Perelman's entropy and optimal degeneration

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Contents

1. Model story: optimal degeneration along Kähler–Ricci flow

Optimal degeneration along Kähler–Ricci flow

Theorem (Chen–Sun–Wang '15)

For any Fano manifold X and any initial Kähler metric ω, the Gromov–Hausdorff limit along Kähler–Ricci flow exists as a Fano variety $\hat{X}(\omega)$ metrized by a Kähler–Ricci soliton $\hat{\omega}$: $\mathrm{Ric}(\hat{\omega}) - L_\xi \hat{\omega} = \hat{\omega}$.

Moreover, there exist the following two step degenerations:

1. $X \rightsquigarrow \bar{X}(\omega)$ via filtration $\mathcal{F}_v$, which generates $\bar{X}(\omega) \circlearrowleft \mathbb{R}$
2. $\bar{X}(\omega) \rightsquigarrow \hat{X}(\omega)$ via $\mathbb{R}$-equivariant special degeneration, on which $\hat{X}(\omega) \circlearrowleft \mathbb{R} = \exp(\mathbb{R}\xi)$.

$$\bar{X}(\omega) = \mathrm{Proj}\Big(\bigoplus_{m \geq 0} \bigoplus_{\lambda \in \mathbb{R}} t^{-\lambda} \mathcal{F}^\lambda H^0(X, -mK_X) / \mathcal{F}^{\lambda+} H^0(X, -mK_X) \Big)$$

$\bar{X}(\omega)$ is a modified K-semistable Fano variety.

cf. I.'18: moduli of mK-ss Fano manifolds (+ thesis $\rightsquigarrow$ algebraic moduli)

Non-archimedean H-entropy

- Dervan–Székelyhidi '16: The H-entropy

$$H_{\mathrm{NA}}(\mathcal{F}) := -\inf_{w}(A_X(w) + \varphi_{\mathcal{F}}(w)) - \log \int_{\mathbb{R}} e^{-t}\mathrm{DH}_{\mathcal{F}}$$

is maximized by CSW's degeneration $\mathcal{F}_v : X \rightsquigarrow \bar{X}(\omega)$ (for any ω).
Here $\varphi_{\mathcal{F}}(w) = \inf\{\sigma \in \mathbb{R} \mid \mathcal{F}^\lambda \subset \mathcal{F}_w^\lambda[\sigma] = \mathcal{F}_w^{\lambda+\bullet\sigma}\}$.
Moreover, for $H(\omega_\varphi) = \int_X h_\varphi e^{h_\varphi}\omega_\varphi^n/n!$, we have

$$\sup_{\mathcal{F}:\text{f.g.}} H_{\mathrm{NA}}(\mathcal{F}) = \inf_{\varphi} H(\omega_\varphi).$$

- Han–Li '20: The maximizers of H_{NA} are unique, hence $\bar{X} = \bar{X}(\omega)$ is independent of the initial metric ω: the degeneration $X \rightsquigarrow \bar{X}$ is canonically attached to each Fano manifold X.

Even when X is singular,

- Han–Li '20: The maximizer of H_{NA} is realized by a q.m. valuation v.
- Blum–Liu–Xu–Zhuang '21: $\mathrm{gr}_v R$ is f.g. $\rightsquigarrow$ $\bar{X}$ is mK-ss Fano variety.

2. Brief introduction to μ-cscK metrics

μ-cscK metric

CscK metric is a natural generalization of Kähler–Einstein metric.

What about Kähler–Ricci soliton?

- In view of K-stability, we want moment map picture.
- In view of optimal degeneration, we want volume minimization.

$\rightsquigarrow$ $\boxed{\mu\text{-cscK metric}}$

Definition

A Kähler metric ω is called $\check{\mu}_\xi^\lambda$-cscK metric if

$$s_{\mu_\xi} := (s(\omega) + \bar{\square}\mu_\xi) + (\bar{\square}\mu_\xi + |\partial^\sharp \mu_\xi|^2) = \lambda \mu_\xi + \mathrm{const}.$$

- $\check{\mu}_{-2\xi}^{2\pi\lambda}$-cscK metric in $\lambda^{-1} c_1(X) \Leftrightarrow$ KRs $\mathrm{Ric}(\omega) - L_\xi \omega = \lambda \omega$.
- Extremal metric $\Rightarrow \mu^\lambda$-cscK metric for $\lambda \ll 0$.

Moment map picture for μ-cscK metric

$(M,\omega) \curvearrowleft T$: Hamiltonian torus action. For $\xi \in \mathfrak{t}$ and $\lambda \in \mathbb{R}$, the map

$$\mathcal{S}_\xi^\lambda : \mathcal{J}_T \to \mathfrak{ham}_T^\vee : J \mapsto (s_{\mu_\xi}(g_J) - \lambda\mu_\xi)e^{\mu_\xi}\omega^n$$

gives a moment map for a symplectic manifold $(\mathcal{J}_T, \Omega_\xi) \curvearrowleft \mathrm{Ham}_T$.

μ_ξ^λ-Futaki invariant

$$\mathrm{Fut}_\xi^\lambda(\zeta) := -\int_X \mu_\zeta(s_{\mu_\xi}(\omega) - \lambda\mu_\xi)e^{\mu_\xi}\omega^n.$$

$\rightsquigarrow$ generalized to test configuration via equivariant intersection formula

$\Rightarrow$ μ_ξ^λK-stability $\supset$ modified K-stability

Theorem (Lahdili '19, I. '20, Apostolov–Jubert–Lahdili '21)

If (X,L) admits a $\check{\mu}_\xi^\lambda$-cscK metric, then it is $\check{\mu}_\xi^\lambda$K-semistable.

Volume minimization: μ-Futaki invariant vanishing

Proposition (A generlaization of Tian–Zhu's result. I. '19)

$(X, \omega) \curvearrowleft K$: Hamiltonian action. There is a functional

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}} : \mathfrak{k} \longrightarrow \mathbb{R}$$
$$\xi \mapsto \check{W}^{\lambda}(\omega, \mu^{\omega}_{\xi})$$

satisfying the following.

1. Its derivative at $\xi \in \mathfrak{k}$ is $\mathrm{Fut}^{\lambda}_{\xi}$.
2. It is proper and bounded from above, hence attains a maximum, which implies the existence of ξ with $\mathrm{Fut}^{\lambda}_{\xi} = 0$.
3. The value

$$\lambda_{\mathrm{ice}}(X, L) := \sup\{\lambda \in \mathbb{R} \mid \check{\boldsymbol{\mu}}^{\lambda'}_{\mathrm{NA}} \text{ has a unique crit. pt. for } \forall \lambda' < \lambda\}$$

is finite (never $\pm\infty$). $\rightsquigarrow$ phase transition

3. μK-stability and non-archimedean μ-entropy

Characteristic μ-entropy

For a test configuration $(\mathcal{X}, \mathcal{L})$, we have the expansion

$$\int_{\mathbb{R}} e^{-\tau.t} \nu_m(t) = \int_{\mathbb{R}} e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})} - \frac{1}{2} (\kappa^{\mathbb{C}^\times}_{\mathcal{X}_0} . e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau). m^{-1} + O(m^{-2})$$

for

$$\nu_m = -\frac{1}{m^n} \frac{d}{dt} \dim \mathcal{F}^t_{(\mathcal{X},\mathcal{L})} R_m \quad \xrightarrow{m \to \infty} \quad \mathrm{DH}_{(\mathcal{X},\mathcal{L})}.$$

Definition (Characteristic μ-entropy)

For $\tau \geq 0$,

$$\check{\boldsymbol{\mu}}(\mathcal{X}, \mathcal{L}; \tau) := 2\pi \frac{(\kappa^{\mathbb{C}^\times}_{\mathcal{X}_0} . e^{\mathcal{L}_{\mathbb{C}^\times}|\mathcal{X}_0}; \tau)}{\int_{\mathbb{R}} e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}} = 2\pi \frac{(K_X . e^L) - \tau.(K^{\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1} . e^{\mathcal{L}_{\mathbb{C}^\times}}; \tau)}{\int_{\mathbb{R}} e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}}$$

$$\check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \tau) := \frac{\int_{\mathbb{R}} (n - \tau.t) e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}}{\int_{\mathbb{R}} e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}} - \log \int_{\mathbb{R}} e^{-\tau.t} \mathrm{DH}_{(\mathcal{X},\mathcal{L})}$$

$$\check{\boldsymbol{\mu}}^\lambda(\mathcal{X}, \mathcal{L}; \tau) := \check{\boldsymbol{\mu}}(\mathcal{X}, \mathcal{L}; \tau) + \lambda \check{\boldsymbol{\sigma}}(\mathcal{X}, \mathcal{L}; \tau)$$

Non-archimedean μ-entropy

Definition (Non-archimedean μ-entropy)

$$\check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X},\mathcal{L};\tau) := 2\pi \frac{(K_X.e^L) - \tau.(K^{\log,\mathbb{C}^\times}_{\bar{\mathcal{X}}/\mathbb{P}^1}.e^{\mathcal{L}_{\mathbb{C}^\times}};\tau)}{\int_{\mathbb{R}} e^{-\tau.t}\mathrm{DH}_{(\mathcal{X},\mathcal{L})}} \geq \check{\boldsymbol{\mu}}(\mathcal{X},\mathcal{L};\tau)$$

$$\check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X},\mathcal{L};\tau) := \check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X},\mathcal{L};\tau) + \lambda\check{\boldsymbol{\sigma}}(\mathcal{X},\mathcal{L};\tau)$$

We have $\check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X}_d,\mathcal{L}_d;\tau) = \check{\boldsymbol{\mu}}_{\mathrm{NA}}(\mathcal{X},\mathcal{L};d\tau)$.

Non-archimedean μ-entropy for toric test configuration

Let P be the moment polytope of a toric variety (X, L). For a convex function q normalized as $\int_P e^q d\mu = 1$, we put

$$\boldsymbol{\mu}^{\lambda}_{\mathrm{NA}}(q) := \int_{\partial P} e^q d\sigma - \lambda \int_P (n+q)e^q d\mu.$$

Then we have $\boldsymbol{\mu}^{\lambda}_{\mathrm{NA}}(\mathcal{X},\mathcal{L};\tau) = \boldsymbol{\mu}^{\lambda}_{\mathrm{NA}}(\tau q_{(\mathcal{X},\mathcal{L})})$.

Perturbation of test configurations

$\sigma \subset N \otimes \mathbb{R}$: a strictly convex full-dim $\mathbb{Q}$-polyhedral cone

$\rightsquigarrow B_\sigma = \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap M] \curvearrowleft T = N \otimes \mathbb{C}^\times$.

Definition (Polyhedral configuration)

A polyhedral configuration consists of a triple $(\mathcal{X}/B_\sigma, \mathcal{L}; \xi)$ of

- a T-equivariant projective flat family $\mathcal{X}$ over B_σ,
- a relatively ample $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}/B_\sigma$,
- a vector $\xi \in \sigma$

Polyhedral configuration defines a f.g. filtration

$$\mathcal{F}^t R_m := \sum_{\langle \mu, \xi \rangle \geq \lambda} \{s \in H^0(X, L^{\otimes m}) \mid s.\chi^{-\mu} \in H^0(\mathcal{X}, \mathcal{L}^{\otimes m})\}.$$

We can define $\check{\boldsymbol{\mu}}^\lambda(\mathcal{F})$ similarly as $\check{\boldsymbol{\mu}}^\lambda(\mathcal{X}, \mathcal{L}; \tau)$.

Characteristic μ-entropy and μK-semistability

- Suppose $(X, L) \curvearrowleft T$. Take $\xi \in \mathfrak{t}$ and $\sigma \subset \mathfrak{t}$ so that $\xi \in \sigma$.

  $\rightsquigarrow (B_\sigma \times X / B_\sigma, p_X^* L; \xi)$ endowed with the diagonal $B_\sigma \times X \curvearrowleft T$.

- For a T-equiv. tc $(\mathcal{X}, \mathcal{L})$, $(B_\sigma \times \mathcal{X} / B_{\sigma \times [0,\infty)}, p_{\mathcal{X}}^* \mathcal{L}; \xi + \tau.\eta)$ endowed with the diagonal $B_\sigma \times \mathcal{X} \curvearrowleft T \times \mathbb{C}^\times$ is a pc for $\sigma' = \sigma \times [0, \infty)$.

Let $\mathcal{F}_{\xi + \tau(\mathcal{X}, \mathcal{L})}$ denote the associated filtration.

Proposition (I. '21 (to appear), essentially proved in I. '20)

$$\frac{d}{d\tau}\Big|_{\tau=0} \check{\boldsymbol{\mu}}^\lambda (\mathcal{F}_{\xi + \tau(\mathcal{X}, \mathcal{L})}) = -\mathrm{Fut}_\xi^\lambda (\mathcal{X}, \mathcal{L}).$$

Cororally (I. '21)

If $\check{\boldsymbol{\mu}}^\lambda$ is maximized by $\xi \in \mathfrak{t}$, then (X, L) is $\check{\mu}_\xi^\lambda$K-semistable.
This is the case when (X, L) admits a $\check{\mu}_\xi^\lambda$-cscK metric for $\lambda \leq 0$.

Optimal degeneration and μK-semistability

Theorem (I. '21, to appear)

If a polyhedral configuration $(\mathcal{X}/B_\sigma, \mathcal{L}; \xi)$ maximizes $\check{\boldsymbol{\mu}}^\lambda$, then the central fibre $(\mathcal{X}_o, \mathcal{L}|_{\mathcal{X}_o})$ is $\check{\mu}^\lambda_\xi$K-semistable.

Question: Show the existence and uniqueness of maximizers of $\check{\boldsymbol{\mu}}^\lambda$.

Remark: Uniqueness seems difficult since $\check{\boldsymbol{\mu}}^\lambda$ is not concave even on $\mathfrak{t}$.

Proposal :

- Consider a 'completion' of the space of f.g. filtrations and extend $\check{\boldsymbol{\mu}}^\lambda_{\mathrm{NA}}$ to the completion as an usc function.
- Show compactness maximizing sequences $\rightsquigarrow$ existence.
- Prove the 'regularity' of the maximizer.

cf. Similar approach by M. Xia in the context of extremal metric.

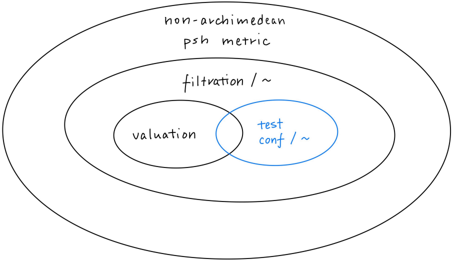
Completion approach for optimal degeneration

Bouckson–Jonsson: a completion $\mathcal{E}^1_{\mathrm{NA}}(X, L)$ of the space of TCs.

Theorem (I. '21, to appear)

There is a complete metric space $\mathcal{E}^{\exp}_{\mathrm{NA}}(X, L) \subset \mathcal{E}^1_{\mathrm{NA}}(X, L)$ and a functional $\boldsymbol{\mu}^\lambda_{\mathrm{NA}} : \mathcal{E}^{\exp}_{\mathrm{NA}}(X, L) \to [-\infty, \infty)$ which extends $\boldsymbol{\mu}^\lambda_{\mathrm{NA}}$ for test configurations.

We can also extend the H-entropy as $H_{\mathrm{NA}} : \mathcal{E}^{\exp}(X, L) \to \mathbb{R}$. We have $\check{\boldsymbol{\mu}}^{2\pi}_{\mathrm{NA}} \leq 2\pi H_{\mathrm{NA}}$ for $(X, L) = (X, -K_X)$.



4. μ-cscK metric and Perelman's entropy

Perelman's W-entropy

Perelman's W-entropy

For a Kähler metric $\omega \in \mathcal{H}(X, L)$ and a smooth function $f \in C^\infty(X, \mathbb{R})$ normalized as $\int_X e^f \omega^n/n! = 1$, we put

$$\check{W}(\omega, f) := -\int_X (s(\omega) + |\partial^\sharp f|^2) e^f \omega^n/n!,$$

$$\check{S}(\omega, f) := \int_X (n + f) e^f \omega^n/n!,$$

$$\check{W}^\lambda(\omega, f) := \check{W}(\omega, f) + \lambda \check{S}(\omega, f).$$

$$\check{W}^\lambda : T\mathcal{H}(X, L) = \mathcal{H}(X, L) \times C^\infty(X)/\mathbb{R} \to \mathbb{R}.$$

μ-cscK metric is critical points of W-functional

Theorem (I. '21)

A state $(\omega, f) \in T\mathcal{H}(X, L)$ is a critical point of $\check{W}^\lambda$ if and only if $\xi = \nabla f$ is real holomorphic and ω is a $\check{\mu}^\lambda_\xi$-cscK metric.

We put

$$\check{\boldsymbol{\mu}}^\lambda(\omega) := \sup_{\int_X e^f \omega^n/n! = 1} \check{W}^\lambda(\omega, f).$$

Theorem (Rothaus '81, I. '21)

For each λ and ω, there exists f attaining the maximum of $\check{W}^\lambda(\omega, \cdot)$. When $\lambda \leq 0$, such f is unique. Consequently, $\check{\boldsymbol{\mu}}^\lambda$ is smooth on $\mathcal{H}(X, L)$ for $\lambda \leq 0$. The critical points are precisely μ^λ-cscK metrics in this case.

The Perelman μ-entropy is analogous to Calabi functional $\int_X s^2 \omega^n/n!$ for extremal metric in a Kähler class or H-entropy $\int_X h e^h \omega^n/n!$ for KRs.

Perelman μ-entropy

For $(X, L) = (X, -K_X)$, we have $\check{\boldsymbol{\mu}}^{2\pi} \geq 2\pi H$.

$$\check{\boldsymbol{\mu}}^{2\pi} : H \overset{analogy}{\sim} \textit{Mabuchi} : \textit{Ding}$$

Theorem (I. '21)

$$\sup_{(\mathcal{X}, \mathcal{L}; \tau)} \check{\boldsymbol{\mu}}^{\lambda}_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) \leq \inf_{\omega \in \mathcal{H}(X, L)} \check{\boldsymbol{\mu}}^{\lambda}(\omega)$$

cf. Dervan–Székelyhidi (and Donaldson, Hisamoto, Xia):

$$\sup_{\mathcal{F}:\text{f.g.}} H_{\mathrm{NA}}(\mathcal{F}) = \inf_{\varphi} H(\omega_{\varphi}).$$

Sketch of proof

- $\check{W}^\lambda(\omega_{\phi_{\tau.t}}, -\dot{\phi}_{\tau.t})$ is monotonically decreasing.
  Consider the action functional $\mathcal{A}(t) :=$ "$-\int_0^t \check{W}^\lambda(\omega_{\phi_{\tau.s}}, -\dot{\phi}_{\tau.s}) ds$" and show its convexity by a tensor calculus on equivariant differential forms and Berman–Berndtsson's subharmonicity argument. (cf. the convexity of Mabuchi functional)
- $\lim_{t\to\infty} \check{W}^\lambda(\omega_{\phi_{\tau.t}}, -\dot{\phi}_{\tau.t}) = \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau)$
  The equivariant Stokes theorem and the regularity of geodesic ray across the central fibre (Chu–Tosatti–Weinkove).

$$\check{\boldsymbol{\mu}}^\lambda(\omega) \geq \check{W}^\lambda(\omega_{\phi_{\tau.t}}, -\dot{\phi}_{\tau.t}) \to \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau)$$

Corolally

For $\lambda \leq 0$, all the critical points of $\boldsymbol{\mu}^\lambda$ are global minimizers, which are precisely μ^λ-cscK metrics. If there is a $\check{\mu}^\lambda_\xi$-cscK metric ω, we have

$$\sup_{(\mathcal{X}, \mathcal{L}; \tau)} \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = \boldsymbol{\mu}^\lambda_{\mathrm{NA}}(\phi_\xi) = \boldsymbol{\mu}^\lambda(\omega) = \inf_{\omega_\varphi \in \mathcal{H}(X,L)} \boldsymbol{\mu}^\lambda(\omega_\varphi).$$

Universal aspect

$$\boxed{H\text{-entropy}} \quad H(\omega) = \frac{\int_X h e^h \omega^n}{\int_X e^h \omega^n} - \log \int_X e^h \omega^n / n!$$

Introduce

$$\check{\mathcal{L}}(\omega, f) = \frac{\int_X f e^h \omega^n}{\int_X e^h \omega^n} - \log \int_X e^f \omega^n / n!,$$

then $H(\omega) = \sup_f \check{\mathcal{L}}(\omega, f)$ and $H_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = \lim_{t \to \infty} \check{\mathcal{L}}(\omega_{\phi_{\tau.t}}, \dot{\phi}_{\tau.t})$

$$\boxed{\text{Calabi functional}} \qquad C(\omega) = \frac{\int_X (s - \bar{s})^2 \omega^n}{2 \int_X \omega^n}$$

Introduce

$$\mathcal{W}_{\mathrm{ext}}(\omega, f) = -\frac{\int_X ((s - f) - (\bar{s} - \bar{f}))^2 \omega^n}{2 \int_X \omega^n} + \frac{\int_X (s - \bar{s})^2 \omega^n}{2 \int_X \omega^n},$$

then $C(\omega) = \sup_f \mathcal{W}_{\mathrm{ext}}(\omega, f)$ and $C_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) := \lim_{t \to \infty} \mathcal{W}_{\mathrm{ext}}(\omega_{\phi_{\tau.t}}, \dot{\phi}_{\tau.t})$, for which we have $\max_{\tau \geq 0} C_{\mathrm{NA}}(\mathcal{X}, \mathcal{L}; \tau) = \frac{2\pi^2}{(L^{\cdot n})} \frac{DF(\mathcal{X}, \mathcal{L})^2}{\|(\mathcal{X}, \mathcal{L})\|^2}$ when $DF(\mathcal{X}, \mathcal{L}) \leq 0$.

Thank you for listening!