Canonical metrics and Perelman's entropy in Kähler geometry

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Contents

1. Canonical metrics and K-stability in Kähler geometry

Extremal metric and Calabi functional

The Calabi functional $C : \mathcal{H}(X, L) \to \mathbb{R}$ is given by

$$C(\omega) := \frac{1}{2} \int_X \hat{s}(\omega)^2 \omega^n, \quad \hat{s}(\omega) = s(\omega) - \bar{s}$$

The critical points are extremal metrics: $\partial^\sharp s(\omega) = g^{i\bar{j}} s_{\bar{j}} \partial_i$ is holomorphic.

- The notion generalizes cscK metric $s(\omega) = \mathrm{const.}$, hence also Kähler–Einstein metric $\mathrm{Ric}(\omega) = \lambda\omega$.
- For a compact complex manifold X and a Kähler class $L = [\omega]$, $\omega \in L$ can be KE metric only when $c_1(X) \propto L$ or $= 0$. Conversely, cscK metric in such Kähler class is automatically KE.
- Extremal metric in L is cscK metric iff $\mathrm{Fut}_{(X,L)} = -\int_X \hat{s}\mu_\bullet \omega^n = 0$.

Berman–Berndtsson, et al. Given (X, L), extremal metrics are unique modulo $\mathrm{Aut}^0(X)$ if it exists.

(Relative) K-stability

A text configuration $(\mathcal{X}, \mathcal{L})$ is a $\mathbb{C}^\times$-equivariant flat family of polarized schemes $(\mathcal{X}, \mathcal{L}) \xrightarrow{\varpi} \mathbb{C}$ endowed with an isomorphism $X \times \mathbb{C}^* \cong \varpi^{-1}(\mathbb{C}^*)$.

Donaldson–Futaki invariant $\Leftarrow$ moment map picture:

$$\mathrm{DF}(\mathcal{X}, \mathcal{L}) := (K_{\bar{\mathcal{X}}/\mathbb{P}^1}.\bar{\mathcal{L}}^{\cdot n}) - \frac{(K_X.L^{\cdot n-1})}{(n+1)(L^{\cdot n})}(\bar{\mathcal{L}}^{\cdot n+1}).$$

Relative Donaldson–Futaki invariant: when $(X, L) \circlearrowleft T$, for $\xi \in \mathfrak{t}$

$$\mathrm{DF}^{\mathrm{rel}}_\xi(\mathcal{X}, \mathcal{L}) := \mathrm{DF}(\mathcal{X}, \mathcal{L}) + \frac{1}{2\pi}\binom{n+2}{2}^{-1} \int_{\bar{\mathcal{X}}} c_{1,T}(\bar{\mathcal{L}}; \xi)^{\smile n+2}.$$

Székelyhidi, et al. If (X, L) has an extremal metric with $\xi = \mathrm{Im}(\partial^\sharp s)$, then we have

$$\mathrm{DF}^{\mathrm{rel}}_\xi(\mathcal{X}, \mathcal{L}) \geq 0 \text{ for } \forall (\mathcal{X}, \mathcal{L}),$$

i.e. (X, L) is relatively K-semistable.

Donaldson's inequality

Donaldson's lower bound:

$$\sup_{(\mathcal{X},\mathcal{L})} -\frac{2\pi \mathrm{DF}(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|} \leq \inf_{\omega \in L} (2C(\omega))^{1/2},$$

where we put

$$\|(\mathcal{X},\mathcal{L})\|^2 = \int_{\mathbb{R}} (t-b)^2 \mathrm{DH}(\mathcal{X},\mathcal{L}) = -\binom{n+2}{2}^{-1} (\bar{\mathcal{L}}^{\cdot n+2}_{\mathbb{C}^\times}; \eta) - \frac{(\bar{\mathcal{L}}^{\cdot n+1})^2}{(n+1)^2}$$

with the barycenter $b := \int_{\mathbb{R}} t \mathrm{DH}(\mathcal{X},\mathcal{L})$.

optimal destabilization conjecture

We have the equality

$$\sup_{(\mathcal{X},\mathcal{L})} -\frac{2\pi \mathrm{DF}(\mathcal{X},\mathcal{L})}{\|(\mathcal{X},\mathcal{L})\|} = \inf_{\omega \in \mathcal{H}(X,L)} (2C(\omega))^{1/2}$$

and the maximum of LHS is achieved by some test configuration.

Kähler–Ricci soliton

Kähler–Ricci soliton is a self-similar solution of Kähler–Ricci flow:

$$\mathrm{Ric}(\omega) - L_{J\xi}\omega = \lambda\omega$$

for some $\xi \in \mathfrak{iso}(X, \omega)$ with holomorphic $\xi^J = J\xi + \sqrt{-1}\xi$.

- The notion generalizes Kähler–Einstein metric $\mathrm{Ric}(\omega) = \lambda\omega$.
- $\omega \in L$ can be KRs only when $c_1(X) \propto L$ or $= 0$.
- KRs is KE metric iff $\mathrm{Fut}_{(X,L)} = 0$, which holds for $\lambda \leq 0$.

Usually, $L = 2\pi c_1(X)$, hence X is Fano.

| Tian–Zhu, et al. | Uniqueness modulo $\mathrm{Aut}^0(X)$ (volume minimization)

| Berman–Witt-Nyström, et al. | If X admits a KRs with ξ, then

$$\mathrm{Fut}_\xi^{\mathrm{mod}}(\mathcal{X}, \mathcal{L}) \geq 0 \text{ for } \forall(\mathcal{X}, \mathcal{L}),$$

i.e. X is modified K-semistable.

H-entropy

The H-entropy for $\omega \in L$ is defined by

$$H(\omega) = \int_X h e^h \omega^n / \int_X e^h \omega^n - \log \int_X e^h \frac{\omega^n}{n!},$$

where h is the Ricci potential $\sqrt{-1}\partial\bar{\partial} h = \mathrm{Ric}(\omega) - \omega$. KRs are the critical points of H.
The following is a generalization of Tian–Zhu's volume functional:

$$H_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) := -\min_{E \subset \mathcal{X}_0} (A_X(v_E) + \frac{\mathrm{ord}_E(\mathcal{L} - L)}{\mathrm{ord}_E \mathcal{X}_0}) - \log \int_{\mathbb{R}} e^{-t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}$$

Theorem (Chen–Sun–Wang, Dervan–Székelyhidi, Han–Li, et al.)

We have the equality

$$\sup_{(\mathcal{X},\mathcal{L})} H_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}) = \inf_{\omega \in L} H(\omega)$$

and the maximum of LHS is achieved by a unique f.g. filtration.

2. Brief introduction to μ-cscK metrics

μ-cscK metric

CscK metric is a natural generalization of Kähler–Einstein metric.

What about Kähler–Ricci soliton?

- In view of K-stability, we want moment map picture.
- In view of optimal degeneration, we want volume minimization.

$\rightsquigarrow$ $\boxed{\mu\text{-cscK metric}}$

Definition

A Kähler metric ω is called $\check{\mu}^{\lambda}_{\xi}$-cscK metric if

$$s_{\mu_\xi} := (s(\omega) + \bar{\Box}\mu_\xi) + (\bar{\Box}\mu_\xi + |\partial^\sharp \mu_\xi|^2) = \lambda \mu_\xi + \mathrm{const}.$$

- $\mu^{2\pi\lambda}_{\xi} = \check{\mu}^{2\pi\lambda}_{-2\xi}$-cscK metric in $\lambda^{-1} c_1(X) \Leftrightarrow$ KRs $\mathrm{Ric}(\omega) - L_\xi \omega = \frac{\lambda}{2\pi}\omega$.
- Extremal metric = the limit of μ^{λ}-cscK metrics as $\lambda \to -\infty$.

Moment map picture for μ-cscK metric

$(M, \omega) \curvearrowleft T$: Hamiltonian torus action. For $\xi \in \mathfrak{t}$ and $\lambda \in \mathbb{R}$, the map

$$\mathcal{S}_\xi^\lambda : \mathcal{J}_T \to \mathfrak{ham}_T^\vee : J \mapsto (s_{\mu_\xi}(g_J) - \lambda \mu_\xi) e^{\mu_\xi} \omega^n$$

gives a moment map for a symplectic manifold $(\mathcal{J}_T, \Omega_\xi) \curvearrowleft \mathrm{Ham}_T$.

μ_ξ^λ-Futaki invariant

$$\mathrm{Fut}_\xi^\lambda(\zeta) := -\int_X \mu_\zeta (s_{\mu_\xi}(\omega) - \lambda \mu_\xi) e^{\mu_\xi} \omega^n.$$

$\rightsquigarrow$ generalized to test configuration via equivariant intersection formula

$\Rightarrow$ μ_ξ^λK-stability $\supset$ modified K-stability

Theorem (Lahdili '19, I. '20, Apostolov–Jubert–Lahdili '21)

If (X, L) admits a $\check{\mu}_\xi^\lambda$-cscK metric, then it is $\check{\mu}_\xi^\lambda$K-semistable.

Volume minimization: μ-Futaki invariant vanishing

Proposition (A generlaization of Tian–Zhu's result. I. '19)

$(X, \omega) \curvearrowleft K$: Hamiltonian action. There is a functional

$$\check{\boldsymbol{\mu}}^\lambda : \mathfrak{k} \longrightarrow \mathbb{R}$$
$$\xi \mapsto \check{W}^\lambda(\omega, \mu_\xi^\omega)$$

satisfying the following.

1. Its derivative at $\xi \in \mathfrak{k}$ is Fut_ξ^λ.
2. It is proper and bounded from above, hence attains a maximum, which implies the existence of ξ with $\mathrm{Fut}_\xi^\lambda = 0$.
3. The value

$$\lambda_{\mathrm{ice}}(X, L) := \sup\{\lambda \in \mathbb{R} \mid \check{\boldsymbol{\mu}}^{\lambda'} \text{ has a unique crit. pt. for } \forall \lambda' < \lambda\}$$

is finite (never $\pm\infty$). $\rightsquigarrow$ phase transition

3. μ-cscK metric and Perelman's entropy

Perelman's W-entropy

Perelman's W-entropy

For a Kähler metric $\omega \in \mathcal{H}(X, L)$ and a smooth function $f \in C^\infty(X, \mathbb{R})$ normalized as $\int_X e^f \omega^n/n! = 1$, we put

$$\check{W}(\omega, f) := -\int_X (s(\omega) + |\partial^\sharp f|^2) e^f \omega^n/n!,$$

$$\check{S}(\omega, f) := \int_X (n + f) e^f \omega^n/n!,$$

$$\check{W}^\lambda(\omega, f) := \check{W}(\omega, f) + \lambda \check{S}(\omega, f).$$

$$\check{W}^\lambda : T\mathcal{H}(X, L) = \mathcal{H}(X, L) \times C^\infty(X)/\mathbb{R} \to \mathbb{R}.$$

Original: $\dfrac{1}{(4\pi\tau)^n} \displaystyle\int_X \big(\tau(R(\omega) + |\nabla f|^2) - (2n - f)\big) e^{-f} \mathrm{vol}_g$

μ-cscK metric is critical points of W-functional

Theorem (I. '21)

A state $(\omega, f) \in T\mathcal{H}(X, L)$ is a critical point of $\check{W}^\lambda$ if and only if $\xi = \nabla f$ is real holomorphic and ω is a $\check{\mu}^\lambda_\xi$-cscK metric.

We put

$$\check{\boldsymbol{\mu}}^\lambda(\omega) := \sup_{\int_X e^f \omega^n/n! = 1} \check{W}^\lambda(\omega, f).$$

Theorem (Rothaus '81, I. '21)

For each λ and ω, there exists f attaining the maximum of $\check{W}^\lambda(\omega, \cdot)$. When $\lambda \leq 0$, such f is unique. Consequently, $\check{\boldsymbol{\mu}}^\lambda$ is smooth on $\mathcal{H}(X, L)$ for $\lambda \leq 0$. The critical points are precisely μ^λ-cscK metrics in this case.

The Perelman μ-entropy is analogous to Calabi functional $\int_X \hat{s}^2 \omega^n/n!$ for extremal metric in a Kähler class or H-entropy $\int_X h e^h \omega^n/n!$ for KRs.

Algebraic bound on Perelman μ-entropy

For $(X, L) = (X, -K_X)$, we have $\check{\boldsymbol{\mu}}^{2\pi} \geq 2\pi H$.

$$\check{\boldsymbol{\mu}}^{2\pi} : H \overset{analogy}{\sim} Mabuchi : Ding$$

Theorem (I. '21)

$$\sup_{(\mathcal{X},\mathcal{L};\tau)} \check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X},\mathcal{L};\tau) \leq \inf_{\omega \in \mathcal{H}(X,L)} \check{\boldsymbol{\mu}}^{\lambda}(\omega)$$

Cororally

For $\lambda \leq 0$, if there is a $\check{\mu}^{\lambda}_{\xi}$-cscK metric ω, we have

$$\sup_{(\mathcal{X},\mathcal{L};\tau)} \boldsymbol{\mu}^{\lambda}_{\mathrm{NA}}(\mathcal{X},\mathcal{L};\tau) = \boldsymbol{\mu}^{\lambda}_{\mathrm{NA}}(\xi) = \boldsymbol{\mu}^{\lambda}(\omega) = \inf_{\omega_\varphi \in \mathcal{H}(X,L)} \boldsymbol{\mu}^{\lambda}(\omega_\varphi).$$

In particular, all the critical points of $\boldsymbol{\mu}^{\lambda}$ are global minimizers, which are precisely μ^{λ}-cscK metrics.

Characteristic μ-entropy

For a test configuration $(\mathcal{X}, \mathcal{L})$, we have the expansion

$$\int_{\mathbb{R}} e^{-\tau.t} \nu_m(t) = \int_{\mathbb{R}} e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})} - \frac{1}{2} (\kappa_{\mathcal{X}_0}^{\mathbb{C}^\times} . e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau). m^{-1} + O(m^{-2})$$

for

$$\nu_m = -\frac{1}{m^n} \sum_{\lambda \in \mathbb{Z}} \dim H^0(\mathcal{X}_0, \mathcal{L}_0^{\otimes m})_\lambda \quad \xrightarrow{m \to \infty} \quad \mathrm{DH}_{(\mathcal{X},\mathcal{L})}.$$

For $\tau \geq 0$,

$$\check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) := 2\pi \frac{(\kappa_{\mathcal{X}_0}^{\mathbb{C}^\times} . e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)}{\int_{\mathbb{R}} e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}} = 2\pi \frac{(K_X . e^L) - \tau.(K_{\bar{\mathcal{X}}/\mathbb{P}^1}^{\log, \mathbb{C}^\times} . e^{\mathcal{L}_{\mathbb{C}^\times}}; \tau)}{\int_{\mathbb{R}} e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}}$$

$$\check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \tau) := \frac{\int_{\mathbb{R}} (n - \tau.t) e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}}{\int_{\mathbb{R}} e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}} - \log \int_{\mathbb{R}} e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}$$

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) := \check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) + \lambda \check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \tau)$$

Sketch of proof

- $\check{W}^\lambda(\omega_{\phi_{\tau.t}}, -\dot{\phi}_{\tau.t})$ is monotonically decreasing.
  Consider the action functional $\mathcal{A}(t) :=$ "$- \int_0^t \check{W}^\lambda(\omega_{\phi_{\tau.s}}, -\dot{\phi}_{\tau.s}) ds$" and show its convexity by a tensor calculus on equivariant differential forms and Berman–Berndtsson's subharmonicity argument. (cf. the convexity of Mabuchi functional)
- $\lim_{t \to \infty} \check{W}^\lambda(\omega_{\phi_{\tau.t}}, -\dot{\phi}_{\tau.t}) = \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau)$
  The equivariant Stokes theorem and the regularity of geodesic ray across the central fibre (Chu–Tosatti–Weinkove).

$$\check{\boldsymbol{\mu}}^\lambda(\omega) \geq \check{W}^\lambda(\omega_{\phi_{\tau.t}}, -\dot{\phi}_{\tau.t}) \to \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau)$$

Universal aspect

$$\boxed{H\text{-entropy}} \quad H(\omega) = \frac{\int_X h e^h \omega^n}{\int_X e^h \omega^n} - \log \int_X e^h \omega^n / n!$$

Introduce

$$\check{\mathcal{L}}(\omega, f) = \frac{\int_X f e^h \omega^n}{\int_X e^h \omega^n} - \log \int_X e^f \omega^n / n!,$$

then $H(\omega) = \sup_f \check{\mathcal{L}}(\omega, f)$ and $H_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = \lim_{t \to \infty} \check{\mathcal{L}}(\omega_{\phi_{\tau.t}}, \dot{\phi}_{\tau.t})$

$$\boxed{\text{Calabi functional}} \qquad C(\omega) = \frac{1}{2} \frac{\int_X \hat{s}^2 \omega^n}{\int_X \omega^n}$$

Introduce

$$\mathcal{W}_{\mathrm{ext}}(\omega, f) = -\frac{\int_X (\hat{s} - \hat{f})^2 \omega^n}{2 \int_X \omega^n} + \frac{\int_X \hat{s}^2 \omega^n}{2 \int_X \omega^n},$$

then $C(\omega) = \sup_f \mathcal{W}_{\mathrm{ext}}(\omega, f)$ and $C_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) := \lim_{t \to \infty} \mathcal{W}_{\mathrm{ext}}(\omega_{\phi_{\tau.t}}, \dot{\phi}_{\tau.t})$, for which we have $\max_{\tau \geq 0} C_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = \frac{2\pi^2}{(L^{\cdot n})} \frac{DF(\mathcal{X}, \mathcal{L})^2}{\|(\mathcal{X}, \mathcal{L})\|^2}$ when $DF(\mathcal{X}, \mathcal{L}) \leq 0$.

4. μK-stability and non-archimedean μ-entropy

Characteristic μ-entropy and μK-semistability

We can extend $\check{\boldsymbol{\mu}}^\lambda$ to f.g. filtration of $R = \bigoplus_m H^0(X, L^{\otimes m})$.

Proposition (I. '21 (to appear), essentially proved in I. '20)

There is a family of f.g. filtrations $\{\mathcal{F}_{\xi+\tau(\mathcal{X},\mathcal{L})}\}_{\tau \in [0,\infty)}$ satisfying $\check{\boldsymbol{\mu}}^\lambda(\mathcal{F}_\xi) = \check{\boldsymbol{\mu}}^\lambda(\xi)$ and

$$\frac{d}{d\tau}\Big|_{\tau=0} \check{\boldsymbol{\mu}}^\lambda(\mathcal{F}_{\xi+\tau(\mathcal{X},\mathcal{L})}) = -\mathrm{Fut}^\lambda_\xi(\mathcal{X},\mathcal{L}).$$

Corolally (I. '21)

If $\check{\boldsymbol{\mu}}^\lambda$ is maximized by $\xi \in \mathfrak{t}$, then (X, L) is $\check{\mu}^\lambda_\xi$K-semistable.
This is the case when (X, L) admits a $\check{\mu}^\lambda_\xi$-cscK metric for $\lambda \leq 0$.

Optimal degeneration and μK-semistability

Theorem (I. '21, to appear)

If a f.g. filtration $\mathcal{F}$ maximizes $\check{\boldsymbol{\mu}}^\lambda$, then the central fibre $(\mathcal{X}_o, \mathcal{L}|_{\mathcal{X}_o}) = \mathrm{Proj} \bigoplus_m \bigoplus_{\lambda \in \mathbb{R}} t^{-\lambda}.\mathcal{F}^\lambda R_m / \mathcal{F}^{\lambda+} R_m$ is $\check{\mu}_\xi^\lambda$K-semistable.

Question: Show the existence and uniqueness of maximizers of $\check{\boldsymbol{\mu}}^\lambda$.

Proposal :

- Consider a 'completion' of the space of f.g. filtrations and extend $\check{\boldsymbol{\mu}}_{\mathrm{NA}}^\lambda$ to the completion as an usc function.
- Show compactness maximizing sequences $\rightsquigarrow$ existence.
- Prove the 'regularity' of the maximizer.

cf. Similar approach by M. Xia in the context of extremal metric.

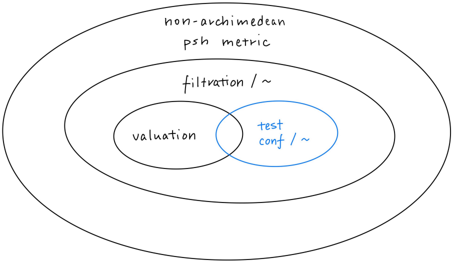
Completion approach for optimal degeneration

Boucksom–Jonsson: a completion $\mathcal{E}^1_{\mathrm{NA}}(X, L)$ of the space of TCs.

Theorem (I. '21, to appear)

There is a complete metric space $\mathcal{E}^{\mathrm{exp}}_{\mathrm{NA}}(X, L) \subset \mathcal{E}^1_{\mathrm{NA}}(X, L)$ and a functional $\boldsymbol{\mu}^\lambda_{\mathrm{NA}} : \mathcal{E}^{\mathrm{exp}}_{\mathrm{NA}}(X, L) \to [-\infty, \infty)$ which extends $\boldsymbol{\mu}^\lambda_{\mathrm{NA}}$ for test configurations.

We can also extend the H-entropy as $H_{\mathrm{NA}} : \mathcal{E}^{\mathrm{exp}}(X, L) \to \mathbb{R}$. We have $\check{\boldsymbol{\mu}}^{2\pi}_{\mathrm{NA}} \leq 2\pi H_{\mathrm{NA}}$ for $(X, L) = (X, -K_X)$.



Thank you for listening!