t = t\omega_t + (1-t)\alpha$$

for every $t \in [0,1)$ and any initial metric $\alpha$.

**Lemma 3.4.13.** Let $\mathcal{X} \to \bar{\Delta}$ be a family of Fano $T$-manifolds with $R_{\xi'}(\mathcal{X}_\sigma) = 1$ for the K-optimal vector $\xi$ over a compact disc $\bar{\Delta}$ and $\boldsymbol{\alpha}$ be a smooth family of $T_{\mathbb{R}}$-invariant Kähler metrics $\alpha_\sigma$ on $\mathcal{X}_\sigma$. Then there is a sufficiently divisible $k \in \mathbb{N}$ and a positive constant $c > 0$ which depend only on the pair $(\mathcal{X}, \boldsymbol{\alpha})$ such that for any $\sigma \in \bar{\Delta}$ and $t \in [0,1)$ the following uniform partial $C^0$-estimate holds.

$$\rho_{X_\sigma, \omega_t(\alpha_\sigma), k} \geq c,$$

where $\rho_{X_\sigma, \omega_t(\alpha_\sigma), k}$ denotes the function on $X_\sigma$ defined by

$$\rho_{X_\sigma, \omega_t(\alpha_\sigma), k}(x) := \max |s(x)|_{h_{X_\sigma, \omega_t(\alpha_\sigma), k}},$$

where $s$ runs over $s \in H^0(X_\sigma, \mathcal{O}(-kK_X))$ with $\int_{X_\sigma} |s|^2_{h_{X_\sigma, \omega_t(\alpha_\sigma), k}} (k\omega_t(\alpha_\sigma))^n = 1$ and $h_{X_\sigma, \omega_t(\alpha_\sigma), k}$ denotes a metric on $-kK_{X_\sigma}$ whose curvature is $k\omega_t(\alpha_\sigma)$.

*Proof.* This follows from estimates in the proof of Lemma 5.6, Lemma 5.7 and Lemma 5.8 in [F. Wang, X. Zhu]. Note that we can uniformly take constants $C$ in Lemma 5.6, $B$ in Lemma 5.7 and $c_1, C$ in Lemma 5.8 independent of $\alpha_\sigma$, since the constants of Theorem A in [Mab] and of Corollary 5.3 in [Zhu] can be uniformly taken. Then it follows that any sequence $(X_{\sigma_i}, \omega_{t_i}(\alpha_{\sigma_i}))$ $(t_i \to 1)$ is a sequence of almost Kähler–Ricci solitons in the sense of [F. Wang, X. Zhu, Definition 5.1]. Now we can deduce our estimate from [JWZ, Corollary 1.4], [DoSu1, Lemma 3.4] and the argument after the lemma. $\square$

Now we can apply the arguments in [DoSu1] to the metric family

$$\{(X, \omega_t(\alpha_s))\}_{(t,s) \in [0,1) \times [0,1]},$$

under the above partial $C^0$-estimate. Thus we have a sufficiently divisible number $k \in \mathbb{N}$ with the following properties.

1. The pair $(X_\sigma, \omega_t(\alpha_\sigma))$ defines a point $\mathrm{Hilb}(X_\sigma, \omega_t(\alpha_\sigma))$ in the compact Hausdorff topological space $\mathrm{Hilb}_T/U_T$ by embedding $X_\sigma$ into $\mathbb{C}P^N$ using a unitary basis of $H^0(X_\sigma, \mathcal{O}(-kK_{X_\sigma}))$ with respect to the metric $\omega_t(\alpha_\sigma)$.

2. For any sequence $(\sigma_i, t_i) \in \Delta \times [0,1]$, we have a subsequence such that $(X_{\sigma_i}, \omega_{t_i}(\alpha_{\sigma_i}))$ converges in the 'complexified' Gromov–Hausdorff topology to some $\mathbb{Q}$-Fano variety $X_\infty$ with a Kähler–Ricci soliton $(\omega_\infty, \xi'_\infty)$.

3. After taking a further subsequence, the sequence $\mathrm{Hilb}(X_{\sigma_i}, \omega_{t_i}(\alpha_{\sigma_i})) \in \mathrm{Hilb}_T/U_T$ converges in $\mathrm{Hilb}_T/U_T$ to the point $\mathrm{Hilb}(X_\infty, \omega_\infty)$ which is similarly defined using a unitary embedding $X_\infty \hookrightarrow \mathbb{C}P^N$.

***Proof of Proposition 3.2.18***. Let $\{(X, \omega_t(\alpha))\}_{t\in[0,1)}$ be the family of solutions of the continuity method with an initial metric $\alpha$. Suppose there is a smooth Fano $T$-manifold with Kähler–Ricci soliton $(X_\infty, \omega_\infty)$ which is the limit of a Gromov–Hausdorff convergent subsequence $(X, \omega_{t_i}(\alpha))$. First we show that the limit $(X_\infty, \omega_\infty)$ is uniquely determined independent of the choice of the initial metrics $\alpha$ and the subsequences $(X, \omega_{t_i}(\alpha))$. Suppose $\alpha'$ is another Kähler metric on $X$ and $(X, \omega_{t'_i}(\alpha')) \to (X'_\infty, \omega'_\infty)$ be a convergent subsequence to a $\mathbb{Q}$-Fano $T$-variety with Kähler–Ricci soliton. Set $\alpha_s := s\alpha' + (1-s)\alpha$. As we noted right before this proof, we can find a sufficiently divisible number $k_{\boldsymbol{\alpha}} \in \mathbb{N}$ so that all $(X, \omega_t(\alpha_s))$ can be uniformly embedded using the unitary basis of $H^0(X, \mathcal{O}(-k_{\boldsymbol{\alpha}} K_X))$ with respect to $\omega_t(\alpha_s)$, which defines a point $\mathrm{Hilb}(X, \omega_t(\alpha_s)) \in \mathrm{Hilb}_T/U_T$. Moreover, we can assume $(X_\infty, \omega_\infty)$ and $(X'_\infty, \omega'_\infty)$ also define points $\mathrm{Hilb}(X_\infty, \omega_\infty) \in \mathrm{Hilb}_T/U_T$, $\mathrm{Hilb}(X'_\infty, \omega'_\infty) \in \mathrm{Hilb}_T/U_T$ respectively, and $\mathrm{Hilb}(X, \omega_{t_i}(\alpha)) \to \mathrm{Hilb}(X_\infty, \omega_\infty) \in \mathrm{Hilb}_T/U_T$, $\mathrm{Hilb}(X, \omega_{t'_i}(\alpha')) \to \mathrm{Hilb}(X'_\infty, \omega'_\infty) \in \mathrm{Hilb}_T/U_T$. These embeddings clearly define a continuous map $[0,1) \times [0,1] \to \mathrm{Hilb}/U_T : (t,s) \mapsto \mathrm{Hilb}(X, \omega_t(\alpha_s))$.

Suppose $X_\infty \ncong X'_\infty$. If $\overline{\mathrm{Hilb}(X_\infty, \omega_\infty) PGL_T} \cap \mathrm{Hilb}(X'_\infty, \omega'_\infty) PGL_T \neq \emptyset$, then we obtain a test configuration of $X_\infty$ with the central fiber $X'_\infty$ from the reductivity of the stabilizer $\mathrm{Aut}_T(X'_\infty)$, which allows to apply the étale slice theorem [AHR, Theorem 2.1] and the Hilbert-Mumford theorem. Since the central fiber admits a Kähler–Ricci soliton, the modified algebraic Futaki invariant of this test configuration is zero. However, as $X_\infty$ has Kähler–Ricci soliton and hence K-polystable, $X'_\infty$ must be isomorphic to $X_\infty$. This contradicts to our assumption.

So we have $\overline{\mathrm{Hilb}(X_\infty,\omega_\infty)PGL_T} \cap \mathrm{Hilb}(X'_\infty,\omega'_\infty)PGL_T = \emptyset$. Then in particular we can take open neighbourhoods $B_\varepsilon(\overline{\mathrm{Hilb}(X_\infty,\omega_\infty)PGL_T})$, $B_{\varepsilon'}(\mathrm{Hilb}(X'_\infty,\omega'_\infty)U_T)$ separating the two closed subsets $\overline{\mathrm{Hilb}(X_\infty,\omega_\infty)PGL_T}$ and $\mathrm{Hilb}(X'_\infty,\omega'_\infty)U_T$. Here we use a $U_T$-invariant distance on $\mathrm{Hilb}_T$ to consider $B_\varepsilon$ and fix this distance. Take $U_T$-invariant open neighbourhoods $V \Subset V' \subset \mathrm{Hilb}_T$ of $\mathrm{Hilb}(X_\infty,\omega_\infty)U_T$ so that $\mathcal{U}|_{V'} \to V'$ parametrizes Fano $T$-manifolds appearing in the family $\varpi : \mathcal{X} \to B$ with central fiber $\mathcal{X}_0 = X_\infty$. We can assume $\mathrm{Hilb}(X, \omega_{t_i}(\alpha)) \in V/U_T$. From the finiteness of the fibers of the morphism $[B/R] \to \boldsymbol{\mathcal{K}}_{T,\chi}$ in Proposition 3.4.7, there are only finitely many isomorphism classes of Fano $T$-manifolds with Kähler–Ricci solitons in this family that can be the central fiber of some gentle degeneration of $X$. Putting $\omega_i(\sigma) := \omega_{\sigma t'_i + (1-\sigma)t_i}(\alpha_\sigma)$, we have a continuous curve $\mathrm{Hilb}(X, \omega_i(-)) : [0,1] \to \mathrm{Hilb}_T/U_T$. Furthermore, putting

$$\sigma_i := \sup\{\sigma \in [0,1] \mid \mathrm{Hilb}(X,\omega_i(-))|_{[0,\sigma)} \subset B_\varepsilon(\overline{\mathrm{Hilb}(X_\infty,\omega_\infty)PGL_T}/U_T)\},$$

we obtain a sequence of almost Kähler–Ricci solitons in the sense of [F. Wang, X. Zhu]. So after taking a subsequence, we have a sequence $(X, \omega_i(\sigma_i))$ converging to some $\mathbb{Q}$-Fano $T$-variety admitting Kähler–Ricci soliton $(X''_\infty, \omega''_\infty)$ with the convergent corresponding sequence $\mathrm{Hilb}(X, \omega_i(\sigma_i)) \to \mathrm{Hilb}(X''_\infty, \omega''_\infty)$ in $\mathrm{Hilb}_T/U_T$. Replacing $\varepsilon$ with $\varepsilon/2^k$, we can construct $\sigma_{i,k}$ and $X''_{\infty,k}$ by the same process.

Suppose there is infinitely many $i$ for each $k$ such that $\mathrm{Hilb}(X, \omega_i(\sigma_{i,k})) \notin VPGL_T/U_T$. After taking subsequence, we know that

$$\mathrm{Hilb}(X,\omega_i(\sigma'_{i,k})) \in \partial(\overline{VPGL_T}/U_T) \cap B_{\varepsilon/2^{k-1}}(\overline{\mathrm{Hilb}(X_\infty,\omega_\infty)PGL_T}/U_T) \tag{3.15}$$

for

$$\sigma'_{i,k} := \sup\{\sigma \in [0,\sigma_{i,k}] \mid \mathrm{Hilb}(X,\omega_i(-))|_{[0,\sigma)} \subset VPGL_T/U_T\}.$$

Since $(X, \omega_i(\sigma'_{i,k}))$ is a sequence of almost Kähler–Ricci solitons for each $k$, we can assume $(X, \omega_i(\sigma'_{i,k})) \to (X'''_{\infty,k}, \omega'''_{\infty,k})$ for some $\mathbb{Q}$-Fano $T$-variety with Kähler–Ricci soliton $(X'''_{\infty,k}, \omega'''_{\infty,k})$. The diagonal argument shows that there is a subsequence $\{(X, \omega_{i_k}(\sigma'_{i_k,k}))\}_{k=1}^\infty$ of $\{(X, \omega_i(\sigma'_{i,k}))\}_{i,k}$ and a $\mathbb{Q}$-Fano $T$-variety $X'''_{\infty,\infty}$ such that $(X, \omega_{i_k}(\sigma'_{i_k,k})) \to (X'''_{\infty,\infty}, \omega'''_{\infty,\infty})$ and $\mathrm{Hilb}(X, \omega_{i_k}(\sigma'_{i_k,k})) \to \mathrm{Hilb}(X'''_{\infty,\infty}, \omega'''_{\infty,\infty})$. Now from the property (3.15), we conclude $\mathrm{Hilb}(X'''_{\infty,\infty}, \omega'''_{\infty,\infty}) \in \overline{\mathrm{Hilb}(X_\infty,\omega_\infty)PGL_T} \setminus \mathrm{Hilb}(X_\infty,\omega_\infty)PGL_T$. But this is absurd in the same way as we have seen before.

Therefore we can assume that for any large $k$, $\mathrm{Hilb}(X, \omega_i(\sigma_{i,k}))$ is in the neighbourhood $VPGL_T/U_T$ except for only finitely many $i$. In this case, the convergent sequence $(X, \omega_i(\sigma_{i,k})) \to (X''_{\infty,k}, \omega''_{\infty,k})$ defines a convergent sequence $\mathrm{Hilb}(X, \omega_i(\sigma_{i,k})) \to \mathrm{Hilb}(X''_{\infty,k}, \omega''_{\infty,k})$ in $\mathrm{Hilb}_T/U_T$ that is uniformly away from $\mathrm{Hilb}(X_\infty, \omega_\infty)PGL_T$ because $\mathrm{Hilb}(X, \omega_i(\sigma_{i,k})) \in \partial B_{\varepsilon/2^k}(\overline{\mathrm{Hilb}(X_\infty, \omega_\infty)PGL_T}/U_T)$. It follows that $X''_{\infty,k} \ncong X_\infty$. Since $\mathrm{Hilb}(X''_\infty, \omega''_{\infty,k}) \in V'PGL_T/U_T$ and each there is a gentle degeneration of $X$ with its central fiber $X''_{\infty,k}$, there is only finitely many isomorphism classes in $\{X''_{\infty,k}\}_{k=1}^\infty$. So we can assume $X''_{\infty,k}$ is all isomorphic after taking subsequence. From the uniqueness of Kähler–Ricci soliton, the sequence $(X''_{\infty,k}, \omega''_{\infty,k})$ is constant and hence converges to the limit $(X''_{\infty,\infty}, \omega''_{\infty,\infty}) \cong (X''_{\infty,k}, \omega''_{\infty,k})$. It follows that $\mathrm{Hilb}(X''_{\infty,\infty}, \omega''_{\infty,\infty}) \in \overline{\mathrm{Hilb}(X_\infty, \omega_\infty)PGL_T}$ from the fact

$$\mathrm{Hilb}(X''_{\infty,k}, \omega''_{\infty,k}) \in \overline{B_{\varepsilon/2^k}}(\overline{\mathrm{Hilb}(X_\infty, \omega_\infty)PGL_T}).$$

This is the last contradiction in this argument, which is now familiar to us. Finally, we conclude $X'_\infty \cong X_\infty$, so the limit is independent of the choice of the initial metrics $\alpha$ and the subsequences $t_i$.

Now we proceed to prove the uniqueness of the central fibers of gentle degenerations of $X$. Let $\mathcal{X} \to \Delta$ be a gentle degeneration. We have a smooth family of Kähler metrics $\alpha_s$ on $\mathcal{X}_s$ which extends the Kähler–Ricci soliton $\alpha_0$ on the central fiber $\mathcal{X}_0$, thanks to the stability argument of the Kähler condition in any sufficiently small deformation (see the last chapter of [KM-book]). The uniqueness of the continuity path, proved in [TZ1], shows that $\omega_t(\alpha_0) = \alpha_0$, so we can find a sequence $t_i \to 1$ and $s_i \to 0 \in \Delta$ so that $(X, \omega_{t_i}(\alpha_{s_i}))$ converges to $(\mathcal{X}_0, \alpha_0)$. We can show that the sequence $(X, \omega_{t'_i}(\alpha_{s_i}))$ also converges to $(\mathcal{X}_0, \alpha_0)$ for any sequence $t'_i \to 1$ by a similar argument as above (compare [LWX1, Lemma 6.9. (1)]). Consider some convergent sequence $(X, \omega_{t_m}(\alpha_{s_i})) \xrightarrow{t_m \to 1} (X_{\infty,i}, \omega_{\infty,i})$ and a sequence $t'_i \to 1$ so that $d_{GH}((X_{\infty,i}, \omega_{\infty,i}), (X, \omega_{t'_i}(\alpha_{s_i}))) < 1/i$. The diagonal argument shows that $(X_{\infty,i}, \omega_{\infty,i}) \to \mathcal{X}_0$. Since $\mathcal{X}_0$ is a smooth Fano $T$-manifold, $X_{\infty,i}$ is also smooth for large $i$. From what we have shown in the above argument, it follows that for any fixed Kähler metric $\alpha$ on $X$, we obtain $(X, \omega_t(\alpha)) \xrightarrow{t \to 1} (X_{\infty,i}, \omega_{\infty,i})$ for each $i$, so especially $(X_{\infty,i}, \omega_{\infty,i})$ are all isomorphic to each other. Now we conclude $(X, \omega_t(\alpha)) \xrightarrow{t \to 1} (\mathcal{X}_0, \alpha_0)$ where the limit is independent of the choice of the initial metrics $\alpha$ and the sequence is also independent of the choice of the central fibers $(\mathcal{X}_0, \alpha_0)$ of the gentle degenerations. So for another central fiber $(\mathcal{X}_0', \alpha_0')$ of another gentle degeneration $\mathcal{X}' \to \Delta$ of $X$, we also have $(X, \omega_t(\alpha)) \xrightarrow{t\to 1} (\mathcal{X}_0', \alpha_0')$. It follows that $(\mathcal{X}_0', \alpha_0')$ is isomorphic to $(\mathcal{X}_0, \alpha_0)$ from the uniqueness of the limit. This is what we expected. □

## 3.5 Discussions

### 3.5.1 On some examples

Here we observe step by step some known examples of Fano manifolds admitting Kähler–Ricci solitons. Although the existence is known, as far as the author knows, even the associated holomorphic vector fields $\xi'$ are not explicitly given in almost all examples.

*Example* 3.5.1. The blowing-up of $\mathbb{C}P^2$ at one point is a typical example of Fano manifold admitting non-Einstein Kähler–Ricci solitons. This seems the first example of a compact complex manifold proved to admit Kähler–Ricci solitons, which was found by Koiso [Koi] and Cao [Cao], independently.

*Example* 3.5.2 (toric Fano manifolds). It is shown in [X-J. Wang, X. Zhu] and reproved by [DaSz] from the K-stability viewpoint that every toric Fano manifold admit Kähler–Ricci soliton and it is Kähler–Einstein if and only if the barycenter of the canonical polytope coincides with the origin. Note that the maximal torus action on a toric Fano manifold is not necessarily K-optimal.

Every toric Fano manifold is rigid, i.e. $H^1(X, \Theta_X) = 0$, where $\Theta_X$ denotes the tangent sheaf ([BieBri, Proposition 4.2.]). It follows that toric Fano manifolds give discrete points in the moduli space $\mathcal{KR}_{GH}(n)$.

*Example* 3.5.3 (Fano homogeneous toric bundles). It is shown in [PS] and recovered in [Hua] that Fano homogeneous toric bundles have Kähler–Ricci solitons. This is a generalization of the main result in [X-J. Wang, X. Zhu]. It is again proved in [BieBri, Proposition 4.2.] that Fano homogeneous toric bundles are rigid (see also [BieBri, Proposition 2.2.1.], [T. Del, Example 3.10.]).

*Example* 3.5.4 (horospherical Fano manifolds). It is shown in [T. Del] from the K-stability viewpoint and reproved by [F. Del] that every horospherical Fano manifold admits Kähler–Ricci soliton. This is a generalization of one of the main results in [PS].

Horospherical Fano manifolds with Picard number one ($b^2 = 1$) are classified in [Pas]. There is a unique horospherical Fano manifold (with an action of the complex $G_2$ group) in this classified class which admits a non-trivial small deformation. We can see as follows (or just by checking the criterion in [T. Del]) that the Kähler–Ricci soliton on this horospherical Fano manifold $X_0$ is not Kähler–Einstein. It is shown in [PP] that the Kuranishi family of this horospherical Fano manifold $X_0$ is given by an iso-trivial degeneration $\mathcal{X} \to \mathbb{C}$ of the orthogonal Grassmanian $Gr_q(2,7)$. As the Grassmanian $Gr_q(2,7)$ is homogeneous, it admits Kähler–Einstein metric ([Mat2]). If $X_0$ admits Kähler–Einstein metric, then we get a contradiction by the separation property of Kähler–Einstein Fano manifolds ([SSY, LWX1]) as the deformation $\mathcal{X} \to \mathbb{C}$ is iso-trivial and the general fibre admits Kähler–Einstein metric. Thus we conclude that $X_0$ cannot admit Kähler–Einstein metrics, while it admits Kähler–Ricci soliton explained as above.

This example shows that the family $\mathcal{X} \to \mathbb{C}$ is not in the category $\boldsymbol{\mathcal{K}}(n)$, though any fibers in the family, which are isomorphic to either $Gr_q(2,7)$ or $X_0$, admit Kähler–Ricci solitons. We have to separate them into two pieces $\mathcal{X}^* \to \mathbb{C}^*$ and $X_0 \to \{0\}$ as the associated holomorphic vector fields jump at the origin.

It seems interesting to study whether any horospherical Fano manifolds are K-rigid, which means $H^1_T(X, \Theta_X) = 0$ for a K-optimal action $X \curvearrowleft T$.

*Example* 3.5.5 (Fano manifolds with complexity one). It is shown in [IS] and [CabSüs] that complexity one Fano threefolds of type 2.30, 2.31, 3.8*, 3.18, 3.21, 3.22, 3.23, 3.24, 4.5* and 4.8 from Mori and Mukai's classification [MM] admit non-Einstein Kähler–Ricci soliton.

Especially 3.8 and 4.5 admit deformations, so $H^1_T(X, \Theta) /\!\!/ \mathrm{Aut}_T(X)$ might be not mere a point.

The product $X \times Y$ of two Fano manifolds $X, Y$ with Kähler–Ricci solitons admits Kähler–Ricci solitons. So for instance, suppose $X$ is a Del Pezzo surface of degree $1 \leq d \leq 4$ and $Y$ is the blowing-up of $\mathbb{C}P^2$ at one point, then $X \times Y$ admits non-Einstein Kähler–Ricci solitons. By deforming $X$ while fixing $Y$, we get a $T$-equivariant deformation of $X \times Y$ where $X \times Y \curvearrowleft T$ is induced from the K-optimal action $Y \curvearrowleft T$. So $X \times Y$ provides a non discrete point in the moduli space $\mathcal{KR}_{GH}(n)$ outside of the subset $\mathcal{K}_{0,GH}(n)$ consisting of Kähler–Einstein Fano manifolds.

Dancer–Wang's examples [DW] may also provide non discrete points in the moduli space.

### 3.5.2 Future studies

#### Questions on the structure of the moduli space

*Question* 3.5.6. Study explicit examples of $(T, \chi)$ or $(M, \omega, T)$ with non-trivial $T$ whose moduli space $\mathcal{K}_{T,\chi}$ or $\mathcal{K}(M, \omega, T)$ has positive dimension and has a concrete description on its structure.

The author does not have any concrete description of positive dimensional moduli spaces $\mathcal{K}(M, \omega, T)$ so far. Related studies in the Kähler–Einstein case (i.e. $T = 0$) are explored by [OSS, SS, LiuXu].

*Question* 3.5.7. Is the complex analytic space $\mathcal{K}_{T,\chi}$ actually quasi-projective?

This question is related to the result in [LWX2] where the quasi-projectivity of the moduli space of Fano manifolds with Kähler–Einstein metrics is proved.

When $T$ is non-trivial, even the finiteness of the number of the connected components of $\mathcal{K}_{T,\chi}$ is still unknown, even though it has a natural topological compactification as a moduli space.

*Question* 3.5.8. Is there a canonical complex analytic structure on the compact topological space $\overline{\mathcal{KR}_{GH}}(n)$? How about on the space $\overline{\mathcal{KR}_{GH}}(n) \setminus \mathcal{KR}_{GH}(n)$? Can we identify them with algebraic spaces, or moreover with projective schemes?

This is related to the work of [Oda3, LWX1]. The techniques in this chapter do not work, at least directly, in the singular setting.

*Question* 3.5.9. Is there a canonical complex analytic (or algebraic) moduli space of $\mathbb{Q}$-Fano varieties with Kähler–Ricci solitons?

In all questions, it seems better to investigate modified K-stability from more algebro-geometric perspectives, possibly with some help of differential geometry.

#### Questions related to the extent of the moduli space

*Question* 3.5.10. Are there any non-gentle/modified K-unstable examples of Fano manifolds with Picard number one? How about birationally rigid Fano manifolds with Picard number one?

This is a refined question related to the Odaka-Okada conjecture [OO]. Two modified K-unstable examples are given in [T. Del], but both have the Picard number greater than one. The following is an optimistic conjecture towards a framework for classification of K-unstable $\mathbb{Q}$-Fano varieties.

**Conjecture 3.5.11.** Let $X$ be a $\mathbb{Q}$-Fano variety.

1. If $X$ is not modified K-semistable, there is a (non-equivariant) $\mathbb{R}$-degeneration (cf. [DeSz, CSW]) of $X$ whose central fiber is a modified K-semistable $\mathbb{Q}$-Fano variety whose H-invariant attains the infimum of the H-invariants over all $\mathbb{R}$-degenerations. Moreover, these degenerations are unique up to isomorphisms.

2. If $X$ is modified K-semistable with respect to a torus action $T$, then there is a $T$-equivariant degeneration $\mathcal{X}' \to \Delta$ of $X$ whose central fiber $X_0'$ is a K-polystable $\mathbb{Q}$-Fano $T$-variety (modified K-polystable with respect to the $T$-action). Moreover, any two such $T$-equivariant degenerations $\mathcal{X}_1' \to \Delta, \mathcal{X}_2' \to \Delta$ are equivalent up to scaling in the sense of the $T$-equivariant version of [BHJ, Definition 6.1.], not only they have isomorphic central fibres.

This conjecture is related to [CSW, Conjecture 3.7.] and is an analogy of the Harder-Narasimhan filtration for torsion-free coherent sheaves and the Jordan-Hölder filtration for semistable coherent sheaves (see [HL-book]) as already observed in [DeSz, Remark 3.6.]. We include the singular case for the future application to the Question 3.5.9.

For the first item, [CSW] shows that every smooth Fano manifold $X$ has an $\mathbb{R}$-degeneration with the $\mathbb{Q}$-Fano central fiber $X_0$ and there is an another degeneration $\mathcal{X}' \to \Delta$ of $X_0$ with the modified K-polystable $\mathbb{Q}$-Fano central fiber $X_0'$ with the K-optimal vector $\xi'$, which can be extended to $X_0$ with the vanishing modified Futaki invariant (see also [DeSz]). So as for the existence, it suffices to prove the modified K-semistability of $X_0$. Since $(X_0', \xi')$ is K-polystable, the problem is reduced to the 'stability of K-semistability in small deformations', which is related to the Artinness of the $\mathbb{C}$**an**-stack consisting of K-semistable $\mathbb{Q}$-Fano $T$-varieties, as in Proposition 3.4.4. It is remarkable that if $X$ is K-unstable (with respect to the trivial torus action), then $X_0'$ must be endowed with non-Einstein Kähler–Ricci solitons ([CSW, p. 17]).

The existence part of the second item is confirmed in [DaSz] for smooth modified K-semistable Fano $T$-manifolds. The uniqueness of the central fiber in this case could be demonstrated by the same methods in [LWX1], which is a role model of our proof of Proposition 3.2.18. (We worked with the smooth central fiber because the author felt that it makes arguments clear. )

The uniqueness assertion in the second item is stronger than the uniqueness of the central fiber. This stronger uniqueness (for every smooth gentle Fano $T$-manifold $X$) has the following application.

**Corollary 3.5.12** (of the uniqueness statement of Conjecture 3.5.11 (2))**.** The moduli space $\boldsymbol{\mathcal{K}}_{T,\chi} \to \mathcal{K}_{T,\chi}$ we constructed in Theorem 3.4.8 is good in the sense of Alper [Alp1]. (In our case, the cohomological affineness should be defined as the exactness of the push-forward functor $Coh(\boldsymbol{\mathcal{K}}_{T,\chi}) \to Mod(\mathcal{K}_{T,\chi})$. )

Actually, using the uniqueness of the degeneration in the sense of [BHJ], we can show that the étale morphism $[B/R] \to \boldsymbol{\mathcal{K}}_{T,\chi}$ is an open embedding. Then the corollary follows from the fact that $[BK^c/K^c] \to BK^c /\!\!/ K^c$ is a good moduli space. Recall that we have already shown the central fiber of the degeneration is unique, which we used to prove that the morphism $[B/R] \to BK^c /\!\!/ K^c$ factors through $\mathrm{Im}[B/R] \subset \boldsymbol{\mathcal{K}}_{T,\chi}$. There may be other ways to show this naturally expected corollary.

## 3.6 Appendix: Complex analytic stacks

In this Appendix A, we briefly review some general notions and examples of stacks which we used in section 4. As we work only over the category (or more precisely, the site) $\mathbb{C}$**an** of complex spaces, we do not introduce stacks in full generality, which actually work over any site such as the étale sites of schemes or algebraic spaces, the site of $C^\infty$-manifolds and so on. The interested readers should also refer to [SPA, FGA-book] for stacks in full generality.

### 3.6.1 Fibred category

Recall that we denote by $\mathbb{C}$**an** the category of complex spaces, which are not assumed to be reduced nor irreducible. The set of holomorphic morphisms between complex spaces $U$ and $V$ is denoted by $\mathrm{Holo}(U, V)$.

**Definition 3.6.1** (fibred category)**.** Let $\boldsymbol{\mathcal{F}}$ be a category and $p : \boldsymbol{\mathcal{F}} \to \mathbb{C}\mathbf{an}$ be a functor to the category of complex spaces. The functor $p : \boldsymbol{\mathcal{F}} \to \mathbb{C}\mathbf{an}$ is called a *fibred category* over $\mathbb{C}$**an** if it satisfies the following properties. For any holomorphic morphism $f : X \to Y$ between complex spaces and any object $\eta \in \mathrm{Obj}(\boldsymbol{\mathcal{F}})$, there exists an object $\xi \in \mathrm{Obj}(\boldsymbol{\mathcal{F}})$ and a strongly cartesian morphism $\phi : \xi \to \eta \in \mathrm{Mor}(\boldsymbol{\mathcal{F}})$ over $f$.

Here the morphism $\phi : \xi \to \eta$ is called *strongly cartesian* if it enjoys the following universal property: for any complex space $X'$, any holomorphic morphism $g : X' \to X$, any object $\xi' \in \mathrm{Obj}(\boldsymbol{\mathcal{F}})$ with $p(\xi') = X'$ and any morphism $\phi' : \xi' \to \eta \in \mathrm{Mor}(\boldsymbol{\mathcal{F}})$ with $p(\phi') = f \circ g$, there exists a unique morphism $\chi : \xi' \to \xi$ such that $\phi' = \phi \circ \chi$ and $p(\chi) = g$.

$$\begin{tikzcd}
\boldsymbol{\mathcal{F}} \arrow[d, "p"] & \xi' \arrow[r, dashed, "\chi"] \arrow[rr, bend left, "\phi'"] & \xi \arrow[r, "\phi"] & \eta \\
\mathbb{C}\mathbf{an} & X' \arrow[r, "g"] \arrow[rr, bend left, "f \circ g"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Let $X$ be a complex space and $p : \boldsymbol{\mathcal{F}} \to \mathbb{C}\mathbf{an}$ be a fibred category. We denote by $\boldsymbol{\mathcal{F}}(X)$ the subcategory of $\boldsymbol{\mathcal{F}}$ consisting of objects $\xi \in \mathrm{Obj}(\boldsymbol{\mathcal{F}})$ with $p(\xi) = X$ and morphisms $\phi$ with $p(\phi) = \mathrm{id}_X$. We call $\boldsymbol{\mathcal{F}}$ (or more precisely $\boldsymbol{\mathcal{F}} \to \mathbb{C}\mathbf{an}$) a *category fibred in groupoids* if morphisms in $\boldsymbol{\mathcal{F}}(X)$ are all invertible for any complex space $X$.

A functor $f : \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{G}}$ between two fibred categories is called a *morphism* of fibred categories if $p_{\boldsymbol{\mathcal{F}}} = p_{\boldsymbol{\mathcal{G}}} \circ f$ (strictly) and $f$ maps strongly cartesian morphisms in $\boldsymbol{\mathcal{F}}$ to strongly cartesian morphisms in $\boldsymbol{\mathcal{G}}$. We can also consider 2-*morphisms* between two (1-)morphisms $f, g : \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{G}}$ which are just natural transformations $t : f \to g$ satisfying $p_{\boldsymbol{\mathcal{G}}}(t_\xi : f(\xi) \to g(\xi)) = \mathrm{id}_{p_{\boldsymbol{\mathcal{F}}}(\xi)}$ for all $\xi \in \mathrm{Obj}(\boldsymbol{\mathcal{F}})$.

The functor $\mathbb{C}\mathbf{an}_X \to \mathbb{C}\mathbf{an} : (\xi : S \to X) \mapsto S$, where $\mathbb{C}\mathbf{an}_X$ denotes the category of holomorphic morphisms $\xi : S \to X$, is a typical example of category fibred in groupoids (actually in sets). A holomorphic morphism of complex spaces $f : X \to Y$ gives the morphism $\mathbb{C}\mathbf{an}_X \to \mathbb{C}\mathbf{an}_Y$ which maps an object $\xi : S \to X$ to the object $f \circ \xi : S \to Y$. On the other hand, a morphism $f : \mathbb{C}\mathbf{an}_X \to \mathbb{C}\mathbf{an}_Y$ as fibred categories gives a holomorphic morphism $f(\mathrm{id}_X) : X \to Y$. Therefore, we have a canonical fully faithful embedding of $\mathbb{C}\mathbf{an}$ to the (2-)category of fibred categories. So we often abbreviate $\mathbb{C}\mathbf{an}_X$ as $X$.

*Example* 3.6.2. Let $a : X \times G \to X$ be a holomorphic action of a complex Lie group $G$ to a complex space $X$. We denote by $[X/G]_p$[1] the fibred category

[1]The symbol $_p$ means that this fibred category is not a stack in general; it is just a pre-stack.

(in groupoids) defined as follows.

1. Its objects are holomorphic morphisms $\xi : S \to X$ from some complex spaces $S$.

2. Its morphisms $\xi_S \to \eta_T$ are the pairs $(f, \phi)$ of holomorphic morphisms $f : S \to T$ and $\phi : S \to X \times G$ satisfying $p_1 \circ \phi = \xi$ and $a \circ \phi = \eta \circ f$.

3. Its functor $[X/G]_p \to \mathbb{C}\mathbf{an}$ maps objects $\xi_S$ to $S$ and morphisms $(f, \phi) : \xi_S \to \eta_T$ to $f : S \to T$.

Objects in the fibred category $[X/G]_p$ coincide with $X = \mathbb{C}\mathbf{an}_X$, but morphisms are different. For instance, two objects $x, y : \mathrm{pt} \to X$ in $[X/G]_p$ are isomorphic if and only if there exists an element $g \in G$ with $xg = y$. We have the morphism $X \to [X/G]_p$ of fibred categories defined by $\xi_S \mapsto \xi_S$.

There is another related fibred category $[\![X/G]\!]$ with a good geometric feature.

*Example* 3.6.3. We denote by $[\![X/G]\!]$ the fibred category (in groupoids) defined as follows.

1. An object consists of a triple $(S, P, \xi)$ where $S$ is a complex space, $P$ is a principal $G$-holomorphic bundle over $S$ and $\xi : P \to X$ is a $G$-equivariant holomorphic morphism.

2. A morphism $(S, P, \xi) \to (T, Q, \eta)$ is a pair $(f, \phi)$ where $f : S \to T$ is a holomorphic morphism and $\phi : P \to Q$ is a $G$-equivariant holomorphic morphism over $f$ which induces an biholomorphism $P \cong S \times_T Q$, and satisfies $\xi = \xi' \circ \phi$.

3. Its functor $[\![X/G]\!] \to \mathbb{C}\mathbf{an}$ maps objects $(S, P, \xi)$ to $S$ and morphisms $(f, \phi) : (S, P, \xi) \to (T, Q, \eta)$ to $f : S \to T$.

We have the morphism $[X/G]_p \to [\![X/G]\!]$ of fibred categories which maps an object $\xi : S \to X$ to the object $(S, S \times G, a \circ (\xi \times \mathrm{id}_G))$ and a morphism $(f, \phi) : \xi_S \to \eta_T$ to the morphism $(f, f \times \phi)$. This is a typical example of the 'stackification' we treat in the next subsection. The fibred category $[\![X/G]\!]$ is called a *quotient stack*.

When the action is proper free, then there exists a complex space $X/G$, a holomorphic morphism $X \to X/G$ and an isomorphism $X/G \cong [\![X/G]\!]$ of fibred categories, which is compatible with $X \to X/G$ and $X \to [\![X/G]\!]$.

Let us see another example generalizing $[X/G]_p$. A *holomorphic groupoid* consists of the following data $(X, R, s, t, c)$:

1. $X$ and $R$ are complex spaces.

2. $s$ and $T$ are holomorphic morphisms from $R$ to $X$.

3. $c : R \times_{s,X,t} R \to R$ is a holomorphic morphism.

These data are to satisfy the following rules for any complex space $S$:

1. For every holomorphic morphism $\xi \in \mathrm{Holo}(S, X)$, there exists a holomorphic morphism $e_\xi \in \mathrm{Holo}(S, R)$ such that $c \circ (e_\xi \times \phi) = \phi$ and $c \circ (\psi \times e_\xi) = \psi$ for any pairs $(e_\xi, \phi)$, $(\psi, e_\xi)$ with $s \circ e_\xi = t \circ \phi$ and $s \circ \psi = t \circ e_\xi$.

2. The equality $c \circ ((c \circ (\phi \times \psi)) \times \chi) = c \circ (\phi \times (c \circ (\psi \times \chi)))$ holds for any $\phi, \psi, \chi \in \mathrm{Holo}(S, R)$ with $s \circ \phi = t \circ \psi$ and $s \circ \psi = t \circ \chi$.

3. For any $\phi \in \mathrm{Holo}(S, R)$, there exists a $\psi \in \mathrm{Holo}(S, R)$ such that $s \circ \phi = t \circ \psi = \xi$, $s \circ \psi = t \circ \phi = \eta$ and $c \circ (\phi \times \psi) = e_\eta$, $c \circ (\psi \times \phi) = e_\xi$.

This condition is equivalent to say that $\mathrm{Holo}(S, X)$ forms an abstract groupoid whose morphisms $\xi \to \eta$ are $\phi \in \mathrm{Holo}(S, R)$ with $s \circ \phi = \xi$ and $t \circ \phi = \eta$, and composition is given by $c$.

A holomorphic group action $a : X \times G \to X$ gives an example of holomorphic groupoid with $R = X \times G$, $s = p_1$, $t = a$ and $c = \mathrm{id} \times \mu : X \times G \times G \to X \times G$. If $u : U \to X$ is a holomorphic morphism, then we can consider the pull-back holomorphic groupoid $(U, (U \times U) \times_{u \times u, X \times X, s \times t} R, s', t', c')$.

*Example* 3.6.4. We denote by $[X/R]_p$ the fibred category (in groupoids) defined as follows.

1. Its objects are holomorphic morphisms $\xi : S \to X$ from some complex spaces $S$.

2. Its morphisms $\xi_S \to \eta_T$ are the pairs $(f, \phi)$ of holomorphic morphisms $f : S \to T$ and $\phi : S \to R$ satisfying $s \circ \phi = \xi$ and $t \circ \phi = \eta \circ f$.

3. Its functor $[X/R]_p \to \mathbb{C}\mathbf{an}$ maps objects $\xi_S$ to $S$ and morphisms $(f, \phi) : \xi_S \to \eta_T$ to $f : S \to T$.

Here is our interested fibred category from Definition 3.4.1.

**Lemma 3.6.5.** The category $\boldsymbol{\mathcal{K}}_{T,\chi}$ and $\boldsymbol{\mathcal{K}}^s_{T,\chi}$ forms a fibred category by the functor $\boldsymbol{\mathcal{K}}^{(s)}_{T,\chi} \to \mathbb{C}\mathbf{an} : (\pi : \mathcal{M} \to S, \alpha) \mapsto S$.

This is just because the following cartesian diagram gives a cartesian morphism for any holomorphic morphism $f : X \to Y$ between complex spaces and any object $(\pi : \mathcal{M} \to Y, \alpha) \in \boldsymbol{\mathcal{K}}^{(s)}_{T,\mathcal{X}}$.

$$\begin{array}{ccc} X \times_Y \mathcal{M} & \longrightarrow & \mathcal{M} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ X & \xrightarrow{f} & Y \end{array}$$

The correspondence $(\pi : \mathcal{M} \to Y, \alpha) \mapsto (f^*\pi : X \times_Y \mathcal{M} \to X, f^*\alpha)$ gives a functor $\boldsymbol{\mathcal{K}}_{T,\mathcal{X}}(Y) \to \boldsymbol{\mathcal{K}}_{T,\mathcal{X}}(X)$. It looks like that this provides a functor $X \mapsto \boldsymbol{\mathcal{K}}_{T,\mathcal{X}}(X)$ from the category $\mathbb{C}\mathbf{an}$ to the "category" of groupoids, but actually does not. This nuisance comes from the set theoretical fact that $X \times_{f,Y} (Y \times_{g,Z} \mathcal{M}) \neq X \times_{g\circ f,Z} \mathcal{M}$; they are not exactly the same objects but just naturally isomorphic. This is the reason why we should formulate things in terms of fibred category.

### 3.6.2 Descent data

We introduce descent data of a fibred category over $\mathbb{C}\mathbf{an}$.

**Definition 3.6.6** (descent data)**.** Let $p : \boldsymbol{\mathcal{F}} \to \mathbb{C}\mathbf{an}$ be a fibred category, $X$ be a complex space, $\mathcal{U} := \{i_\alpha : U_\alpha \hookrightarrow X\}_{\alpha\in A}$ be an open cover of $X$ (in the real topology). We denote by $u_{\alpha,\beta} : U_\alpha \cap U_\beta \hookrightarrow U_\alpha$ the inclusion morphism to the first factor and by $u_{\alpha\beta,\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \hookrightarrow U_\alpha \cap U_\beta$ the inclusion morphism to the intersection of the first and second factor ($u_{\alpha\beta,\gamma} = u_{\beta\alpha,\gamma}$). Put $A_2 := A \times A/\mathfrak{S}_2$, $A_3 := A \times A \times A/\mathfrak{S}_3$.

A *descent datum* of $\boldsymbol{\mathcal{F}}$ over $(X, \mathcal{U})$ consists of the following data $\mathfrak{D} = (\Xi_1, \Xi_2, \Xi_3, \Theta_2, \Theta_3)$:

$$\begin{aligned} \Xi_1 &:= \{\xi_\alpha \in \boldsymbol{\mathcal{F}}(U_\alpha)\}_{\alpha\in A} \\ \Xi_2 &:= \{\xi_{\alpha\beta} \in \boldsymbol{\mathcal{F}}(U_\alpha \cap U_\beta)\}_{\{\alpha,\beta\}\in A_2} \\ \Xi_3 &:= \{\xi_{\alpha\beta\gamma} \in \boldsymbol{\mathcal{F}}(U_\alpha \cap U_\beta \cap U_\gamma)\}_{\{\alpha,\beta,\gamma\}\in A_3} \end{aligned}$$

are sets of objects in $\boldsymbol{\mathcal{F}}$ and

$$\begin{aligned} \Theta_2 &:= \{\theta_{\alpha,\beta} : \xi_{\alpha\beta} \to \xi_\alpha \mid \theta_{\alpha,\beta} \text{ is cartesian over } u_{\alpha,\beta}\}_{(\alpha,\beta)\in A^2} \\ \Theta_3 &:= \{\theta_{\alpha\beta,\gamma} : \xi_{\alpha\beta\gamma} \to \xi_{\alpha\beta} \mid \theta_{\alpha\beta,\gamma} \text{ is cartesian over } u_{\alpha\beta,\gamma}\}_{(\{\alpha,\beta\},\gamma)\in A_2\times A} \end{aligned}$$

are sets of cartesian morphisms in $\boldsymbol{\mathcal{F}}$. These data must satisfy

$$\theta_{\alpha,\beta} \circ \theta_{\alpha\beta,\gamma} = \theta_{\alpha,\gamma} \circ \theta_{\gamma\alpha,\beta}$$

for any $\alpha, \beta, \gamma \in A$.

$$\begin{tikzcd} \boldsymbol{\mathcal{F}} \arrow[r, "p"] & \mathbb{C}\mathbf{an} \end{tikzcd}$$

$$\begin{tikzcd}
\xi_{\alpha\beta\gamma} \arrow[rr, "\theta_{\beta\gamma,\alpha}"] \arrow[dr, "\theta_{\alpha\beta,\gamma}"] \arrow[dd, "\theta_{\gamma\alpha,\beta}"'] & & \xi_{\beta\gamma} \arrow[dr, "\theta_{\beta,\gamma}"] \arrow[dd] & \\
& \xi_{\alpha\beta} \arrow[rr, "\theta_{\beta,\alpha}"] \arrow[dd, "\theta_{\alpha,\beta}"] & & \xi_{\beta} \\
\xi_{\gamma\alpha} \arrow[rr] \arrow[dr, "\theta_{\alpha,\gamma}"'] & & \xi_{\gamma} & \\
& \xi_{\alpha} & &
\end{tikzcd}
\qquad
\begin{tikzcd}
U_{\alpha\beta\gamma} \arrow[rr, "u_{\beta\gamma,\alpha}"] \arrow[dr, "u_{\alpha\beta,\gamma}"] \arrow[dd, "u_{\gamma\alpha,\beta}"'] & & U_{\beta\gamma} \arrow[dr, "u_{\beta,\gamma}"] \arrow[dd] & \\
& U_{\alpha\beta} \arrow[rr, "u_{\beta,\alpha}"] \arrow[dd, "u_{\alpha,\beta}"] & & U_{\beta} \arrow[dd, "i_\beta"] \\
U_{\gamma\alpha} \arrow[rr] \arrow[dr, "u_{\alpha,\gamma}"'] & & U_{\gamma} \arrow[dr, "i_\gamma"] & \\
& U_{\alpha} \arrow[rr, "i_\alpha"] & & X
\end{tikzcd}$$

A descent datum $\mathfrak{D} = (\Xi_1, \Xi_2, \Xi_3, \Theta_2, \Theta_3)$ is called *effective* if there exists an object $\xi \in \boldsymbol{\mathcal{F}}(X)$ and a set of morphisms

$$\Theta_1 := \{\theta_\alpha : \xi_\alpha \to \xi \mid \theta_\alpha \text{ is cartesian over } i_\alpha\}_{\alpha \in A}$$

satisfying

$$\theta_\alpha \circ \theta_{\alpha,\beta} = \theta_\beta \circ \theta_{\beta,\alpha}$$

for any $\alpha, \beta \in A$. We define an *effective descent datum* of $\boldsymbol{\mathcal{F}}$ over $(X, \mathcal{U})$ to be an object consisting of data $\mathfrak{D}_+ = (\mathfrak{D}, \xi, \Theta_1) = (\xi, \Xi_1, \Xi_2, \Xi_3, \Theta_1, \Theta_2, \Theta_3)$ as above.

*Remark* 3.6.7. Note that

- Every descent datum of $\mathbb{C}\mathbf{an}_X$ is effective.
- There are descent data of $[X/G]_p$ which are not effective, in general.
- Every descent datum of $[\![X/G]\!]$ is effective.

As for the second item, consider the fibred category $[(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*]_p$ for example. More explicitly, let $\mathcal{U}$ be an open cover of $\mathbb{C}P^1$ defined by two open subsets $U_\alpha := \{(z_1 : z_2) \mid z_2 \neq 0\}, U_\beta := \{(z_1 : z_2) \mid z_1 \neq 0\}$ and let $\xi_\alpha : U_\alpha \to \mathbb{C}^2 \setminus \{0\}, \xi_\beta : U_\beta \to \mathbb{C}^2 \setminus \{0\}$ be morphisms defined by $\xi_\alpha(z_1 : z_2) := (z_1/z_2, 1), \xi_\beta(z_1 : z_2) := (1, z_2/z_1)$, respectively. Consider a descent datum over $(\mathbb{C}P^1, \mathcal{U})$ with $\Xi_1 := \{\xi_\alpha, \xi_\beta\}$ given by an obvious way. In order to be effective, this descent datum should define a non-constant morphism $\mathbb{C}P^1 \to \mathbb{C}^2 \setminus \{0\}$ which is *isomorphic* (not equal) to $\xi_\alpha, \xi_\beta$ when restricted to each open set, but this is impossible as every holomorphic map $\mathbb{C}P^1 \to \mathbb{C}^2 \setminus \{0\}$ is constant. So this descent datum is not effective in this fibred category.

On the other hand, the corresponding descent datum in the fibred category $[\![(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*]\!]$ becomes effective, completed by the object $\mathbb{C}P^1 \xleftarrow{\mathbb{C}^*} (\mathbb{C}^2 \setminus \{0\}) \xrightarrow{\mathrm{id}} (\mathbb{C}^2 \setminus \{0\})$ in $[\![(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*]\!]$. Actually, $[\![(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*]\!]$ is isomorphic to $\mathbb{C}P^1$ as fibred categories.

**Definition 3.6.8** (morphism of descent data)**.** Let $\mathfrak{D} = (\Xi_1, \Xi_2, \Xi_3, \Theta_2, \Theta_3)$, $\mathfrak{D}' = (\Xi_1', \Xi_2', \Xi_3', \Theta_2', \Theta_3')$ be two descent data of $\boldsymbol{\mathcal{F}}$ over $(X, \mathcal{U})$. A *morphism* from $\mathfrak{D}$ to $\mathfrak{D}'$ is a triple $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ of sets of morphisms

$$\Phi_1 := \{\phi_\alpha : \xi_\alpha \to \xi_\alpha' \in \boldsymbol{\mathcal{F}}(U_\alpha) \mid \xi_\alpha \in \Xi_1, \xi_\alpha' \in \Xi_1'\}_{\alpha \in A}$$
$$\Phi_2 := \{\phi_{\alpha\beta} : \xi_{\alpha\beta} \to \xi_{\alpha\beta}' \in \boldsymbol{\mathcal{F}}(U_\alpha \cap U_\beta) \mid \xi_{\alpha\beta} \in \Xi_2, \xi_{\alpha\beta}' \in \Xi_2'\}_{\{\alpha,\beta\} \in A_2}$$
$$\Phi_3 := \{\phi_{\alpha\beta\gamma} : \xi_{\alpha\beta\gamma} \to \xi_{\alpha\beta\gamma}' \in \boldsymbol{\mathcal{F}}(U_\alpha \cap U_\beta \cap U_\gamma) \mid \xi_{\alpha\beta\gamma} \in \Xi_3, \xi_{\alpha\beta\gamma}' \in \Xi_3'\}_{\{\alpha,\beta,\gamma\} \in A_3}$$

in $\boldsymbol{\mathcal{F}}$ satisfying

$$\phi_\alpha \circ \theta_{\alpha,\beta} = \theta_{\alpha,\beta}' \circ \phi_{\alpha\beta} \quad \text{and} \quad \phi_{\alpha\beta} \circ \theta_{\alpha\beta,\gamma} = \theta_{\alpha,\beta,\gamma}' \circ \phi_{\alpha\beta\gamma}$$

for all $\alpha, \beta, \gamma \in A$. Descent data naturally form a category with these morphisms. We denote by $\boldsymbol{\mathcal{F}}_{\mathrm{des}}(X, \mathcal{U})$ the category of descent data.

Let $\mathfrak{D}_+ = (\mathfrak{D}, \xi, \Theta_1)$, $\mathfrak{D}_+' = (\mathfrak{D}', \xi', \Theta_1')$ be two effective descent data of $\boldsymbol{\mathcal{F}}$ over $(X, \mathcal{U})$. A *morphism* from $\mathfrak{D}_+$ to $\mathfrak{D}_+'$ is a quadruple $\Phi_+ = (\phi, \Phi_1, \Phi_2, \Phi_3)$ where $(\Phi_1, \Phi_2, \Phi_3)$ gives a morphism of corresponding descent data and $\phi : \xi \to \xi'$ is a morphism in $\boldsymbol{\mathcal{F}}(X)$ satisfying

$$\phi \circ \theta_\alpha = \theta_\alpha' \circ \phi_\alpha$$

for any $\alpha \in A$. We denote by $\boldsymbol{\mathcal{F}}_{\mathrm{eff}}(X, \mathcal{U})$ the category of effective descent data.

### 3.6.3 Stacks over the complex analytic site $\mathbb{C}\mathbf{an}$

We can consider the forgetful functors $\boldsymbol{\mathcal{F}}_{\mathrm{eff}}(X,\mathcal{U}) \to \boldsymbol{\mathcal{F}}_{\mathrm{des}}(X,\mathcal{U})$ defined by $\mathfrak{D}_+ = (\mathfrak{D}, \xi, \Theta_1) \mapsto \mathfrak{D}$ and $\boldsymbol{\mathcal{F}}_{\mathrm{eff}}(X,\mathcal{U}) \to \boldsymbol{\mathcal{F}}(X)$ defined by $\mathfrak{D}_+ = (\mathfrak{D}, \xi, \Theta_1) \mapsto \xi$. The latter functor $\boldsymbol{\mathcal{F}}_{\mathrm{eff}}(X,\mathcal{U}) \to \boldsymbol{\mathcal{F}}(X)$ is fully faithful and essentially surjective. Therefore there is an inverse functor $\boldsymbol{\mathcal{F}}(X) \to \boldsymbol{\mathcal{F}}_{\mathrm{eff}}(X,\mathcal{U})$ (assuming the axiom of global choice). As for our fibred category $\boldsymbol{\mathcal{K}}_{T,\chi}$, there is a canonical choice[2] of the inverse functor defined by

$$(\pi : \mathcal{M} \to S) \mapsto (\mathcal{M} \to S, \{\pi^{-1}(U_\alpha) \to U_\alpha\}_\alpha, \{\pi^{-1}(U_\alpha \cap U_\beta) \to U_\alpha \cap U_\beta\}_{\alpha,\beta}, \ldots).$$

However, in general there is no canonical choice of this inverse functor; there needs an additional choice of $(\mathfrak{D}, \Theta_1)$ compatible to $\xi$, which is not unique as object but unique only up to isomorphisms.

**Definition 3.6.9** ($\mathbb{C}\mathbf{an}$-stack)**.** A fibred category $p : \boldsymbol{\mathcal{F}} \to \mathbb{C}\mathbf{an}$ is called a *stack over* $\mathbb{C}\mathbf{an}$ or just $\mathbb{C}\mathbf{an}$-*stack* if it satisfies the following two conditions for any complex space $X$ and any open cover $\mathcal{U}$ of $X$.

1. The functor $\boldsymbol{\mathcal{F}}_{\mathrm{eff}}(X,\mathcal{U}) \to \boldsymbol{\mathcal{F}}_{\mathrm{des}}(X,\mathcal{U})$ is fully faithful.
2. The functor $\boldsymbol{\mathcal{F}}_{\mathrm{eff}}(X,\mathcal{U}) \to \boldsymbol{\mathcal{F}}_{\mathrm{des}}(X,\mathcal{U})$ is essentially surjective.

*Remark* 3.6.10. If we have a choice of pull back $f \mapsto f^*\xi$ with a morphism $f^*\xi \to \xi$ so that it is cartesian over $f : S \to X$, we can consider a contravariant functor defined by

$$Mor_X(\xi, \eta) : \mathbb{C}\mathbf{an}_X^{\mathrm{op}} \to \mathbf{Sets} : (f : S \to X) \mapsto \mathrm{Hom}_{\boldsymbol{\mathcal{F}}(S)}(f^*\xi, f^*\eta),$$

where $\mathbb{C}\mathbf{an}_X^{\mathrm{op}}$ stands for the opposite category of $\mathbb{C}\mathbf{an}_X$. Then the first condition of the above definition is equivalent to say that the functor $Mor_X(\xi, \eta)$ is a sheaf on the site $\mathbb{C}\mathbf{an}_X$.

It is customary to denote by $Isom_X(\xi, \eta)$ the functor $Mor_X(\xi, \eta)$ when $\boldsymbol{\mathcal{F}}$ is a category fibred in groupoids, as every morphism in $\boldsymbol{\mathcal{F}}(X)$ is an isomorphism.

We can consider a related fibred category $\boldsymbol{\mathcal{M}or}_X(\xi, \eta)$ (in setoids) without a choice of pull back. The category consists of objects $(f_S, \phi_\xi, \phi_\eta)$ where

[2]This is well-defined because $U_\alpha \cap U_\beta = U_\beta \cap U_\alpha$ as complex spaces, in particular as sets. On the other hand, $U_\alpha \times_X U_\beta \neq U_\beta \times_X U_\alpha$ as sets, though they are canonically isomorphic, because of the set theoretical fact $(a, b) = \{\{a\}, \{a, b\}\} \neq \{\{b\}, \{b, a\}\} = (b, a)$.

$f : S \to X$ is a holomorphic morphism of complex spaces and $\phi_\xi : \xi_S \to \xi, \phi_\eta : \eta_S \to \eta \in \mathrm{Mor}(\boldsymbol{\mathcal{F}})$ are cartesian arrows over $f$. Its morphisms $(f_S, \phi_\xi, \phi_\eta) \to (f_{S'}, \phi'_\xi, \phi'_\eta)$ are triples $(g, \psi_\xi, \psi_\eta)$ where $g : S \to S'$ is a holomorphic morphism of complex spaces and $\psi : \xi_S \to \xi_{S'}, \psi_\eta : \eta_S \to \eta_{S'}$ are cartesian arrows over $g$ satisfying $f_S = f_{S'} \circ g$, $\phi_\xi = \phi'_\xi \circ \psi_\xi$ and $\phi_\eta = \phi'_\eta \circ \psi_\eta$.

For any fibred category, we can always associate a stack in a canonical way. Here is the fact from [SPA, TAG 02ZN, 0435].

**Proposition 3.6.11.** Let $p : \boldsymbol{\mathcal{F}} \to \mathbb{C}\mathbf{an}$ be a fibred category. Suppose we have a choice of pull back $(f, \xi) \mapsto f^*\xi$ (just for simplicity). Then there exists a $\mathbb{C}\mathbf{an}$-stack $\boldsymbol{\mathcal{F}}'$ (with a choice of pull back) and a morphism $s : \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{F}}'$ of fibred categories with the following properties.

1. For every complex space $X$ and any $\xi, \eta \in \mathrm{Obj}(\boldsymbol{\mathcal{F}}(X))$, the morphism of presheaf $Mor_X(\xi, \eta) \to Mor_X(s(\xi), s(\eta))$ is a sheafification of $Mor_X(\xi, \eta)$.

2. For every complex space $X$ and any $\xi' \in \mathrm{Obj}(\boldsymbol{\mathcal{F}}'(X))$, there exists an open cover $\mathcal{U} = \{i_\alpha : U_\alpha \to X\}_\alpha$ of $X$ such that $i_\alpha^*\xi'$ is isomorphic to $s(\xi_\alpha)$ for some $\xi_\alpha \in \mathrm{Obj}(\boldsymbol{\mathcal{F}})$ for every $\alpha$.

3. Given a $\mathbb{C}\mathbf{an}$-stack $\boldsymbol{\mathcal{G}}$ and a morphism $g : \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{G}}$ of fibred categories, there exists a morphism $g' : \boldsymbol{\mathcal{F}}' \to \boldsymbol{\mathcal{G}}$ of fibred categories such that there exists a 2-isomorphism between $g$ and $g' \circ s$.

The last property actually follows from the first two properties. A stack $\boldsymbol{\mathcal{F}}'$ with the last property is called a *stackification* of $\boldsymbol{\mathcal{F}}$ and the stack $\boldsymbol{\mathcal{F}}'$ constructed in the proof of this proposition as *the stackification* of $\boldsymbol{\mathcal{F}}$ (a fixed construction is in mind). We denote by $[X/G]$, $[X/R]$ the stackification of the fibred category $[X/G]_p$, $[X/R]_p$ respectively and call them *the quotient stack*. The stack $[\![X/G]\!]$ is a stackification of the fibred category $[X/G]_p$, so it is (canonically) isomorphic to the stackification $[X/G]$.

A 2-fibre product [SPA, 003Q] of fibred categories can be calculated as follows. We refer to this construction as *the 2-fibre product of fibred categories.*

**Proposition 3.6.12.** Let $\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{H}}$ be fibred categories over $\mathbb{C}\mathbf{an}$ and $f : \boldsymbol{\mathcal{F}} \to \boldsymbol{\mathcal{H}}$, $g : \boldsymbol{\mathcal{G}} \to \boldsymbol{\mathcal{H}}$. The fibred category $\boldsymbol{\mathcal{E}}$ defined as follows enjoys the universal property of 2-fibre product.

1. An object of $\boldsymbol{\mathcal{E}}$ is a quadruple $(X, \xi, \eta, \phi)$ where $X$ is a complex space, $\xi$ is an object in $\boldsymbol{\mathcal{F}}(X)$, $\eta$ is an object in $\boldsymbol{\mathcal{G}}(X)$ and $\phi : f(\xi) \to g(\eta)$ is an isomorphism in $\boldsymbol{\mathcal{H}}(X)$.

2. A morphism $(X, \xi, \eta, \phi) \to (Y, \xi', \eta', \phi')$ is a pair $(\sigma, \tau)$ where $\sigma : \xi \to \xi'$ is a morphism in $\boldsymbol{\mathcal{F}}$ and $\tau : \eta \to \eta'$ is a morphism in $\boldsymbol{\mathcal{G}}$ satisfying $p(\sigma) = p(\tau) : X \to Y$ and $g(\tau) \circ \phi = \phi' \circ f(\sigma)$.

When $\boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{H}}$ are all $\mathbb{C}$**an**-stacks, the stackification of the fibred category $\boldsymbol{\mathcal{E}}$ is denoted by $\boldsymbol{\mathcal{F}} \times_{f,\boldsymbol{\mathcal{H}},g} \boldsymbol{\mathcal{G}}$. The $\mathbb{C}$**an**-stack $\boldsymbol{\mathcal{F}} \times_{f,\boldsymbol{\mathcal{H}},g} \boldsymbol{\mathcal{G}}$ satisfies the universal property of 2-fibre product in the 2-category of $\mathbb{C}$**an**-stacks.

The following verifying process might help the readers' better understanding of the notion of descent. See Definition 3.4.1 for the definition of $\boldsymbol{\mathcal{K}}_{T,\chi}, \boldsymbol{\mathcal{K}}^s_{T,\chi}$.

**Lemma 3.6.13.** The fibred categories $\boldsymbol{\mathcal{K}}_{T,\chi}$, $\boldsymbol{\mathcal{K}}^s_{T,\chi}$ are $\mathbb{C}$**an**-stacks.

*Proof.* For abbreviation, we let $\mathcal{M}$ stand for $(\pi : \mathcal{M} \to S, \alpha) \in \boldsymbol{\mathcal{K}}_{T,\chi}$. Let $S$ be a complex space, $\mathcal{U} = \{U_\alpha\}_\alpha$ be an open cover of $S$ and $\mathfrak{D} = (\Xi_1, \Xi_2, \Xi_3, \Theta_2, \Theta_3)$ be a descent datum of $\boldsymbol{\mathcal{K}}_{T,\chi}$ over $(X, \mathcal{U})$. Since $\theta_{\alpha,\beta} \in \Theta_2$ is cartesian, it induces an isomorphism

$$\tilde{\theta}_{\alpha,\beta} : \mathcal{M}_{\alpha\beta} \xrightarrow{\sim} \mathcal{M}_\alpha|_{U_\alpha \cap U_\beta}.$$

So we obtain an isomorphism

$$\theta'_{\beta\alpha} := \tilde{\theta}_{\beta,\alpha} \circ \tilde{\theta}^{-1}_{\alpha,\beta} : \mathcal{M}_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} \mathcal{M}_\beta|_{U_\alpha \cap U_\beta}.$$

Similarly, we obtain an isomorphism

$$(\tilde{\theta}_{\beta,\alpha}|_{U_{\alpha\beta\gamma}} \circ \tilde{\theta}_{\beta\alpha,\gamma}) \circ (\tilde{\theta}_{\alpha,\beta}|_{U_{\alpha\beta\gamma}} \circ \tilde{\theta}_{\alpha\beta,\gamma})^{-1} : \mathcal{M}_\alpha|_{U_{\alpha\beta\gamma}} \xrightarrow{\sim} \mathcal{M}_\beta|_{U_{\alpha\beta\gamma}}$$

which we denote by $\theta'_{\beta\alpha,\gamma}$, from the cartesian arrow $\theta_{\alpha\beta,\gamma} \in \Theta_3$.

From the condition $\theta_{\alpha,\beta} \circ \theta_{\alpha\beta,\gamma} = \theta_{\alpha,\gamma} \circ \theta_{\gamma\alpha,\beta}$, we obtain $\theta'_{\gamma\beta,\alpha} \circ \theta'_{\beta\alpha,\gamma} = \theta'_{\gamma\alpha,\beta}$ and $\theta'_{\beta\alpha}|_{U_\alpha \cap U_\beta \cap U_\gamma} = \theta'_{\beta\alpha,\gamma}$. So we can glue $\mathcal{M}_\alpha$ together by gluing maps $\theta'_{\beta\alpha}$ and obtain a complex space $\mathcal{M}$ with a natural set of morphisms $\Theta_1 := \{\theta_\alpha : \mathcal{M}_\alpha \to \mathcal{M}\}_{\alpha \in A}$ such that $(\mathcal{M}, \Xi_1, \Xi_2, \Xi_3, \Theta_1, \Theta_2, \Theta_3)$ is an effective descent datum. Therefore the forgetful functor $\mathcal{F}_{\text{eff}}(S, \mathcal{U}) \to \mathcal{F}_{\text{des}}(S, \mathcal{U})$ is essentially surjective.

It is easy to see that $Mor_S(\mathcal{M}, \mathcal{M}')$ is a sheaf on the site $\mathbb{C}\mathbf{an}_X$. □

*Example* 3.6.14. This example is cited from [Alp1, Example 8.2] and must help the readers to understand that gluing good moduli spaces is a non-trivial task. Consider the $\mathbb{C}^*$-action on $\mathbb{C}^2$ by the scalar multiplication. The quotient stack $[(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*]$ is naturally an open sub-stack of the quotient stack $[\mathbb{C}^2/\mathbb{C}^*]$. Both stacks admit good moduli spaces $[(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*] \to \mathbb{C}P^1$, $[\mathbb{C}^2/\mathbb{C}^*] \to \mathbb{C}^2 /\!\!/ \mathbb{C}^* = \mathrm{pt}$ respectively. In spite of the openness of the morphism $[(\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*] \to [\mathbb{C}^2/\mathbb{C}^*]$, the induced morphism $\mathbb{C}P^1 \to \mathrm{pt}$ of good moduli spaces is not open.

# Chapter 4

# Application of Theorem G: Algebraic moduli problems on $\mathbb{Q}$-Fano varieties with KRs

Theorem G in chapter 3 enables us to approach the problem on compactification of the moduli space of Fano manifolds with Kähler–Ricci solitons. When the author wrote [Ino1], Theorem G was missing, so that the author was unable to apply established methods in [Oda2, Oda3, LWX1] on the moduli space of Kähler–Einstein Fano varieties. This is why the author developed another method to construct the moduli space of Fano manifolds with Kähler–Ricci solitons, making use of moment map picture on $\mu$-scalar curvature. Now we have filled the missing piece and so are able to apply the methods in [Oda2, Oda3, LWX1]. We will prove the algebraicity of the moduli space constructed in the previous section. We also outline how we can reduce the moduli problem to a fundamental task on $\mu$K-stability; Conjecture I.

The content is based on the article [Ino3].

## 4.1 Algebraic moduli space of Fano manifolds with KRs

### 4.1.1 Constructibility

#### The YTD conjecture on $\mu$K-stability of Fano varieties

We firstly prepare some terminologies. For a $\mathbb{Q}$-Fano $T$-variety $X$, we call a vector $\xi \in \mathfrak{t}$ *K-optimal* if we have $\mathrm{Fut}^{2\pi}_\xi|_{\mathfrak{t}} \equiv 0$. Such a vector $\xi$ always exists and is unique for each $T$-action by [TZ2, Lemma 2.2]. A torus action on a $\mathbb{Q}$-Fano variety $X$ is called *K-optimal* if the torus is generated by the K-optimal vector $\xi$ of some maximal torus action in $X$. All K-optimal actions on $X$ are conjugate to each other since the K-optimal vector associated to a maximal torus is unique and all maximal tori are conjugate to each other. We call a $\mathbb{Q}$-Fano $T$-variety $X$ *$\mu$K-semistable (resp. $\mu$K-polystable, $\mu$K-stable)* if $T$ is K-optimal and the polarized variety $(X, -K_X)$ is $\mu^{2\pi}_\xi$K-semistable (resp. $\mu^{2\pi}_\xi$K-polystable, $\mu^{2\pi}_\xi$K-stable) with respect to the K-optimal vector $\xi$.

**Conjecture I.** Let $X$ be a $\mathbb{Q}$-Fano variety admitting Kähler–Ricci soliton with a K-optimal torus action $T$. Then it is $\mu$K-polystable with respect to general $T$-equivariant test configurations.

It is already known by [BW] that a $\mathbb{Q}$-Fano variety with Kähler–Ricci soliton is $\mu$K-polystable with respect to *special degenerations*. There are two possible approaches to this conjecture from different perspectives:

- Algebraic approach: As Li–Xu [LX] for the usual K-stability, we show that the $\mu$K-polystability of a $\mathbb{Q}$-Fano variety with respect to general test configurations and with respect to special degenerations are equivalent.

- Analytic approach: As Berman–Darvas–Lu [BDL] for the usual K-stability with smooth $X$, studying $\mu^\lambda_\xi$-Mabuchi functional, we show that any test configuration with vanishing $\mu^\lambda_\xi$-Futaki invariant is equivalent to a product configuration under the existence of $\mu^\lambda_\xi$-cscK metric.

For our interest in this section, it suffices to deal with smoothable $\mathbb{Q}$-Fano variety. We already know that a Fano manifold with Kähler–Ricci soliton is $\mu$K-semistable with respect to general test configurations. So in the smooth case, it is a matter of dealing with test configurations with vanishing $\mu$-Futaki invariant.

The converse claim 'from polystability to the existence of canonical metric' is in general regarded as a difficult direction of the Yau–Tian–Donaldson conjecture. However, for Fano manifolds, it is already proved by Datar–Székelyhidi [DaSz] and [Y. Li] that a $\mu$K-polystable K-smoothable $\mathbb{Q}$-Fano variety admits a Kähler–Ricci soliton, as a variant of the Kähler–Einstein case [CDS, Tian2, SSY], using twisted Kähler–Ricci soliton instead of log Kähler–Einstein metric. The argument heavily employs an argument on Gromov–Hausdorff limit under a uniform estimate on Kähler metrics with Ricci lower bound (the partial $C^0$-estimate), which is not suited for general polarization (cf. [DoSu1]).

### The moduli stack

A *K-family over $B$ of $\mathbb{Q}$-Fano varieties* is a $T$-equivariant proper flat family $\pi : \mathcal{X} \to B$ of $\mathbb{Q}$-Fano varieties with K-optimal $T$-action which enjoys Kollár condition. Here *Kollár condition* means that some reflexive power $\omega_{\mathcal{X}/B}^{[m]}$ of the relative canonical sheaf is $T$-equivariantly isomorphic to a $T$-equivariant line bundle and every reflexive power $\omega_{\mathcal{X}/B}^{[m]}$ commutes with arbitrary base change (cf. [Kov], [BX]). A non-equivariant family may not be a K-optimal torus equivariant family, however, we can stratify the base so that the family restricted on each stratum admits a K-optimal torus action (cf. [Ino1]). We call a K-family $\pi : \mathcal{X} \to B$ *$\mu$K-semistable family* if every fibre is a $\mu$K-semistable $\mathbb{Q}$-Fano $T$-variety.

A *K-smoothable* $\mathbb{Q}$-Fano variety $X$ is a $\mathbb{Q}$-Fano variety which admits a K-family $\pi : \mathcal{X} \to \Delta$ of $\mathbb{Q}$-Fano varieties with an isomorphism $\mathcal{X}_0 \cong X$ whose fibres away from the origin $0 \in \Delta$ are smooth Fano manifolds.

Define the moduli stack $\boldsymbol{\mathcal{M}}^n$ by putting its fibre over $B$ as

$$\boldsymbol{\mathcal{M}}^n(B) := \left\{ \begin{array}{c} \mu K\text{-semistable families over } B \text{ of} \\ K\text{-smoothable } \mathbb{Q}\text{-Fano varieties of dimension } n \end{array} \right\}. \tag{4.1}$$

Let us recall a terminology in [Ino1]. We call a Fano manifold $X$ *gentle* if it admits a special degeneration to a smooth Fano manifold with Kähler–Ricci solitons which is equivariant with respect to a K-optimal action of $X$.

**Proposition 4.1.1.** Let $\alpha \in c_1(X)$ be a Kähler form on $X$. If there exists a twisted Kähler–Ricci soliton $\omega_t$ on $X$ for every $0 \leq t < 1$, which satisfies

$\mathrm{Ric}\omega_t - L_{\xi^J}\omega_t = t\omega_t + (1-t)\alpha$, then $X$ is $\mu$K-semistable. In particular, every gentle Fano manifold is $\mu$K-semistable. (cf. [Ino1, Section 2])

*Sketch of proof.* Gentle Fano manifold admits a twisted Kähler–Ricci soliton for every $t < 1$. Similarly to [Sze-book], we can see that the following twisted $\mu$Mabuchi functional is bounded from below for each $t < 1$:

$$\mathcal{M}_{\alpha,1-t,\xi}(\phi) = \mathcal{M}_\xi(\phi) + (1-t)\mathcal{J}_{\alpha,\xi}(\phi),$$

where we put

$$\mathcal{M}_\xi(\phi) := -\int_0^1 dt \int_X \dot{\phi}_t \hat{s}_\xi^{2\pi} e^{\theta_\xi(\phi_t)} \omega_{\phi_t}^n,$$
$$\mathcal{J}_{\alpha,\xi}(\phi) := n\int_0^1 dt \int_X \dot{\phi}_t (\mathrm{tr}_{\omega_{\phi_t}}(\alpha - \omega_{\phi_t}) + \xi^J \varphi_t) e^{\theta_\xi(\phi_t)} \omega_{\phi_t}^n,$$

where $\varphi_t$ is a function satisfying $\sqrt{-1}\partial\bar{\partial}\varphi_t = \alpha - \omega_t$. On the other hand, as in [DR, Theorem 6.4] (cf. [Lah]), we can see that the following twisted $\mu$Futaki invariant is the slope of the twisted $\mu$Mabuchi functional along any smooth subgeodesic associated to a smooth test configuration $(\mathcal{X}, \mathcal{L})$ dominating the trivial test configuration:

$$\mathrm{Fut}_{1-t,\xi}(\mathcal{X}, \mathcal{L}) = \mathrm{Fut}_\xi(\mathcal{X}, \mathcal{L}) + (1-t)\frac{\mathrm{Ev}_\xi(-K_X - \mathcal{L}.e^{\mathcal{L}})}{\mathrm{Ev}_\xi(e^L)},$$

independent of $\alpha \in c_1(X)$. It follows that $\mathrm{Fut}_{1-t,\xi}(\mathcal{X}, \mathcal{L})$ is non-negative for every $t < 1$. Since $\mathrm{Fut}_{1-t,\xi}(\mathcal{X}, \mathcal{L})$ is continuous on $t$, $\mathrm{Fut}_\xi(\mathcal{X}, \mathcal{L}) = \mathrm{Fut}_{0,\xi}(\mathcal{X}, \mathcal{L})$ is also non-negative. □

We also put $\boldsymbol{\mathcal{M}}^{n,\circ}$ the substack of $\boldsymbol{\mathcal{M}}^n$ consisting of gentle Fano manifolds. It is shown in [Ino1] that $\boldsymbol{\mathcal{M}}^{n,\circ}$ is Artin in analytic category and admits the moduli space.

**Constructibility**

We firstly note the following counterpart of [Oda2, Lemma 2.10] for general Kähler class.

**Lemma 4.1.2.** Let $B$ be a projective manifold with a Hamiltonian action by a torus $T$. Let $c = [\alpha + \mu] \in NS_T(B, \mathbb{R})$ be an equivariant cohomology class. For $x \in B$ and $\eta \in \mathfrak{t}$, we put $f(x,\eta) := -\mu_\eta(\lim_{t\to\infty} x.e^{t\sqrt{-1}\eta})$. Then there is a finite collection $\{B_i\}$ of disjoint constructible subsets of $B$ and a continuous piecewise linear function $\varphi_i$ such that

$$\varphi_i(\eta) = f(x,\eta)$$

for every $x \in B_i$ and $\eta \in \mathfrak{t}$. Each function $\varphi_i$ is convex if moreover $\alpha$ is a Kähler metric.

*Proof.* Since the equivariant class $c$ is of the form $c = \sum_{k=1}^l a_k c_1^T(L_k)$ for some $a_j \in \mathbb{R}$ and some equivariant (holomorphic) line bundles $L_j$, the claim on the lattice $N(T) \subset \mathfrak{t}$ is a consequence of [Oda2, Lemma 2.10] (cf. [LWX1, Lemma A.3]). Indeed, since the limit $\lim_{t\to\infty} x.e^{t\sqrt{-1}\eta}$ is independent of the equivariant class $c$, we have $f_c(x,\eta) = \sum_k a_k f_{c_1^T(L_k)}(x,\eta)$. For integral $\eta$ associated to a one parameter subgroup $\Lambda : \mathbb{C}^* \to T$, we have $f(x,\eta) = -\bar{\Lambda}_x^* c/\eta^\vee$, where $\bar{\Lambda}_x : \mathbb{C} \to B$ is the extension of the morphism $\Lambda_x : \mathbb{C}^* \to B : \tau \mapsto x.\tau$. We omit the proof for general $\eta \in \mathfrak{t}$. $\square$

We call a collection $\mathfrak{F}$ of $\mathbb{Q}$-Fano varieties is of fixed character if $-mK_X$ is Cartier for a fixed integer $m$, the K-optimal tori $T_X$ are isomorphic to a fixed $T$ and the Hilbert characters $\chi$ of $-mK_X$ with respect to $T$ are fixed for all $X$ in $\mathfrak{F}$. Here the Hilbert character $\chi$ of a line bundle $L$ is the function $\chi : \mathbb{Z} \mapsto R(T)$ given by $\chi(k) = \sum_{i=0}^n (-1)^i H^i(X, kL)$, where $H^i(X, kL)$ is regarded as $T$-representations. For such collection $\mathfrak{F}$, we have a uniform $k_0$ such that for each $X \in \mathfrak{F}$ there is a $T$-equivariant anti-canonical embedding of $X$ into $\mathbb{C}P^{\chi(k_0)-1} \circlearrowleft T$. We call a collection $\mathfrak{F}$ of $\mathbb{Q}$-Fano varieties has bounded characters if $\mathfrak{F}$ is a finite union $\bigcup_{i=1}^N \mathfrak{F}_i$ of collections $\mathfrak{F}_i$ of fixed characters. Note that the K-optimal vector $\xi \in \mathfrak{t}$ of $X$ is determined by its Hilbert character.

Let $\mathrm{Hilb}_T(\chi(\cdot))$ denote the Hilbert scheme of $T$-invariant subschemes of $\mathbb{P}(\chi(1))$ with Hilbert polynomial $\chi(\cdot)$, identifying the character $\chi(1)$ with $T$-representation. Let $\mathcal{U}_T(\chi(\cdot))$ denote the universal family over $\mathrm{Hilb}_T(\chi(\cdot))$ and $\mathcal{L}$ denote the restriction of $p_2^*\mathcal{O}(1)$ on $\mathrm{Hilb}_T(\chi(\cdot)) \times \mathbb{P}(\chi(1))$ to the closed subscheme $\mathcal{U}_T(\chi(\cdot)) \subset \mathrm{Hilb}_T(\chi(\cdot)) \times \mathbb{P}(\chi(1))$. For a $\mathbb{Q}$-Fano manifold $X$ with the Hilbert character $\chi(\ell) = \sum_{i=0}^n (-1)^i H^i(X, \ell(-mK_X))$, we denote by $[X]$ a point of $\mathrm{Hilb}_T(\chi(k\cdot))$ representing an anti-canonically embedded $X$ by $|-kmK_X|$, which is unique modulo $PGL_T(\chi(k))$.

The following is essentially an application of deep analysis in [CW] and arguments in [CSW].

**Proposition 4.1.3.** Let $\mathfrak{F}$ be a collection of K-smoothable $\mathbb{Q}$-Fano varieties of fixed character. Then there is a uniform positive integer $k_1$ such that if $X \in \mathfrak{F}$ is $\mu$K-unstable, there exists a one parameter subgroup $\Lambda : \mathbb{C}^* \to PGL_T(\chi(k_1))$ such that the induced test configuration $(\bar{\Lambda}_X^* \mathcal{U}_T(\chi(k_1 \cdot)), \frac{1}{k_1 m} \bar{\Lambda}_X^* \mathcal{L})$ has negative $\mu_\xi^\lambda$K-Futaki invariant with respect to $\lambda = 2\pi$ and the K-optimal vector $\xi$.

Moreover, assuming Conjecture I, we can take $k_1$ so that if $X \in \mathfrak{F}$ is $\mu$K-semistable but not $\mu$K-polystable, then there exists a one parameter subgroup $\Lambda : \mathbb{C}^* \to PGL_T(\chi(k_1))$ such that the induced test configuration $(\bar{\Lambda}_X^* \mathcal{U}_T(\chi(k_1 \cdot)), \frac{1}{k_1 m} \bar{\Lambda}_X^* \mathcal{L})$ is special and has a $\mu$K-polystable central fibre.

*Proof.* Firstly, we note that there is a uniform constant $\delta \in (0,1)$ such that every smooth Fano manifold $X$ admits a Kähler metric $\omega_X \in 2\pi c_1(X)$ with $\mathrm{Ric}(\omega_X) \geq \delta \omega_X$. Indeed, thanks to the equivalence of Székelyhidi's $R$-invariant $R(X) = \beta(X)$ and the delta invariant $\delta(X)$ proved in [BBJ, Corollary 7.6], it suffices to bound the delta invariants of Fano manifolds from below. The bound holds by the finiteness of delta invariants for smooth Fano manifolds [BLZ, Theorem 1.4]. Otherwise we can show the bound via log K-stability as in [Oda2], translating log K-stability into the existence of twisted KE by [CDS, I].

Since we have a uniform bound of the Sobolev constant for such $\omega_X$, we have a uniform bound in the assumption of [CW, Theorem 6.8 (6.5)]. Thus we get the partial $C^0$-estimate for Kähler–Ricci flow on $t \geq 1$ with the initial metric $\omega_X$ by [CW, Theorem 1.3]. Then the argument in [CSW] shows that there is a uniform integer $k_1$ such that for each $X \in \mathfrak{F}$ there is a vector $\Lambda_X \in \sqrt{-1}\mathrm{Lie}(PU_T(\chi(k_1)))$ and $g_X \in PGL_T(\chi(k_1))$ such that the limit $[\bar{X}] = \lim_{t \to \infty} [X].g_X.e^{t\Lambda_X}$ in the Hilbert scheme $\mathrm{Hilb}_T(\chi(k_1 \cdot))$ is a $\mathbb{Q}$-Fano variety with $\mathrm{Fut}_{\bar{X},\xi} \equiv 0$ for the vector $\xi$ on $\bar{X}$ generated by $\Lambda$. Let $\xi_X \in \mathfrak{t}$ be the K-optimal vector of $X$, which satisfies $\mathrm{Fut}_{X,\xi_X} \equiv 0$. Then since $\Lambda_X$ is $T$-equivariant, we have $\mathrm{Fut}_{\bar{X},\xi_X}|_{\mathfrak{t}} \equiv 0$. Since Tian–Zhu's volume functional $\log \mathrm{Vol}(\eta) = \log \int_{\bar{X}} e^{\theta_\eta} \omega^n$ is strictly convex and $\mathrm{Fut}_{\bar{X},\xi}(\xi_X - \xi) = 0$,

$$\mathrm{Fut}_{\bar{X}, t\xi_X + (1-t)\xi}(\xi_X - \xi) = -\frac{d}{dt} \log \mathrm{Vol}(t\xi_X + (1-t)\xi),$$

is monotonically decreasing. Thus we get

$$\mathrm{Fut}_{\bar{X},\xi_X}(-\xi) = \mathrm{Fut}_{\bar{X},\xi_X}(\xi_X - \xi) < \mathrm{Fut}_{\bar{X},\xi}(\xi_X - \xi) = 0.$$

Now let $T_{\max} \subset PU_T(\chi(k_1))$ be a maximal torus with $\sqrt{-1}\Lambda_X \in \mathfrak{t}_{\max}$. By Theorem G and Lemma 4.1.2, there is a continuous piecewise linear function

$\varphi$ on $\mathfrak{t}_{\max}$ such that $\mathrm{Fut}_{\lim_{t\to\infty}[X].e^{t\eta},\xi_X}(-\eta) = \varphi(\eta)$ for every $\eta \in \mathfrak{t}_{\max}$. Note when $\eta$ is in the lattice $\mathrm{Hom}(U(1), T_{\max}) \subset \mathfrak{t}_{\max}$, we have $\mathrm{Fut}_{\lim_{t\to\infty}[X].e^{t\eta},\xi_X}(-\eta) = \mathrm{Fut}_{\xi_X}(\overline{(\Lambda_\eta)}^*_X \mathcal{U}_T(\chi(k_1\cdot)), \frac{1}{k_1 m}\overline{(\Lambda_\eta)}^*_X \mathcal{L})$ for the one parameter subgroup $\Lambda_\eta$ associated to $\eta$. (The minus sign on $\eta$ comes from the sign inversion of the fundamental vector fields of an $\mathbb{R}$-action and a $\mathbb{C}^*$-action which are related by $\mathbb{R} \to \mathbb{C}^* : t \to e^{-t}$. ) Approximating the ray $\mathbb{R}_+\Lambda \in \mathfrak{t}_{\max}$ by integral rays, we can find a one parameter subgroup $\Lambda' : \mathbb{C}^* \to T_{\max}$ with $\mathrm{Fut}_{\xi_X}(\bar{\Lambda}'^*_X \mathcal{U}_T(\chi(k_1\cdot)), \frac{1}{k_1 m}\bar{\Lambda}'^*_X \mathcal{L}) < 0$. Since we have a convergence of Kähler–Ricci flows [CW, Theorem 6.9], we can also discuss K-smoothable case with a uniform $k_1$ by diagonal argument.

The second case is precisely the case $X$ is $\mu$K-semistable but not admit Kähler–Ricci soliton under Conjecture I. In this case, $X$ is never destabilized by $\Lambda$, so it degenerates in the Hilbert scheme to a $\mathbb{Q}$-Fano variety $X_0$ with Kähler–Ricci soliton which is not isomorphic to $X$ by the argument in [CSW]. $\square$

Theorem G further implies the following key proposition, which works as a translator from analytic result to algebraic result.

**Proposition 4.1.4.** We assume Conjecture I for the claim on $B^{ps}$. Let $(\mathcal{X}, \mathcal{L}) \to B$ be $T$-equivariant family of $T$-polarized schemes Then the subsets

$$B^{ss} = \{b \in B \mid (\mathcal{X}_b, \mathcal{L}_b) \textit{ is a } \mu\textit{K-semistable } \mathbb{Q}\textit{-Fano variety.} \}$$
$$B^{ps} = \{b \in B \mid (\mathcal{X}_b, \mathcal{L}_b) \textit{ is a } \mu\textit{K-polystable } \mathbb{Q}\textit{-Fano variety.} \}$$
$$B^{s} = \{b \in B \mid (\mathcal{X}_b, \mathcal{L}_b) \textit{ is a } \mu\textit{K-stable } \mathbb{Q}\textit{-Fano variety.} \}$$

are constructible sets of $B$.

*Proof.* We can realize the family $(\mathcal{X}, \mathcal{L}) \to B$ as a pull-back of the universal family over a Hilbert scheme Hilb (with a fixed Hilbert polynomial) by embedding $\mathcal{X} \to B$ relatively to $\mathbb{C}P^N$ using sections of a sufficient multiple of $\mathcal{L}$. By the above proposition, we may assume by taking a sufficiently divisible multiple that if a fibre $(\mathcal{X}_b, \mathcal{L}_b)$ is a $\mu$K-unstable $\mathbb{Q}$-Fano variety, then there is a one parameter subgroup $\Lambda : \mathbb{C}^* \to G = \mathrm{Aut}(\mathbb{C}P^N)$ such that $\mathrm{Fut}^\lambda_\xi(\bar{\Lambda}^*_b \mathcal{X}, \bar{\Lambda}^*_b \mathcal{L}) < 0$. Since the inverse image of constructible sets are constructible, we may assume that $B$ is such a Hilbert scheme.

If we take a $G$-equivariant resolution $\beta : \tilde{B} \to B$, then we have $\beta(\tilde{B}^{ps}) = B^{ps}$, so that we may assume $B$ is smooth by Chevalley's theorem on constructible sets. Since Hilbert scheme is projective, we may assume $B$ is smooth projective.

In this situation, the complement $B \setminus B^{ss}$ is equal to the set

$$\left\{ b \in B \;\middle|\; \begin{array}{c} (\mathcal{X}_b, \mathcal{L}_b) \text{ is not} \\ \text{a } \mathbb{Q}\text{-Fano variety} \end{array} \right\} \cup \left\{ b \in B \;\middle|\; \begin{array}{c} \mathrm{Fut}_\xi^\lambda(\bar{\Lambda}_b^* \mathcal{X}, \bar{\Lambda}_b^* \mathcal{L}) < 0 \\ \text{for some } \Lambda : \mathbb{C}^* \to \mathrm{Aut}(\mathbb{C}P^N) \end{array} \right\}.$$

By [HK, 3.11] as in [BX, 3.2], the set $\{ b \in B \mid (\mathcal{X}_b, \mathcal{L}_b) \text{ is a } \mathbb{Q}\text{-Fano variety } \}$ is locally closed in $B$.

On the other hand, the set

$$\left\{ b \in B \;\middle|\; \begin{array}{c} \bar{\Lambda}_b^* \mathcal{D}_\xi \boldsymbol{\mu}_{T \times G}^\lambda (\mathcal{X}/B, \mathcal{L})/\eta^\vee < 0 \\ \text{for some } \Lambda : \mathbb{C}^* \to \mathrm{Aut}(\mathbb{C}P^N) \end{array} \right\}$$

is constructible by Lemma 4.1.2. Therefore, $B \setminus B^{ss}$ is constructible and so is $B^{ss}$. We can similarly show the claim on $B^{ps}, B^s$ (cf. [Oda2]). □

### 4.1.2 Algebraic moduli problems

#### Zariski openness of gentle locus

Now making use of Theorem G, we prove the following. The proof here is independent of the analytic results in section 3.3.

**Theorem 4.1.5.** The moduli stack $\boldsymbol{\mathcal{M}}^{\circ,n} = \boldsymbol{\mathcal{K}}(n)$ (resp. $\boldsymbol{\mathcal{K}}_{T,\chi}$) of gentle Fano manifolds over the étale/fppf site of algebraic schemes is Artin algebraic (resp. Artin algebraic of finite type).

*Sketch of proof.* To see that the moduli stack is Artin algebraic, it suffices to show that the set

$$B^{ss,\circ} := \{ b \in B \mid \mathcal{X}_b \textit{ is a gentle Fano manifold. } \}$$

is Zariski open for every smooth family $\pi : \mathcal{X} \to B$ of Fano manifolds. Since every family $\mathcal{X} \to B$ is Zariski locally isomorphic to the pull-back of the universal family on a $T$-invariant Hilbert scheme along some morphism, we may reduce the problem to the following: suppose a fibre $\mathcal{X}_o$ is a Fano manifold admitting Kähler–Ricci solitons, then there exists a Zariski neighbourhood $U \subset B$ of $o \in B$ such that $\mathcal{X}_b$ is gentle for every $b \in U$.

We firstly show that there is a Zariski neighbourhood $V \subset B$ on which every fibre is a $\mu$K-semistable Fano manifolds. Taking a smaller $B$ if necessary, we may assume the family is anticanonically embedded in a projective space

$\mathcal{X} \hookrightarrow B \times \mathbb{C}P^N$ over $B$. If we cannot take such $V$, then by the constructibility shows that we have a locally closed subset $S \subset B$ such that $\mathcal{X}_b$ is $\mu$K-unstable for every $b \in S$ and the closure $\overline{S}$ meets $o$. Moreover, we may assume that $S$ is taken so that a one parameter subgroup $\mathbb{C}^\times \to \mathrm{Aut}_T(\mathbb{C}P^N)$ destabilizes every $\mathcal{X}_b$ over $S$. As in the previous section, we can see that $\mathcal{X}_b$ over $S$ is $(1-t)$-twisted $\mu$K-unstable for $t \in (t_0, 1]$ some uniform $t_0 < 1$. Since the modified Székelyhidi–Hashimoto invariant $R_\xi(\mathcal{X}_b)$ is a lower semi-continuous function on $b$, we conclude that $\mathcal{X}_o$ is $(1-t_0)$-twisted $\mu$K-unstable. This is a contradiction since we assume $\mathcal{X}_o$ admits Kähler–Ricci solitons, hence it is $(1-t)$-twisted $\mu$K-semistable for every $t \in (0,1]$.

Now we may assume every $\mathcal{X}_b$ is $\mu$K-semistable. By [DaSz], we have a special degeneration to a $\mathbb{Q}$-Fano variety $\mathcal{X}'_b$ with Kähler–Ricci solitons for each $\mathcal{X}_b$. It suffices to show that the central fibres $\mathcal{X}'_b$ are smooth after replacing $B$ with a smaller neighbourhood $U \subset B$. We firstly show that we can take such $U$ as an analytic neighbourhood. Taking a family of Kähler metrics $\alpha_b$ around the fibre $\mathcal{X}_o$ such that $\alpha_o$ is a Kähler–Ricci soliton. Then as in the proof of Proposition 3.2.18, we can see that $\mathcal{X}'_b$, which are obtained as a Gromov–Hausdorff limit along the continuity method $\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\alpha_b$, subconverges to $\mathcal{X}_o$ in a Hilbert scheme as $b \to o$. Since $\mathcal{X}_o$ is smooth, $\mathcal{X}'_b$ must be smooth for $b$ sufficiently close to $o$.

To see that $U$ can be taken as a Zariski open set, we note that any gentle Fano manifold does not admit any equivariant special degeneration to some $\mu$K-semistable *singular* Fano variety. If there exists an equivariant special degeneration of a gentle Fano manifold $X$ to a singular $\mu$K-semistable Fano variety $X_0$, then similarly as in [Y. Li] we can construct an equivariant special degeneration of $X_0$ to a Fano variety $X'_0$ with KRs by the diagonal argument for twisted KRs on $X$. As $X_0$ is singular, $X'_0$ is also singular. We can construct these degenerations equivariant with respect to the maximal torus action, so we also obtain a special degeneration of the gentle Fano manifold $X$ to $X'_0$. However, since the central fibre of special degenerations of $X$ to a Fano variety with KRs are unique by Proposition 3.2.18, the fact that $X'_0$ is singular contradicts to the assumption that $X$ is gentle. Thus we have

$$B \backslash U = \left\{ b \in B \;\middle|\; \begin{array}{c} \mathcal{X}_b \textit{ admits an equivariant special degeneration in } \mathbb{C}P^N \textit{ to} \\ \textit{a } \mu K\textit{-semistable singular Fano variety} \end{array} \right\}.$$

This is a constructible set. It follows that $U$ is a constructible analytic open set, which shows the desired Zariski openness. $\square$

After the publication of [Ino1], the author was asked a question from Yue Fan on the proof of Proposition 3.3.8. He noticed that the proof (as well as Székelyhidi's original proof for cscK metrics) does not show the desired claim but it only shows the following: for each $b \in B$ whose orbit $b.G$ is closed, there exists $t_b > 0$ such that $\mathfrak{J}(t.b)$ admits Kähler–Ricci solitons for every $t \in (0, t_b)$. There is no reason for $t_b \geq t_0$ with a uniform $t_0 > 0$ for every $b \in B$ in a small sphere as we do not know whether we can take a uniform constant $C'$ in the proof independent of $b$. A similar but slightly different proof can be found in [Bro], however, since the pulled-back symplectic form may not be Kähler after we apply the perturbation in Theorem 3.3.7, the proof there also seems to be fail. Yue Fan [Fan] achieved to prove an analogous claim in the context of moduli problem on Higgs bundles, using a result similar to [CS]. Thus the result is recovered for the cscK case.

Now we give an alternative proof in the case of KRs which does not rely on the result in section 3.3.2. Instead of establishing a similar result as in [CS] for the case of KRs, we imitate an argument in [LWX1], applying the above result.

**Proposition 4.1.6.** Let $B \to \mathrm{Hilb}_T$ be an affine étale local slice (with respect to the $PGL_T$-action) at a point $[X] \in \mathrm{Hilb}_T$ representing an anticanonically embedded Fano manifold $X$ with KRs. Then we can take an $\mathrm{Aut}(X)$-invariant analytic open set $B'$ of $B$, such that every closed $\mathrm{Aut}(X)$-orbit of $B'$ parametrizes a Fano manifold with KRs.

*Proof.* If not, there is a sequence $b_k \in B$ converging to the point $o \in B$ corresponding to the point $[X] \in \mathrm{Hilb}_T$ such that $b_k.G \subset B$ is closed but the Fano manifolds $\mathcal{X}_{b_k}$ do not admit KRs. Similarly as in the above proof (or as in [Y. Li]), we can construct special degenerations of $\mathcal{X}_{b_k}$ to Fano manifolds $\mathcal{X}'_{b_k}$ with KRs in $\mathbb{C}P^N$ which converges to $\mathcal{X}_o = X$ in the Hilbert scheme. Since $B \times_G PGL_T \to \mathrm{Hilb}_T$ is open, the degenerations $\mathbb{C} \to \mathrm{Hilb}_T$ are in the image of this map for sufficiently large $k$. Since $B \times_G PGL_T \to \mathrm{Hilb}_T$ is equivariant, we can realize the degenerations in $B \times_G PGL_T$. It follows that $b_k.G \times PGL_T = p^{-1}((b_k, e).PGL_T)$ is not closed in $B \times PGL_T$ as the orbit $(b_k, e).PGL_T$ is not closed in $B \times_G PGL_T$, which contradicts to the fact that $b_k.G$ is closed in $B$. □

Now we prove the following, using Theorem 4.1.5 and Theorem 3.4.8.

**Theorem 4.1.7.** The moduli space $\mathcal{K}(n)$ constructed in section 3.4 is a separated algebraic space.

*Proof.* As proved in section 3.4, we have an analytic moduli space $\boldsymbol{\mathcal{M}}^{\circ,n} \to \mathcal{K}(n)$. Since gentle locus is Zariski open, we can cover the moduli stack by étale morphisms $[B_\alpha/G_\alpha] \to \boldsymbol{\mathcal{M}}^{\circ,n}$ with affine $B_\alpha$. Then by the universality of the morphism $[B_\alpha/G_\alpha] \to B_\alpha /\!\!/ G_\alpha$, we get a collection of morphisms $\phi_\alpha : \mathcal{M}_\alpha \to \mathcal{K}(n)$ from affine normal varieties $\mathcal{M}_\alpha = B_\alpha /\!\!/ G_\alpha$ which covers $\mathcal{K}(n)$. Similarly as in the proof of Theorem 3.4.8, we can see that these morphisms are analytically étale.

For each point $x \in \phi_\alpha(\mathcal{M}_\alpha) \cap \phi_\beta(\mathcal{M}_\beta)$, we can construct algebraic étale morphisms $\mathcal{M}_{\alpha\beta,x} \to \mathcal{M}_\alpha, \mathcal{M}_\beta$ from an affine normal variety $\mathcal{M}_{\alpha\beta,x}$ so that they commute with the morphisms $\phi_\alpha, \phi_\beta$ and the composition $\mathcal{M}_{\alpha\beta,x} \to \mathcal{K}(n)$ covers $x$, similarly as in the proof of Theorem 3.4.8. Here the point is the commutativity of morphisms, however, this easily follows from the analytic construction of the moduli space. □

### Towards the compactification

We can strengthen the result in the previous section under Conjecture I.

**Claim.** We assume Conjecture I. Then the moduli stack $\boldsymbol{\mathcal{M}}^n$ is Artin algebraic and admits a unique proper algebraic good moduli space $\mathcal{M}^n$. Moreover, the moduli space enjoys the following.

- The points of $\mathcal{M}^n$ are exactly smoothable $\mu$K-polystable $\mathbb{Q}$-Fano varieties, which are precisely smoothable $\mathbb{Q}$-Fano varieties with Kähler–Ricci solitons.

- The forgetting morphism $\boldsymbol{\mathcal{M}}^n \to \mathcal{M}^n$ maps $\mu$K-semistable $\mathbb{Q}$-Fano $T$-varieties $X$ and $X'$ to the same point if and only if there are $T$-equivariant test configurations of $X$ and $X'$ with an identical smoothable $\mu$K-polystable $\mathbb{Q}$-Fano $T$-variety.

If the moduli space exists, then the properness of the moduli space is a consequence of the compactness result in [PSS] combined with the boundedness of $\mathrm{Fut}(\xi)$ for K-optimal vectors $\xi$ ([Ino1, Proposition 4.11]). It is recently proved by [Y. Li] that a K-smoothable $\mathbb{Q}$-Fano variety admits a Kähler–Ricci soliton if it is $\mu$K-polystable with respect to special degenerations.

We can show that K-smoothable $\mu$K-semistable $\mathbb{Q}$-Fano varieties are bounded. If we further aim to construct a finite type moduli space for $\mathbb{Q}$-Fano varieties with Kähler–Ricci solitons and with non-smoothable klt singularities, we must impose a bound on Hilbert polynomials since there is an unbounded collection of toric orbifolds with Kähler–Ricci solitons (cf. [PSS]), unlike Kähler–Einstein case. In view of [PSS], the author speculates that it suffices to bound the usual Futaki invariants $\mathrm{Fut}(\xi)$ of the K-optimal vectors $\xi$ and the volumes in order to bound the Hilbert polynomials of $\mu$K-semistable $\mathbb{Q}$-Fano varieties.

Now we outline the proof of the main Claim. More details will be included in a separate paper.

*Outline of the proof of Claim.* As in [Oda3] and [LWX1], we apply Alper's gluing theorem of (local) good moduli spaces [Alp2, Theorem 1.3] (cf. [Alp1, Proposition 7.9]). We will construct étale local moduli stacks $i_\alpha : \boldsymbol{\mathcal{M}}_\alpha \to \boldsymbol{\mathcal{M}}^n$ of the moduli stack $\boldsymbol{\mathcal{M}}^n$ so that each stack $\boldsymbol{\mathcal{M}}_\alpha$ is of the form $[\mathrm{Spec} A/G]$ with reductive $G$ and so that these $\boldsymbol{\mathcal{M}}_\alpha$ cover $\boldsymbol{\mathcal{M}}^n$. Each $\boldsymbol{\mathcal{M}}_\alpha$ admits the good moduli space $\mathcal{M}_\alpha = \mathrm{Spec} A /\!\!/ G$. If we can check the conditions in Alper's gluing theorem, we get the expected good moduli space $\boldsymbol{\mathcal{M}}^n \to \mathcal{M}^n$ by 'gluing' these local moduli spaces $\mathcal{M}_\alpha$ in étale topology.

As we remarked in [Ino1], the concept of good moduli space is still not well established in analytic category. Especially, good moduli spaces of an analytic stack are not ensured to be unique while the uniqueness is essential for gluing theorem. So we must show that global and local moduli stacks are algebraic in order to apply Alper's framework on good moduli space. Now is the time to apply the above proposition.

Construction of local moduli stacks $\boldsymbol{\mathcal{M}}_\alpha$ reduces to the following claim.

- For a $T$-equivariant family $(\mathcal{X}, \mathcal{L}) \to B$ of K-smoothable $\mathbb{Q}$-Fano $T$-varieties, $B^{ss}$ is Zariski open.

We can prove this similarly to [LWX1, Theorem 7.3] using twisted $\mu$K-stability instead of log K-stability as in [Y. Li]. We take the affine scheme $\mathrm{Spec} A$ as an étale local slice of $\mathrm{Hilb}_T$ at $\mu$K-polystable $[X] \in \mathrm{Hilb}_T$ (cf. [AHR, Theorem 2.1]) so that $\mathrm{Spec} A$ parametrizes only $\mu$K-semistable K-smoothable $\mathbb{Q}$-Fano varieties. The group $G$ is the automoprhism group $\mathrm{Aut}(X)$.

Gluing the local moduli spaces $\mathcal{M}_\alpha$ boils down to the following claim, by shrinking $\mathrm{Spec} A$ further if necessary.

- If a K-smoothable $\mathbb{Q}$-Fano variety $X$ admits two equivariant special degenerations to K-smoothable $\mu$K-polystable $\mathbb{Q}$-Fano varieties $X_0, X_0'$, then $X_0$ and $X_0'$ are isomorphic.

- A point $x \in \mathrm{Spec}A$ parametrizes a $\mu$K-polystable $\mathbb{Q}$-Fano variety $X_x$ if the orbit $x.G$ is closed.

- The stabilizer $G_x$ is isomorphic to $\mathrm{Aut}(X_x)$ for every $x \in \mathrm{Spec}A$.

We can show these similarly to [LWX1, Theorem 1.1, Theorem 8.8, Corollary 8.14]. We in particular make use of the constructibility of $B^{ps}$ to prove the second claim. The first two items show that the morphism $\boldsymbol{\mathcal{M}}_\alpha \to \boldsymbol{\mathcal{M}}$ is universally weakly saturated in Alper's sense (cf. [Alp2]). The last one shows the morphism is pointwise stabilizer preserving. Now we can apply [Alp2, Theorem 1.3] and obtain the expected good moduli space. $\square$

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