μ-cscK metric and μK-stability of polarized manifolds

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Contents:

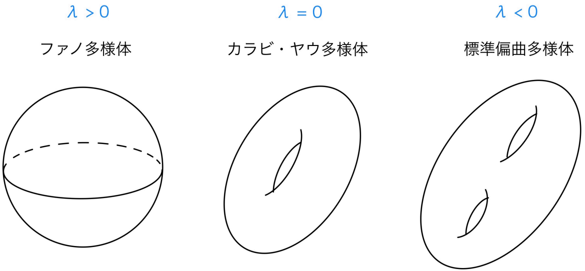
1. Background: cscK metrics & K-stability

cscK metrics and Kähler–Einstein metrics

For a polarized manifold (X, L), when the Kähler class $c_1(L)$ admits a Kähler metric ω with constant scalar curvature (cscK metric)?

Kähler–Einstein metric: When $\lambda c_1(L) = 2\pi c_1(X)$ for some $\lambda \in \mathbb{R}$, then ω is cscK iff it satisfies $\mathrm{Ric}(\omega) = \lambda\omega$.

- $(\lambda < 0)$ $K_X > 0 \Rightarrow \exists$ unique KE metric.
- $(\lambda = 0)$ $K_X \equiv 0 \Rightarrow \exists$ unique Ricci flat metric in any L.
- $(\lambda > 0)$ $K_X < 0 \Rightarrow$ Futaki invariant is an obstruction.



Yau–Tian–Donaldson conjecture

Yau–Tian–Donaldson conjecture

$\exists$ cscK metrics in $c_1(L) \iff (X, L)$ is K-'poly'stable.

cf. Kobayashi–Hitchin correspondence (Donaldson, Uhlenbeck–Yau's theorem)

For a normal test configuration $(\mathcal{X}/\mathbb{C}, \mathcal{L})$ of (X, L), the Donaldson–Futaki invariant is given by

$$DF(\mathcal{X}, \mathcal{L}) := (K_{\bar{\mathcal{X}}/\mathbb{C}P^1}.\mathcal{L}^{\cdot n}) - \frac{n}{n+1}\frac{(K_X.L^{\cdot(n-1)})}{(L^{\cdot n})}(\bar{\mathcal{L}}^{\cdot(n+1)}).$$

The K-(semi)stability of (X, L) is the positivity (non-negativity) of Donaldson–Futaki invariants. cf. Hilbert-Mumford criterion

Donaldson–Fujiki moment map picture

Donaldson-Fujiki moment map picture

(M, ω): C^∞-symplectic manifold. Scalar curvature gives a moment map on $\mathcal{J}(M, \omega)$. Namely, the map $\mathcal{S} : \mathcal{J}(M, \omega) \to \mathrm{Lie}(\mathrm{Ham}(M, \omega))^\vee$ given by

$$\langle \mathcal{S}(J), f \rangle = \int_M (s(g_J) - \bar{s}) f \omega^n$$

is a unique moment map for the symplectic structure Ω on $\mathcal{J}(M, \omega)$:

$$\Omega_J(A, B) = \int_M (JA, B)_{g_J} \omega^n.$$

Kempf–Ness theorem: model of YTD conjecture

Let $(B, \Omega + \nu) \circlearrowleft K$ be a projective manifold with a Hamiltonian action of compact Lie group K.

- $G = K^c$: the complexification
- $L_G := [\Omega + \nu] \in H^2_K(B, \mathbb{R}) = H^2_G(B, \mathbb{R})$
- $\eta^\vee \in H^2_{\mathbb{C}^\times}(\mathbb{C}, \mathbb{R})$: the positive generator

Kempf–Ness theorem (+ Hilbert–Mumford criterion)

For $b \in B$,

- (Semistability) $\nu^{-1}(0) \cap \overline{b.G} \neq \emptyset \iff$ for every $\Lambda : \mathbb{C}^\times \to G$

$$-\Lambda_b^* L_G / \eta^\vee = -\langle \nu(\Lambda_b(0))^c, \Lambda_* \eta \rangle \geq 0.$$

- ('Poly'stability) $\nu^{-1}(0) \cap b.G \neq \emptyset \iff$ if moreover $\Lambda_b^* L_G / \eta^\vee = 0$ only when $\Lambda : \mathbb{C}^\times \to G_x$.

Semistability is Zariski open condition, while polystability is not so.

Uniqueness and Existence

Theorem (Berman-Berndtsson)

CscK metrics in $c_1(L)$ are unique modulo $\mathrm{Aut}^0(X, L)$.

Theorem (Bando–Mabuchi, Stoppa, Berman-Darvas-Lu, et al.)

If the Kähler class $c_1(L)$ admits a cscK metric, then (X, L) is K-‘poly’stable.

Theorem (Chen-Donaldson-Sun, Tian, (Aubin, Yau, Odaka))

When $-K_X \in \mathbb{R}.L$, the Kähler class $c_1(L)$ admits a cscK metric (KE metric) if (and only if) (X, L) is K-‘poly’stable.

Moduli space of Kähler–Einstein Fano varieties

Theorem (Paul-Tian)

For a G-equivariant family $(\mathcal{X}, \mathcal{L}) \to B$ of polarized schemes, there exists a G-equivariant line bundle $CM(\mathcal{X}, \mathcal{L})$ on B such that

$$-c_1^{\mathbb{C}^\times}(f^* CM(\mathcal{X}, \mathcal{L})) = DF(f^* \mathcal{X}, f^* \mathcal{L}).\eta^\vee \in H^2_{\mathbb{C}^\times}(\mathbb{C}, \mathbb{Z}) \cong \mathbb{Z}.\eta^\vee$$

for every $\mathbb{C}^\times$-equivariant morphism $f : \mathbb{C} \to B$.

$+$

Donaldson–Sun, Chen–Donaldson–Sun, Tian

$\Downarrow$

Theorem (Odaka, Li-Wang-Xu)

$\mathbb{Q}$-smoothable Fano varieties with Kähler–Einstein metrics form a proper algebraic moduli space.

2. Background: Kähler–Ricci solitons & modified K-stability

Kähler–Ricci soliton

Examples:

- Fano manifold $X = k$-point blow up of $\mathbb{C}P^n$ $(k = 1, \ldots, n)$ does not admit KE metrics.
- A toric Fano manifold admits a KE metric iff the barycenter of Fano polytope is the origin.
- There are infinitely many toric Fano orbifolds with no KE metrics, while toric Fano orbifolds admitting KE metrics are finite in each dimension.

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Kähler–Ricci soliton: $\mathrm{Ric}(\omega) - L_{J\xi}\omega = \lambda\omega$

cf. normalized Kähler–Ricci flow: $\mathrm{Ric}(\omega_t) - \lambda\omega_t = \dot{\omega}_t$

- Every toric Fano orbifold admits a KRs.
- Every horospherical Fano manifold admits a KRs, which includes infinitely many Fano manifolds with $\rho(X) = 1$ & no KE metrics.

Tian–Zhu's volume minimization and modified K-stability

For a Fano manifold $X \circlearrowleft T$ and $\xi \in \mathfrak{t}$, the modified Futaki invariant $\mathrm{Fut}_\xi \in \mathfrak{t}^\vee$ is defined by

$$\mathrm{Fut}_\xi(\eta) := -\int_X \theta_\eta e^{\theta_\xi} \omega^n,$$

where $\theta_\xi = -2\mu_\xi$ for $[\omega + \mu] \in c_1^T(X)$. Independent of $\omega \in c_1(X)$.

$$\boxed{\exists \text{ KRs} \Rightarrow \mathrm{Fut}_\xi = 0}$$

Proposition (Tian–Zhu)

Regardless of the existence of KRs, $\exists ! \xi \in \mathfrak{t}$ satisfying $\mathrm{Fut}_\xi = 0$.

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Modified K-(semi)stability of X with respect to ξ: For a T-equivariant special degeneration $\mathcal{X} = (\mathcal{X}/\mathbb{C}, -K_{\mathcal{X}/\mathbb{C}})$, the modified Futaki invariant of $\mathcal{X}$ is given by

$$\mathrm{Fut}_\xi(\mathcal{X}) := -\int_{\mathcal{X}_0} \theta_\eta e^{\theta_\xi}.$$

Uniqueness and Existence

Theorem (Tian–Zhu)

Kähler–Ricci solitons on a Fano manifold are unique modulo $\mathrm{Aut}^0(X)$ (and up to scaling).

Theorem (Berman–Witt-Nyström)

If a Fano manifold admits a Kähler–Ricci soliton, then X is modified K-polystable.

Theorem (Datar–Székelyhidi)

A Fano manifold X admits a Kähler–Ricci soliton if (and only if) X is modified K-polystable.

Moduli space of KRs Fano manifolds

Berman–Witt-Nyström

$+$

Moment map picture for KRs + Uniqueness of polystable degeneration

$\Downarrow$

Theorem (I. '19, Adv. Math.)

Fano manifolds with KRs form a complex analytic moduli space.

Moduli space of KRs Fano manifolds

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Theorem (I. '19, Adv. Math.)

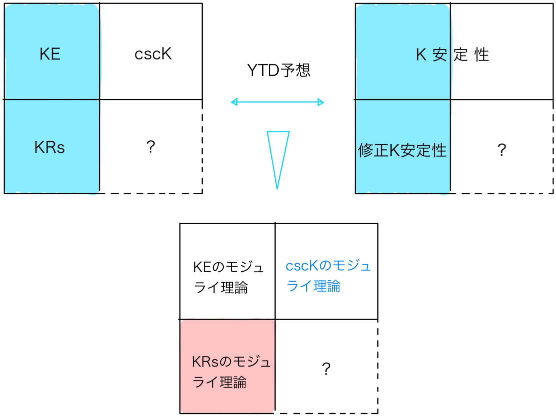
Fano manifolds with KRs form a complex analytic moduli space.

$\downarrow$

Theorem (Dervan–Naumann)

CscK manifolds form a complex analytic moduli space.

Summary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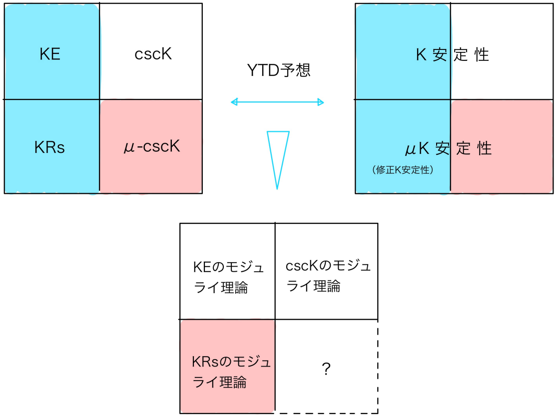
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3. Introduction to μ-cscK – special features

μ-scalar curvature: definition

$X \circlearrowleft T \cong (U(1))^{\times k}$: holomorphic action on a complex (Kähler) manifold

μ-scalar curvature

For $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{t}$ and a T-equivariant Kähler metric $\omega + \mu$, we put

$$s_\xi^\lambda(\omega) := (s(\omega) - \Delta\mu_\xi) - (\Delta\mu_\xi + 2|\nabla\mu_\xi|^2) + 2\lambda\mu_\xi$$
$$= (s(\omega) + \bar{\square}\theta_\xi) + (\bar{\square}\theta_\xi - (J\xi)\theta_\xi) - \lambda\theta_\xi.$$

Definition

A Kähler metric ω is a μ_ξ^λ-cscK metric if $s_\xi^\lambda(\omega)$ is constant.

- Independent of the choice of the moment map μ for ω.
- μ_0^λ-cscK metric $\iff$ cscK metric.
- When $\lambda\omega \in 2\pi c_1(X)$,
  μ_ξ^λ-cscK metric $\iff$ Kähler-Ricci soliton: $\mathrm{Ric}(\omega) - L_{J\xi}\omega = \lambda\omega$.

μ-scalar curvature: "naturality" of the concept

Recall

Donaldson-Fujiki moment map picture

(M,ω): C^∞-symplectic manifold. Scalar curvature gives a moment map on $\mathcal{J}(M,\omega)$. Namely, the map $\mathcal{S} : \mathcal{J}(M,\omega) \to \mathrm{Lie}(\mathrm{Ham}(M,\omega))^\vee$ given by

$$\langle \mathcal{S}(J), f \rangle = \int_M (s(g_J) - \bar{s}) f \omega^n$$

is a moment map for the symplectic structure Ω on $\mathcal{J}(M,\omega)$:

$$\Omega_J(A,B) = \int_M (JA,B)_{g_J} \omega^n.$$

μ-scalar curvature: "naturality" of the concept

Put

$$\bar{s}^\lambda_\xi := \int_M s^\lambda_\xi(g_J)\ e^{\theta_\xi}\omega^n \Big/ \int_M e^{\theta_\xi}\omega^n.$$

Proposition (Moment map picture for μ-cscK, I. '19, Lahdili '19)

$(M, \omega) \circlearrowleft T$: C^∞-symplectic manifold. μ-scalar curvature gives a moment map on $\mathcal{J}_T(M, \omega)$. Namely, the map $\mathcal{S}^\lambda_\xi : \mathcal{J}_T(M, \omega) \to \mathrm{Lie}(\mathrm{Ham}_T(M, \omega))^\vee$ given by

$$\langle \mathcal{S}^\lambda_\xi(J), f \rangle = \int_M (s^\lambda_\xi(g_J) - \bar{s}^\lambda_\xi) f\ e^{\theta_\xi}\omega^n$$

is a moment map for the symplectic structure Ω_ξ on $\mathcal{J}_T(M, \omega)$:

$$\Omega_{\xi,J}(A, B) = \int_M (JA, B)_{g_J}\ e^{\theta_\xi}\omega^n.$$

μ-Futaki invariant and μ-entropy

For $\xi \in \mathfrak{t}$, the μ-Futaki invariant $\mathrm{Fut}^\lambda_\xi \in \mathfrak{t}^\vee$ is defined by

$$\mathrm{Fut}^\lambda_\xi(\eta) := -\langle \mathcal{S}^\lambda_\xi(J), \theta_\eta \rangle = -\int_X (s^\lambda_\xi(\omega) - \bar{s}^\lambda_\xi)\theta_\eta e^{\theta_\xi}\omega^n \Big/ \int_X e^{\theta_\xi}\omega^n.$$

Independent of $\omega \in [\omega]$ and $\mu : X \to \mathfrak{t}^\vee$.

$$\boxed{\exists\ \mu^\lambda_\xi\text{-cscK metric in } [\omega] \Rightarrow \mathrm{Fut}^\lambda_\xi = 0}$$

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$$\boxed{\exists\ \mu^\lambda_\xi\text{-cscK metric in } [\omega] \Rightarrow \mathrm{Fut}^\lambda_\xi = 0}$$

$$\boldsymbol{\mu}^\lambda(-2\xi) = -\frac{\int_X (s + \bar{\square}\theta_\xi) e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} + \lambda \frac{\int_X (n + \theta_\xi) e^{\theta_\xi}\omega^n}{\int_X e^{\theta_\xi}\omega^n} - \lambda \log \int_X e^{\theta_\xi}\frac{\omega^n}{n!}$$

Also independent of $\omega \in [\omega]$ and $\mu : X \to \mathfrak{t}^\vee$.

$$\boxed{\mathcal{D}_\xi \boldsymbol{\mu}^\lambda = \mathrm{Fut}^\lambda_\xi}$$

Properties of μ^λ-entropy

Theorem (I. '19)

- (Existence) Critical points of $\boldsymbol{\mu}^\lambda$ always exist regardless of the existence of μ_ξ^λ-cscK metrics in $[\omega]$.
- (Uniqueness/phase transition) For each $X \circlearrowleft T$,

$$\lambda_{\mathrm{freeze}} := \sup \left\{ \lambda \in \mathbb{R} \;\middle|\; \begin{array}{c} \boldsymbol{\mu}^{\lambda'} \text{ admits a unique} \\ \text{critical point for every } \lambda' \leq \lambda \end{array} \right\}$$

is always **finite** (never $\pm\infty$).

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  is always **finite** (never $\pm\infty$).
- (Extremal limit) Let ξ^λ be the unique critical point of $\boldsymbol{\mu}^\lambda$ for $\lambda < \lambda_{\mathrm{freeze}}$. Then $\lambda \xi^\lambda$ converges to the extremal vector field ξ_{ext} as λ tends to $-\infty$.

The extremal vector field ξ_{ext} is the unique critical point of

$$\int_X (\hat{s}(\omega) - \hat{\theta}_\xi)^2 \omega^n - \int_X \hat{s}^2 \omega^n. \quad (\hat{f} := f - \int_X f\omega^n / \int_X \omega^n)$$

Behavior of μ^λ-entropy: typical example

We can explicitly compute $\boldsymbol{\mu}^\lambda$ of $\mathbb{C}P^1 \circlearrowleft U(1)$. For $\xi = x.\eta \in \mathrm{u}(1)$,

$$\boldsymbol{\mu}^\lambda_{-K_{\mathbb{C}P^1}}(\xi) = 2\pi\Big(1 - \frac{x}{\tanh x}\Big) + \lambda\Big(-1 + \frac{x}{\tanh x}\Big) - \lambda \log \frac{2\sinh x}{x}.$$

- $\lambda_{\mathrm{freeze}}(\mathbb{C}P^1, -K_{\mathbb{C}P^1}) = 4\pi$.

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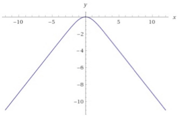
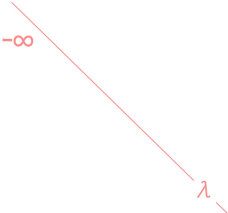
- $\lambda_{\mathrm{freeze}}(\mathbb{C}P^1, -K_{\mathbb{C}P^1}) = 4\pi$.
- There actually exists a μ^λ_ξ-cscK metric for exactly two $\xi \neq 0$ (and $\xi = 0$) when $\lambda > 4\pi$.
- As $\lambda \to \infty$, the family of (non-cscK) μ^λ-cscK metrics ω_λ admits a family of diffeomorphisms $f_\lambda : D^2 \to \mathbb{C} \subset \mathbb{C}P^1$ from a disk of radius $\sqrt{2}$ such that $f_\lambda^* \omega_\lambda$ converges to the flat metric. (while f_λ does not converge to a diffeomorphism onto $\mathbb{C}$.)

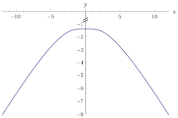
Graphs of $\mu^{\lambda}(x.\ \eta)/2\pi$

plot $1 - \frac{x}{\tanh(x)}$ $\quad \lambda = 0$

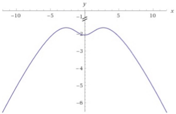




plot $-1 + \frac{x}{\tanh(x)} - 2\log\left(\frac{2\sinh(x)}{x}\right)$ $\quad \lambda_{\text{freeze}} = 4\pi$



plot $-2 + 2 \times \frac{x}{\tanh(x)} - 3\log\left(\frac{2\sinh(x)}{x}\right)$ $\quad \lambda = 6\pi$



Closedness of framework

- (Scaling) ω: μ^{λ}_{ξ}-cscK metric $\Rightarrow$ $c^{-1}\omega$: $\mu^{c\lambda}_{c\xi}$-cscK metric.
- (Product) (X, ω_X), (Y, ω_Y): μ^{λ}-cscK metrics with the same λ and with respect to vector fields ξ_X, ξ_Y, respectively $\Rightarrow$ $(X \times Y, \omega_X \oplus \omega_Y)$: μ^{λ}-cscK metric with respect to $\xi_X \oplus \xi_Y$.

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- (Perturbation of λ) $\exists$ μ^λ-cscK metric in $[\omega]$ with $\lambda < \lambda_1$ for the first eigenvalue λ_1 of $\Delta - \nabla\mu_\xi$ $\Rightarrow$ $\exists$ $\mu^{\tilde{\lambda}}$-cscK metric in the same $[\omega]$ for $\tilde{\lambda} \in (\lambda - \epsilon, \lambda + \epsilon)$.
- (Perturbation of Kähler class) We can also perturb Kähler classes under the above condition.

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- (Perturbation of Kähler class) We can also perturb Kähler classes under the above condition.
- (Propagation) $\exists$ extremal metric in $[\omega]$ $\Rightarrow$ μ^λ-cscK metric in the same $[\omega]$ for $\lambda \ll \lambda_{\mathrm{freeze}}$ and also for $\lambda \gg \lambda_{\mathrm{freeze}}$.
- (Uniqueness) Convexity of weighted Mabuchi functional shows that μ^λ-cscK metrics are unique for $\lambda < \lambda_{\mathrm{freeze}}$. (Lahdili)

Calabi ansatz on $\mathbb{P}_\Sigma(L \oplus \mathcal{O})$

Consider the ruled manifold $\mathbb{P}_\Sigma(L \oplus \mathcal{O})$ for a positive L on an algebraic curve Σ. Let F denote a fibre and $B = \{(x, (0:1)) \mid x \in \Sigma\}$ denote the section at infinity. The Kähler cone is given by

$$\{aF + bB \mid b > 0, \frac{a}{b} > -\frac{\deg L}{2}\}.$$

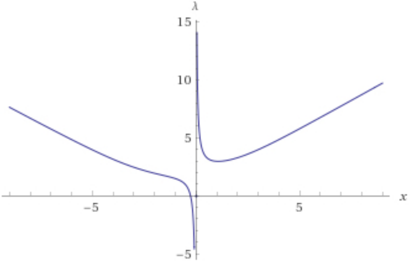
Theorem (I. '20)

Every Kähler class in the cone $\{aF + bB \mid a, b > 0\}$ admits a μ^λ-cscK metric for every $\lambda \geq 0$ (for some ξ).

For $g(\Sigma) \geq 2$ and small $\frac{a}{b}$, the Kähler class $aF + bB$ does not admit extremal metrics. (rel. K-unstable $\Rightarrow$ no μ^λ-cscK metrics for $\lambda \ll 0$.)

Calabi ansatz on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} = \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O}(1) \oplus \mathcal{O})$

- The anti-canonical class $-K_X$ of $X = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} = \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O}(1) \oplus \mathcal{O})$ admits both KRs and extremal metric (no cscK metrics).
- Calabi ansatz: $\exists$ μ^λ-cscK metrics for every $\lambda \in \mathbb{R}$ (with a negative $x_\lambda = \xi^\lambda/\eta = (6/11) \cdot \xi^\lambda/\xi_{\mathrm{ext}}$) connecting KRs and extremal metric.



(x from −9 to 9)

- We can see $2.9 \times 2\pi < \lambda_{\mathrm{freeze}} < 3 \times 2\pi$.

4. How to formulate μK-stability? – equivariant calculus

Remarks

Recall

$$DF(\mathcal{X},\mathcal{L}) := (K_{\bar{\mathcal{X}}/\mathbb{C}P^1}.\mathcal{L}^{\cdot n}) - \frac{n}{n+1}\frac{(K_X.L^{\cdot(n-1)})}{(L^{\cdot n})}(\bar{\mathcal{L}}^{\cdot(n+1)}).$$

- In moduli context, test configurations appear by pulling back the universal family $\mathcal{U}$ on Hilb along $\mathbb{C}^\times$-equivariant morphisms $\mathbb{C} \to \mathrm{Hilb}$, which is not necessarily normal.
- We can define DF also for non-normal $(\mathcal{X},\mathcal{L})$ by using homology Todd class $\tau(\mathcal{O}_{\bar{\mathcal{X}}}) = [\bar{\mathcal{X}}] - \frac{1}{2}\kappa_{\bar{\mathcal{X}}} + \cdots \in A_{\mathbb{Q}}(\bar{\mathcal{X}})$ instead of $K_{\bar{\mathcal{X}}}$. (cf. Fulton, Edidin-Graham)
- The intersection formula is useful to see the behavior of $DF(\mathcal{X},\mathcal{L})$ along the normalization and resolutions of $\mathcal{X}$.

Recall

$$\boldsymbol{\mu}^\lambda = -\frac{\int_X (\mathrm{Ric} + \bar{\Box}\mu)e^{\omega+\mu}}{\int_X e^{\omega+\mu}} + \lambda\frac{\int_X (\omega+\mu)e^{\omega+\mu}}{\int_X e^{\omega+\mu}} - \lambda \log \int_X e^{\omega+\mu}$$

μFutaki invariant of test configuration

(X, L): T-equivariant polarized manifold (scheme)
For $\xi \in \mathfrak{t}$, we define the μ^0_ξ-Futaki invariant of a T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$ by the following equivariant intersection formula:

$$\mathrm{Fut}^0_\xi(\mathcal{X}, \mathcal{L}) := 4\pi \frac{\mathrm{Ev}_\xi \Big((\kappa^T_{\bar{\mathcal{X}}/\mathbb{C}P^1} . e^{\bar{\mathcal{L}}_T}) \cdot (e^{L_T}) - (\kappa^T_X . e^{L_T})(e^{\bar{\mathcal{L}}_T}) \Big)}{(\mathrm{Ev}_\xi (e^{L_T}))^2} \in \mathbb{R}.$$

When $\mathcal{X}$ is smooth, this is equivalent to:

$$-2 \frac{\int_{\bar{\mathcal{X}}} (\mathrm{Ric}^{\mathrm{rel}}_{\tilde{\Omega}} + \bar{\square}_{\tilde{\Omega}} \tilde{\Theta}_\xi) e^{\Omega + \Theta_\xi} \int_X e^{\Omega + \Theta_\xi} - \int_X (\mathrm{Ric}_\omega + \bar{\square} \theta_\xi) e^{\omega + \theta_\xi} \int_{\bar{\mathcal{X}}} e^{\Omega + \Theta_\xi}}{(\int_X e^{\omega + \theta_\xi})^2},$$

where $\mathrm{Ric}^{\mathrm{rel}}_{\tilde{\Omega}} = \mathrm{Ric}(\tilde{\Omega}) - \pi^* \mathrm{Ric}(\omega_{\mathbb{C}P^1})$ for some metrics $\tilde{\Omega}, \omega_{\mathbb{C}P^1}$ on $\bar{\mathcal{X}}, \mathbb{C}P^1$.
We can similarly define

$$\mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}) := \mathrm{Fut}^0_\xi(\mathcal{X}, \mathcal{L}) + \lambda(\text{equiv. intersection on } \bar{\mathcal{L}}).$$

μK-semistability

A T-polarized manifold is μ_ξ^λK-semistable if Fut_ξ^λ is non-negative for any test configuration.

Theorem (I. '20 + Lahdili '19)

- If $c_1(L)$ admits a μ_ξ^λ-cscK metric, then (X, L) is μ_ξ^λK-semistable.
- If (X, L) is μ^λK-semistable for $\lambda \ll 0$, then (X, L) is relatively K-semistable.

Recall:

- If (X, L) admits and extremal metric, then it admits μ^λ-cscK metrics for $\lambda \ll 0$ and $\lambda \gg 0$.
- Even if (X, L) is μ^λK-semistable for $\lambda \gg 0$, (X, L) may not be relatively K-semistable.

Generalization of CM line bundle

Theorem (I. '20)

For $\lambda \in \mathbb{R}$ and $\xi \in \mathfrak{t}$, there exists a characteristic class $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda$ assigning $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}/B, \mathcal{L}) \in H^2_G(B, \mathbb{R})$ for each $T \times G$-equivariant family of polarized schemes $(\mathcal{X}/B, \mathcal{L})$ over smooth G-variety B which enjoys the following:

1. Naturality: $f^* \mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}/B, \mathcal{L}) = \mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}'/B', \mathcal{L}')$ for

$$\begin{array}{ccc} \mathcal{X}' & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

2. μ-Futaki invariant: $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda(\mathcal{X}/\mathbb{C}, \mathcal{L}) = \mathrm{Fut}^\lambda_\xi(\mathcal{X}, \mathcal{L}).\eta^\vee$ for any T-equivariant test configuration $(\mathcal{X}, \mathcal{L})$
3. CM line bundle: $\mathcal{D}_0 \boldsymbol{\mu}^\lambda_G(\mathcal{X}/B, \mathcal{L}) = -\frac{4\pi}{(L^{\cdot n})} c_1^G(CM(\mathcal{X}/B, \mathcal{L}))$

Application

Combining with Chen–Sun's deep analysis on Kähler–Ricci flow and the analytic openness of μK-semistable locus established in the previous work, we can show that μK-semistable locus of a $\mathbb{Q}$-Fano family is Zariski open on the base. Then we get the following result on algebraicity.

Theorem (I. '20)

The moduli space of Fano manifolds with KRs is an algebraic space.

I also have a plan for compactifying the moduli space (in progress).

Idea of construction – the case $\lambda = 0$ (to economize space)

Recall the following expression of μ^λ-entropy:

$$\boldsymbol{\mu}^0 = -\frac{\int_X (\mathrm{Ric} + \bar{\square}\mu) e^{\omega+\mu}}{\int_X e^{\omega+\mu}}.$$

Both $\int_X (\mathrm{Ric} + \bar{\square}\mu) e^{\omega+\mu}$ and $\int_X e^{\omega+\mu}$ are the integration of equivariant forms. In other words, we can regard these as the pushforward of the equivariant cohomology classes

$$K_X^T \frown e^{L_T}, e^{L_T} \in \hat{H}_T(X, \mathbb{R}) := \prod_{k=0}^{\infty} H_T^{2k}(X, \mathbb{R})$$

along $p : X \to \mathrm{pt}$, which are elements of $\hat{H}_T(\mathrm{pt}, \mathbb{R}) \cong \prod_{k=0}^{\infty} S^k \mathfrak{t}^\vee$ and are the Taylor expansion (at $0 \in \mathfrak{t}$) of the functionals $\int_X (\mathrm{Ric} + \bar{\square}\mu) e^{\omega+\mu}$, $\int_X e^{\omega+\mu}$ on $\mathfrak{t}$. For a G-equivariant polarized family $(\mathcal{X}/B, \mathcal{L})$, we put

$$\boldsymbol{\mu}^0_{\mathcal{X}/B,\mathcal{L}} := 2\pi \frac{\pi_*(\kappa_{\mathcal{X}/B}.e^{\mathcal{L}})}{\pi_*(e^{\mathcal{L}})} \in \hat{H}_G(B, \mathbb{R}).$$

Idea of construction – Sketch of equivariant calculus

1. (Differential at ξ along G) For $\xi \in \mathfrak{t}$, we introduce a differential operation
$$\mathcal{D}_\xi : H^\omega_{T\times G}(B,\mathbb{R}) \to H^2_G(B,\mathbb{R})$$
for some subring $H^\omega_{T\times G}(B,\mathbb{R})$ of $\hat{H}_{T\times G}(B,\mathbb{R})$ where T acts on B trivially. When $G = \{1\}$ and $B = \mathrm{pt}$, $H^\omega_{T\times G}(B,\mathbb{R})$ is identified with the ring of real analytic functions on $\mathfrak{t}$.
2. (Convergence result) For $T \times G$-equivariant polarized family $(\mathcal{X}/B,\mathcal{L})$, we can show that $\boldsymbol{\mu}^\lambda_{\mathcal{X}/B,\mathcal{L}}$ is in $H^\omega_{T\times G}(B,\mathbb{R})$, using Cartan model of equivariant deRham current homology. The element $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{\mathcal{X}/B,\mathcal{L}} \in H^2_G(B,\mathbb{R})$ is what we want!
3. (Equivariant Grothendieck-Riemann-Roch) Naturality and the identification with CM line bundle comes from the equivariant Grothendieck-Riemann-Roch theorem by Edidin-Graham.
4. (Localization formula) Using the equivariant localization formula, we can see $\mathcal{D}_\xi \boldsymbol{\mu}^\lambda_{\mathcal{X}/\mathbb{C},\mathcal{L}} = \mathrm{Fut}^\lambda_\xi(\mathcal{X},\mathcal{L}).\eta^\vee$.

Thank you for listening!

5. Bonus talk: Beyond YTD conjecture

Beyond YTD conjecture: Perelman's μ-entropy

Define $\check{\boldsymbol{\mu}}^\lambda : \mathcal{H}(X, [\omega]) \to \mathbb{R}$ by

$$\check{\boldsymbol{\mu}}^\lambda(\omega) := \sup_{f \in C^\infty(X)} \check{W}^\lambda(\omega, f)$$

where

$$\check{W}^\lambda(\omega, f) := \int_X (s(\omega) + \bar{\square} f - \lambda(n+f)) e^f \omega^n \Big/ \int_X e^f \omega^n - \lambda \log \left(\frac{1}{n!} \int_X e^f \omega^n \right).$$

Theorem (to appear)

The functional $\check{\boldsymbol{\mu}}^\lambda$ is smooth when $\lambda \leq 0$ (mountain pass theorem).

- Its critical points of $\check{\boldsymbol{\mu}}^\lambda$ are precisely μ^λ-cscK metrics (for some ξ).
- The critical points are global minimizer.
- A Kähler metric ω is μ_ξ^λ-cscK metric iff $\check{\boldsymbol{\mu}}^\lambda(\omega) = \boldsymbol{\mu}^\lambda(\xi)$.

Beyond YTD conjecture: Non-archimedean μ-entropy

$$\boldsymbol{\mu}^\lambda_{NA} : \mathcal{H}^{NA}(X, L) \to \mathbb{R}$$

Theorem (Essentially proved in I. '20)

If a 'test configuration' $\phi_\xi \in \mathcal{H}^{NA}(X, L)$ associated to a vector field maximizes $\boldsymbol{\mu}^\lambda_{NA}$, then (X, L) is μ^λ_ξK-semistable.

Conjecture

$$\sup \boldsymbol{\mu}^\lambda_{NA} \leq \inf \check{\boldsymbol{\mu}}^\lambda$$

- X admits a μ^λ-cscK metric ω
  $\iff \check{\boldsymbol{\mu}}^\lambda(\omega) = \inf \check{\boldsymbol{\mu}}^\lambda \iff \check{\boldsymbol{\mu}}^\lambda(\omega) = \sup \boldsymbol{\mu}^\lambda_{NA}$.
- X is μ^λK-semistable with respect to ξ
  $\iff \boldsymbol{\mu}^\lambda_{NA}(\phi_\xi) = \sup \boldsymbol{\mu}^\lambda_{NA} \iff \boldsymbol{\mu}^\lambda_{NA}(\phi_\xi) = \inf \check{\boldsymbol{\mu}}^\lambda$.

Remarks: Han-Li's result when $L = -K_X$

- Berman–Witt-Nyström proved that $(X, -K_X)$ is $\mu^{2\pi}$K-polystable with respect to special degenerations if X admits a KRs ($= \mu^{2\pi}$-cscK metric).
- Recently, J. Han and C. Li introduced G-uniform g-Ding stability '$\boldsymbol{D}_g^{NA}(\phi) \geq \gamma \cdot \boldsymbol{J}_g^{NA}(\phi)$' and proved the equivalence of G-uniform g-Ding stability of $(X, -K_X)$ for 'maximal' G is equivalent to the existence of KR g-soliton.
- They also show that the (G-uniform) g-Ding stability of $(X, -K_X)$ is equivalent to that with respect to special degenerations, using MMP with scaling. The proof works also for $\boldsymbol{M}_g^{NA}$. (I guess it works also for Fut_ξ^λ.)
- g-Mabuchi stability for $g = e^{\langle \xi, - \rangle}$ must be equivalent to μ_ξ^λK-stability. (λ is determined from ξ.)
- Thus, $\exists$ KRs on $X \iff (X, -K_X)$ is $\mu^{2\pi}$K-polystable.