A Thermodynamical formalism for Optimal degeneration problem in Kähler geometry

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Entropy and Equilibrium in Stochastic thermodynamics

Ω: a finite set (for simiplicity) called system
$E : \Omega \to \mathbb{R}$: a function called Hamiltonian

For a probability distribution p on Ω (non-equilibrium state), we consider

$$S(p) := -\sum_{i \in \Omega} p(i) \log p(i) \quad \ldots \quad \text{Entropy}$$
$$U(p) := \sum_{i \in \Omega} E(i) p(i) \quad \ldots \quad \text{Internal energy}$$

A state p is called equilibrium if it maximizes S among all p' with the same internal energy $U(p') = U(p)$.

Lagrange multiplier $\beta = 1/T$
$\rightsquigarrow$ Equilibrium is a minimizer of $U - TS$ for a suitable choice of T.
$\rightsquigarrow$ Equilibrium $=$ Canonical distribution $p = e^{-\beta E} / \sum e^{-\beta E}$.

Setup in Kähler geometry

Today, I explain optimal degeneration problem in Kähler geometry can be formalized in an analogous way.

Mathematical setup:
X: a compact (= closed) complex manifold
L: a polarization or a Kähler class $\in H^2(X, \mathbb{R})$

Complex $= \mathbb{C}^n = (\mathbb{R} \oplus \mathbb{R}i)^n \rightsquigarrow X$ is real $2n$-dimensional

- Locally, $X = \{f_1(z_1, \ldots, z_N) = \cdots = f_k(z_1, \ldots, z_N) = 0\}$, using multivariable holomorphic/polynomial functions.
- Global structure (compactness) is important in today's talk

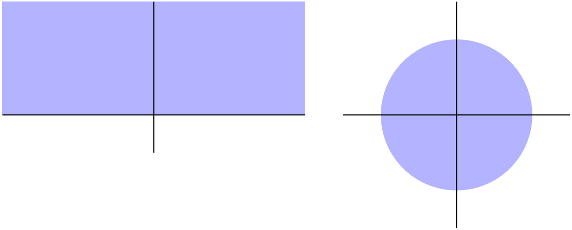
e.g. X can be a Riemann surface, K3 surface, $\mathbb{C}P^n$, toric manifolds ....

Comment: Closely related to Symplectic manifold (phase space).

Holomorphic structure

$X \cong X'$ "$\Leftrightarrow$" $\exists$ a holomorphic change of variables: $z'_i = h_i(z_1, \ldots, z_N)$.

e.g. the upper half space $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ and a disc $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ are biholomorphic, while shapes are different.



"Geometric shape" of X is not yet determined.

$\rightsquigarrow$ Consider "geometric shapes" of X and study the "best shape".

Canonical metric

Consider a Riemannian metric g on X.

Kähler condition: we assume the following compatibility with (X, L)

1. On a holomorphic coordinate, g can be written as

$$g = g_{i\bar{j}} dz^i d\bar{z}^j + \overline{g_{i\bar{j}}} d\bar{z}^i dz^j, \quad dz^i = dx^i + \sqrt{-1} dy^i.$$

2. The 2-form $\sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ is closed and its deRham class is L.

Curvatures of g:

Ricci curvature: $\mathrm{Ric}_{i\bar{j}} := -\partial_i \bar{\partial}_j \log \det g_{k\bar{l}}$

Scalar curvature: $s := g^{i\bar{j}} \mathrm{Ric}_{i\bar{j}}$

g is called

Kähler–Einstein metric: $\mathrm{Ric}_{i\bar{j}} = \lambda g_{i\bar{j}}$

Kähler–Ricci soliton: $\mathrm{Ric}_{i\bar{j}} - \partial_i \bar{\partial}_j f = \lambda g_{i\bar{j}} \quad \rightsquigarrow 2\pi c_1(X) = \lambda L$

cscK metric: $s = \mathrm{const} \quad \rightsquigarrow \mathrm{const} = 2\pi n (c_1(X).L^{\cdot n-1})/(L^{\cdot n})$

Two situations

Given a space (X, L), there are two possibilities:

1. There exists a canonical metric, in which case solution is unique.
2. There is no canonical metric.

Intuitively,

1. Space (X, L) admits a static equilibrium.
2. Space (X, L) admits no static equilibrium.

The case 1 ... Yau–Tian–Donaldson conjecture.

The case 2 ... more important, interesting and challenging.

Dynamics of shape of space

Conjecture (Optimal degeneration)

When there is no canonical metric, there would be dynamics $g(t)$ of space (X, L) which develops to a static equilibrium g_∞ of another space (X_0, L_0) at $t = \infty$.

e.g. Ricci flow (limit: Ricci soliton), Calabi flow, ... "μ-flow"?

How one can find "canonical flow" satisfying the above property?

Idea: Regard dynamics $g(t)$ as a state of spacetime $X \times [0, \infty)$ and characterize dynamics with the above property as "equilibrium of spacetime".

Especially interested in the limit behavior $t = \infty$.
$\rightsquigarrow$ Want to determine boundary condition for equilibrium.

Boundary condition for spacetime: schematic degeneration

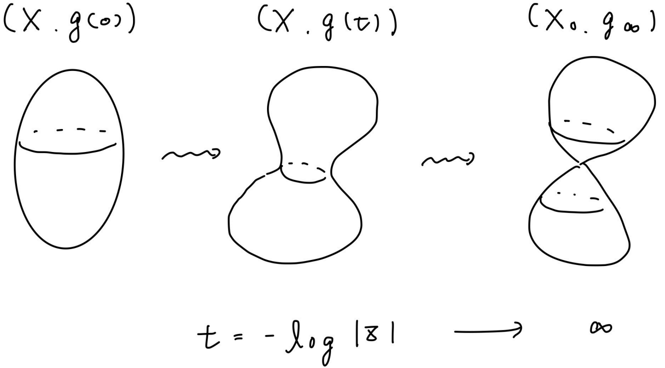
Let's put the vision into a mathematical form.
We will introduce the notion of degeneration of (X, L), assign quantities to it and characterize optimal degeneration by these quantities.

Scheme theoretic definition: a degeneration is a $\mathbb{C}^\times$-equivariant flat family $\varpi : \mathcal{X} \to \mathbb{C}$ of schemes over $\mathbb{C}$ together with a relative polarization $\mathcal{L}$ and the base change diagram

$$\begin{tikzcd} X \times (\mathbb{C} \setminus \{0\}) \sqcup X_0 \arrow[r, "\mathbb{C}^\times\text{-equiv fb-wise isom.}"] \arrow[d] & \mathcal{X} \arrow[d] \\ (\mathbb{C} \setminus \{0\}) \sqcup \{0\} \arrow[r] & \mathbb{C} \end{tikzcd}$$

"Equilibrium of spacetime" $g(t)$ must converge towards a canonical metric on $X_0 = \varpi^{-1}(0)$. (Not precise, for simplicity)
$\rightsquigarrow \mathcal{X}$ represents a boundary condition.

Illustration of degeneration



Toric manifold: for brevity

Before introducing quantities for degeneration...

Let's focus on a convenient class: $\{$ toric manifolds $\} \subset \{$ Kähler mfds $\}$.
It helps to describe the quantities in a simpler way.
(I need 100 pages to explain general case.)

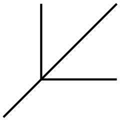


Figure: This represents $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

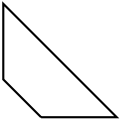


Figure: This represents L on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

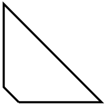


Figure: This represents L' on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

A convex polytope $P \subset \mathbb{R}^n$ spanned by vertices $v_1, \dots, v_N \in \mathbb{Z}^n$.
$\rightsquigarrow (X, L)$ associated to P, which is a real $2n$-dimensional manifold with a torus symmetry $X \circlearrowleft T = U(1)^{\times n}$ and a moment map $\mu : X \to \mathbb{R}^n$.

"Non-archimedean" Equilibrium

A "nice" convex function q on P assigns a degeneration of (X, L).
We put $u := \frac{\int_P d\mu}{\int_P e^q d\mu} e^q$ and

$$S_P(u) := -\frac{1}{\int_P d\mu} \int_P u \log u d\mu \quad \cdots \quad \text{Entropy}$$

$$U_P(u) := \frac{1}{\int_P d\mu} \int_{\partial P} u d\sigma \quad \cdots \quad \text{Internal } \mu\text{-energy}$$

Put $\mathfrak{U}_P := [\min U, U(1_P)]$. For $U \in \mathfrak{U}_P$, we call u equilibrium of internal energy U if

$$U_P(u) = U, \quad S_P(u) = \sup\{S_P(u') \mid U_P(u') = U\}.$$

Theorem ('23)

For each $U \in \mathfrak{U}_P$, there exists a unique equilibrium of internal energy U.
If $\mathfrak{U}_P \neq \{U(1_P)\}$, space (X, L) admits no cscK metric (no static equilibrium).

Free μ-energy

Now we consider

$$F_P(T, u) = U_P(u) - TS_P(u) \quad \cdots \quad \text{Free } \mu\text{-energy.}$$

Theorem ('23)

For $T \geq 0$, there exists a unique minimizer u of $F_P(T, \bullet)$, which is equilibrium of some internal energy $U \in \mathfrak{U}_P^* \cup \{U(1_P)\}$.
Moreover, if the minimizer is a "nice" convex function, then $X_0 = \varpi^{-1}(0)$ of the associated degeneration $\mathcal{X}$ is $\mu^{-2\pi T}$K-semistable.

To show the existence, we use compactness: for a minimizing sequence u_i, we prove that we can take a subsequence which converges to some u_∞ in a suitable topology. Difficult to describe the minimizer explicitly.

μK-semistability is related to the existence of μ-cscK metric: a kind of canonical metric which generalizes cscK metric and Kähler–Ricci soliton.

Perelman entropy and minimax principle

These are related to Perelman's entropy.

$$W(T,g,f) := \frac{1}{\int_X \mathrm{vol}_g} \int_X (s(g) + \frac{1}{2}|\nabla f|^2 + 2\pi T f) e^f \mathrm{vol}_g,$$

$$\mu_{\mathrm{Per}}(T,g) := \inf_f \{ W(T,g,f) \mid \int_X e^f \mathrm{vol}_g = \int_X \mathrm{vol}_g \}.$$

Theorem ('21)

We have

$$\min_q 2\pi F_P(T,q) \geq \sup_g \mu_{\mathrm{Per}}(T,g).$$

For $T \geq 0$, g is a $\mu^{-2\pi T}$-cscK metric if and only if g maximizes $\mu_{\mathrm{Per}}(T,g)$. If its maximizer exists, we have

$$\min_q 2\pi F_P(T,q) = \max_g \mu_{\mathrm{Per}}(T,g).$$

Mathematical questions

1. Prove the equality $\min_q 2\pi F_P(T, q) = \sup_g \mu_{\mathrm{Per}}(T, g)$ for general case.
2. Show the existence of optimal degeneration for general Kähler manifold by establishing the following two steps.
    - Establish a compactness on non-archimedean psh metrics.
    - Establish a regularity theory on non-archimedean psh metrics.
3. Construct moduli theory on $\mu^{-2\pi T}$K-semistable polarized schemes
4. and observe "wall-crossing" of moduli spaces as T varies.

Subtle questions

1. Temperature is a new parameter in Kähler geometry. Can we use it to understand "equilibrium of spacetime" $g(t)$, using Wick rotation?
2. Can we find other physical structure in this framework?
   Statistical mechanics, quantum mechanics, information theory...
   (Since the framework only shares an abstract feature with thermodynamics, we might need to consider some abstraction of these theories.)
3. Is there any relation to quantum entanglement and "emergence of spacetime"?
4. As usual in thermodynamics, free μ-energy can be derived as the difference of entropy of the composition with a heat bath (infinite dimensional space). This implicates the geometry of optimal degeneration is of infinite dimensional nature ("infinite dimensional Yamabe problem"). Perelman finds a heuristic argument on monotonicity of W-entropy which utilizes this picture. Can we find similar heuristics on $\min_q 2\pi F_P(T,q) = \sup_g \mu_{\mathrm{Per}}(T,g)$?

Thank you for listening!