‘Phase transition’ in Kähler geometry

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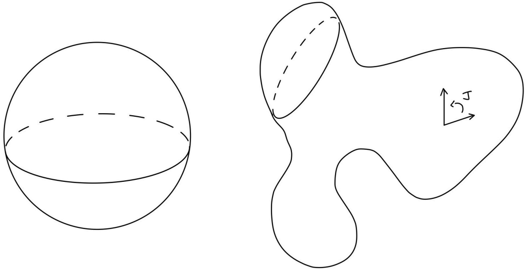
What ‘shape’ of a space is most “natural”?

$X = (M, J)$: a compact complex manifold.

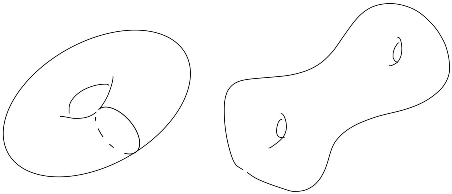


$g = g_{i\bar{j}} dz^i d\bar{z}^j = \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j} dz^i d\bar{z}^j$: a Kähler metric

$\rightsquigarrow L := [g] \in H^2(X, \mathbb{R})$: a polarization

Question: Can we find/define a “canonical metric” g_{can} in L which tells us information on the complex geometry of (X, L)?

Classical examples



$\dim_{\mathbb{C}} X = \frac{1}{2} \dim_{\mathbb{R}} X = 1 \rightsquigarrow \exists\, g \in L$ Kähler–Einstein metric

$$\mathrm{Ric}(g) = \lambda_L g,$$

unique modulo $\mathrm{Aut}(X)$.

$$\mathrm{Ric}(g) := -(\log \det g)_{i\bar{j}} dz^i d\bar{z}^j$$
$$s(g) := -g^{i\bar{j}} (\log \det g)_{i\bar{j}}$$

Various notions of canonical metrics

Definition

Let (X, L) be a polarized manifold. A Kähler metric $g \in L$ is called

- Kähler–Einstein metric if $\mathrm{Ric}(g) = \lambda g$.
- Kähler–Ricci soliton if $\mathrm{Ric}(g) - \mathscr{L}_\xi g = \lambda g$ for a holo. vec. ξ.

  $\rightsquigarrow \lambda L = 2\pi c_1(X)$: a strong restriction on (X, L).

- CscK metric if $s(g) = \mathrm{const}$.

Remark

- (normalized) Kähler–Ricci flow: $\mathrm{Ric}(\omega_t) - \lambda \omega_t = -\dot{\omega}_t$

  $\rightsquigarrow$ Kähler–Ricci soliton.

- KE $\iff$ cscK and $\lambda L = 2\pi c_1(X)$ for some $\lambda \in \mathbb{R}$.
- Unique modulo $\mathrm{Aut}(X, L)$.

Philosophy: Donaldson–Fujiki moment map picture

- $(M, \omega) \circlearrowleft T$: symplectic manifold
- $\mathcal{J} := \mathcal{J}_T(M, \omega)$: the space of T-inv alm cpx strs. comp. with ω
- $\mathrm{Ham} := \mathrm{Ham}_T(M, \omega)$: the group of T-equiv. Hamiltonian diffeomorphisms
- $\mathfrak{h} := \mathrm{Lie}(\mathrm{Ham}) = C^\infty_T(M)/\mathbb{R}$

Proposition (Donaldson–Fujiki '97, I. '18, Lahdili '19)

There are symplectic structures Ω_{cscK}, Ω_{KRs} and corresponding moment maps

$$\mathscr{S}_{\mathrm{cscK}}, \mathscr{S}_{\mathrm{KRs}} : \mathcal{J} \to \mathfrak{h}^\vee$$

s.t. $\mathscr{S}_\bullet(J) = 0$ for integrable $J \in \mathcal{J}$ iff g_J is $\bullet = \mathrm{cscK}/\mathrm{KRs}$.

As for KRs, we assume $\lambda L = 2\pi c_1(M, \omega)$.

$\rightsquigarrow$ applied to a construction of a Hausdorff moduli space of complex structures/bihol. (Keywords: Kempf–Ness, GIT, K-stability, Yau–Tian–Donaldson conjecture...)

μ-cscK metric

For $\lambda \in \mathbb{R}$, we call a Kähler metric $g \in L$ μ^λ-cscK metric if

$$\begin{cases} s(g) + \Delta\mu - \frac{1}{2}|\nabla\mu|^2 & = \lambda\mu \\ \bar\partial\partial^\sharp\mu := (g^{i\bar{j}}\mu_{\bar{j}})_{\bar{k}} \frac{\partial}{\partial z^i} \otimes d\bar{z}^k & = 0. \end{cases}$$

This curvature notion is naturally derived from the moment map picture on KRs. It unifies cscK metric and KRs.

Theorem (I. '19 + Lahdili '20)

For each (X, L), μ^λ-cscK metric is unique mod Aut for $\lambda \ll 0$.

Conjecture: Uniqueness holds as long as $\lambda \leq 0$.

"Phase transition" of μ-cscK metrics

When $\lambda \gg 0$, μ^λ-cscK metric is not unique in general!

We can construct a non-trivial μ^λ-cscK metric on $\mathbb{C}P^1 = S^2$ for $\lambda > 4\pi$.

$$\boldsymbol{\mu}^\lambda(x) := (\lambda - 2\pi)\frac{\int_{-1}^1 (1+xt)e^{xt}\,dt}{\int_{-1}^1 e^{xt}\,dt} - \lambda \log \int_{-1}^1 e^{xt}\,dt$$
$$= (\lambda - 2\pi)\frac{x}{\tanh x} - \lambda \log\left(\frac{2\sinh x}{x}\right).$$

If there exists a μ^λ-cscK metric on $\mathbb{C}P^1$ w.r.t. μ with $\partial^\sharp \mu = x.\sqrt{-1}z\frac{\partial}{\partial z}$, then x must be a critical point of $\boldsymbol{\mu}^\lambda$.

While $x = 0$ is always a critical point, there appears new critical points $x \neq 0$ when $\lambda > 4\pi$.

Thank you for listening!